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The fractional stochastic heat equation on the circle: Time regularity and potential theory

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Abstract

We consider a system of d linear stochastic heat equations driven by an additive infinite-dimensional fractional Brownian noise on the unit circle S^1. We obtain sharp results on the Hölder continuity in time of the paths of the solution $u = \{u(t,x)\}_{t\in\mathbb{R}_+, x\in S^1}$. We then establish upper and lower bounds on hitting probabilities of u, in terms of the Hausdorff measure and Newtonian capacity respectively.

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Keywords: Hitting probabilities; Stochastic heat equation; Fractional Brownian motion; Path regularity

1. Introduction and main results

We consider a system of d stochastic heat equations on the unit circle driven by an infinite-dimensional fractional Brownian motion B^H with Hurst parameter $H \in (0, 1)$. That is,

$$\frac{\partial u_i}{\partial t}(t,x) = \Delta_x u_i(t,x) + \frac{\partial B_i^H}{\partial t}(t,x), \quad t > 0, x \in S^1, \tag{1.1}$$

with initial condition $u_i(0,x) = 0$, for all $i = 1, \ldots, d$. Here Δ_x is the Laplacian on S^1 and B^H a centered Gaussian field on $\mathbb{R}_+ \times S^1$ defined, for all $x, y \in S^1$ and $s, t \geq 0$, by its covariance

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structure

$$\mathrm{E}\left[B_i^H(t,x)\,B_j^H(s,y)\right] = 2^{-1}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right)Q(x,y)\,\delta_{i,j},$$

where Q is an arbitrary covariance function on S^1 and $\delta_{i,j}$ is the Kronecker symbol. To simplify our study, we assume that B^H is spatially homogeneous and separable in space; therefore $Q(x,y)$ depends only on the difference $x-y$, and we denote it abusively $Q(x-y)$.

Note that because Q is positive definite, there exists a sequence of non-negative real numbers $\{q_n\}_{n\in\mathbb{N}}$ such that

$$Q(x-y) = \sum_{n\in\mathbb{N}} q_n \cos(n(x-y)).$$

This expression may be only formal for certain choices of the sequence $\{q_n\}_n$, as these pointwise values may explode, but this Fourier representation is always relevant if one allows Q to be a Schwartz distribution. Examples will be given below where $Q(0)$ is infinite while all other values are finite (Riesz-kernel case); another, also with $Q(0)=\infty$, will show that Q may not be equal to its Fourier series at any point (fractional noise case for small Hurst parameter), but still allows a solution to (1.1). Any case with $Q(0)=\infty$ denotes a distribution-valued noise B^H in space, for which the notation $B^H(t,x)$ is only formal in the parameter x.

The infinite-dimensional fractional Brownian motion B^H, with values in $L^2(S^1)$, can also be defined using its random Fourier series decomposition as

$$B_i^H(t,x) = \sum_{n=0}^{\infty}\sqrt{q_n}\left(\cos(nx)\,\beta_{i,n}^H(t) + \sin(nx)\,\beta_{i,n}'^H(t)\right),$$

where the sequences $\{\beta_{i,n}^H\}_{n\in\mathbb{N}}$ and $\{\beta_{i,n}'^H\}_{n\in\mathbb{N}}$, $i\in\{1,\ldots,d\}$, are independent and each formed of independent one-dimensional standard fractional Brownian motions. Then, the "mild" or "evolution" solution of the stochastic integral formulation of Eq. (1.1) is given by the evolution convolution

$$u_i(t,x) = \sum_{n=0}^{\infty}\sqrt{q_n}\left(\cos(nx)\int_0^t \mathrm{e}^{-n^2(t-s)}\beta_{i,n}^H(\mathrm{d}s) + \sin(nx)\int_0^t \mathrm{e}^{-n^2(t-s)}\beta_{i,n}'^H(\mathrm{d}s)\right). \tag{1.2}$$

[16] showed when such a solution exists, and more specifically, that the necessary and sufficient condition for existence of (1.2) in $L^2\left(\Omega\times[0,T]\times S^1\right)$ (cf. [16, Corollary 1]) is

$$\sum_{n=1}^{\infty} q_n n^{-4H} < \infty.$$

The study of stochastic PDEs similar to (1.2), that is, using fractional Brownian noise in time, is a fairly recent endeavor. Preceding [16] was the particular case where B^H is white in continuous space $\mathbb{R}$ (which would correspond to our case when $q_n = 1$ for all n) which was studied in [7], where the solution exists if and only if $H > \frac{1}{4}$. The topic is very active today; some recent results in directions tangential to ours include: [11] (evolution equations), [8] (solutions of semilinear equations), [13] (on the stochastic wave equation with fBm) and [2] (existence of the stochastic heat equation with colored noise in $\mathbb{R}^d$ and $H > 1/2$.) Our article is closer to the line of [16]; in comparison with this and other papers concerned with regularity (such as [18], see below), our article is the first to manage sharp time-regularity results when $H < 1/2$.

This article goes beyond regularity issues, however. Herein we develop a potential theory for the solution to the system of equation (1.1). In particular, given $A \subset \mathbb{R}^d$, we want to determine whether the process $\{u(t,x),\ t \geq 0,\ x \in S^1\}$ visits, or hits, A with positive probability.

Potential theory for the linear and non-linear stochastic heat equation driven by a space–time white noise was developed in [4,5]. The aim of this paper is to obtain upper and lower bounds on hitting probabilities for the solution of (1.1). For this, following the approach developed in [4], a careful analysis of the moments of the increments of the process $u(t,x)$ is needed. In particular, this will lead us to solve an open question which is the Hölder continuity in time of the solution of (1.1) when $H < \frac{1}{2}$. The Hölder continuity in space for the solution of (1.1) was studied in [17], and the Hölder continuity in time when $H \geq \frac{1}{2}$ is due to [18]. These are generalizations of earlier work done for the stochastic heat equation with time white noise potential: [14,15].

Let us first state, in some detail, the path continuity results we obtain for the solution of the fractional heat equation on the circle (1.1), as these are a valuable immediate consequence of our work. Assume that for all n large enough

$$cn^{4H-2\alpha-1} \leq q_n \leq Cn^{4H-2\alpha-1}, \tag{1.3}$$

for some positive constants c and C and $\alpha \in (0,1]$ with $\alpha \neq 2H$. Our basic quantitative result is the following bounds on the variance of the increments of the solution: for $t_0, T > 0$, for some positive constants c, C, c_{t_0}, C_{t_0}, for all $x, y \in S^1$, and all $s, t \in [t_0, T]$,

$$c_{t_0} |x-y|^{2\alpha} \leq \mathrm{E}\left[\|u(t,x) - u(t,y)\|^2\right] \leq C_{t_0} |x-y|^{2\alpha}$$
$$c\, |t-s|^{\alpha\wedge(2H)} \leq \mathrm{E}\left[\|u(t,x) - u(s,x)\|^2\right] \leq C\, |t-s|^{\alpha\wedge(2H)}.$$

Here and throughout $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

We then immediately get that u is β-Hölder continuous in space for any $\beta \in (0,\alpha)$ and is β-Hölder continuous in time for any $\beta \in (0, \frac{\alpha}{2} \wedge H)$, but not for β equal to the upper values of these intervals. All these results are true for any $H \in (0,1)$. Moreover, these results are sharp for our additive stochastic heat equation (1.1): up to non-random constants, exact moduli of continuity can be found (see the last bullet point below).

Let us consider some examples:

- In the case where B^H is "white noise" in space, then u exists if and only if $H > 1/4$; moreover u is β-Hölder continuous in space for any $\beta \in (0, 2H - \frac{1}{2})$ and β-Hölder continuous in time for any $\beta \in (0, H - \frac{1}{4})$. This follows from the above continuity results because the white noise case is the case $q_n \equiv 1$: the appellation "white" reflects the fact that all spatial Fourier frequencies are equally represented.
- In the case where B^H has a covariance function in space given by the Riesz kernel, that is, $Q(x-y) = |x-y|^{-\gamma}$, $0 < \gamma < 1$, we can prove that q_n is commensurate with $n^{\gamma-1}$. More specifically, we can show that $q_n = n^{\gamma-1} c(n)$ where $c(n)$ is a function bounded between two positive constants, because it can be written as the partial sum of an alternating series with decreasing general term and positive initial term (see Appendix A.1). Therefore, the solution of (1.1) exists if and only if $H > \frac{\gamma}{4}$ and u is β-Hölder continuous in space for any $\beta \in (0, 2H - \frac{\gamma}{2})$ and β-Hölder continuous in time for any $\beta \in (0, H - \frac{\gamma}{4})$. See [13] for the existence, uniqueness and Hölder regularity of the solution of the stochastic wave equation in

$\mathbb{R}$ driven by a multiplicative infinite-dimensional fractional Brownian motion with $H \in (\frac{1}{2}, 1)$ and a space covariance given the Riesz kernel. Note also that the condition $H > \frac{\gamma}{4}$ is the same as the one found in the paper [2] (when $x \in \mathbb{R}^d$ instead of S^1).

- In the case where B^H behaves as "fractional Brownian noise" both in time and space with common Hurst parameter H, then the solution of (1.1) exists if and only if $H > \frac{1}{3}$. Indeed, a wide class of examples fitting the description "fractional Brownian noise" can be defined by assuming that $q_n = c(n)n^{1-2H}$ where the function c only needs to be bounded above and below by positive constants. When $H > 1/2$, if one prefers to work starting from the spatial covariance function Q, one may stipulate that B^H is has a Riesz-kernel covariance, i.e. $Q(x-y) = |x-y|^{2H-2} = |x-y|^{-\gamma}$ with $\gamma = 2(1-H) \in (0,1)$, in which case one is in the situation of the last example, with $q_n = c(n)\, n^{1-2H}$ where the function c was described therein.

  On the other hand, if $H \leq 1/2$, no Riesz-kernel interpretation is possible with $q_n = c(n)n^{1-2H}$ no matter what the choice of c bounded. Appendix A.2 contains another interpretation in this case, which also works for $H \in (1/2, 1)$. This interpretation, which uses a differentiation construction, also allows a justification, for all $H \in (0,1)$, of why we use the appellation "fractional Brownian noise" in the case $q_n = c(n)n^{1-2H}$. In all cases, i.e. for all $H \in (1/3, 1)$, u is β-Hölder continuous in space for any $\beta \in (0, 3H-1)$ and is β-Hölder continuous in time for any $\beta \in (0, \frac{3H-1}{2})$.
- Similarly to the previous example, but more generally, to obtain a B^H that behaves like a fractional Brownian noise with parameter H in time and K in space, we can set $q_n = n^{1-2K}$ (using the same justification as in the Appendix relative to the previous example). This is equivalent to $\alpha = 2H + K - 1$. In other words, the full scale of fractional Brownian noise with $H, K \in (0,1)$ covers the case of Riesz kernels with $\gamma = 2K - 1$, and also extends to the case $K \in (0, 1/2]$ which is not covered by the Riesz kernels. We then get existence of a solution if and only if $2H + K > 1$, and the solution is then β-Hölder continuous in space for any $\beta \in (0, 2H + K - 1)$ and is β-Hölder continuous in time for any $\beta \in (0, \frac{2H+K-1}{2})$.
- In addition to the examples above, which are of Riesz, white noise, or fractional Brownian noise type in space, we mention the classical Ornstein–Uhlenbeck (OU) process, which uses $Q(r) = \exp(-ar)$, i.e. for small r, the squared canonical metric is $2ar + o(r)$, so that the local behavior is very close to standard Brownian motion (note that the corresponding B^H is a bonafide function in space), and corresponds to $q_n \asymp n^{-2}$ (see Lemma 2.1). We note then that hypothesis (1.3) is satisfied with $\alpha = (4H+1)/2$, which can only work if $H < 1/4$. It is the fact that the OU covariance is that of a bonafide function that creates this slight difficulty, but if instead one interprets B^H as an OU noise, i.e as the spatial derivative of a process with spatial OU behavior, then $q_n \asymp 1$ i.e. $\alpha = 2H - 1/2$, and the behavior is like the case of white noise in space (first example above).
- From Gaussian-regularity results such as Dudley's entropy upper bound (see [10]), we can state that if the upper bound in (1.3) holds, then the modulus of continuity random variable

$$\sup_{x,y\in S^1; s,t\in[t_0,T]} \left(\frac{\|u(t,x) - u(t,y)\|}{|x-y|^{\alpha} \log^{1/2}(1 + 1/|x-y|)} + \frac{\|u(t,x) - u(s,x)\|}{|t-s|^{(\alpha/2)\wedge H} \log^{1/2}(1 + 1/|t-s|)} \right)$$

is finite almost surely. Moreover, a (near) converse also holds: if the above random variable (with logarithmic terms moved to the numerators) is finite, then the upper bound in (1.3) holds for some constant $C < \infty$ (see [17, Corollary 1]).

Those examples treat the case where the covariance is "white" or "Riesz type" in space, but other interesting examples such as Bessel type covariance, Poisson kernel, etc. could also be considered.

We now state the results of potential theory that we will prove in this paper. For this, let us first introduce some notation. For all Borel sets $F \subseteq \mathbb{R}^d$ we define $\mathscr{P}(F)$ to be the set of all probability measures with compact support in F. For all $\mu \in \mathscr{P}(\mathbb{R}^d)$, we let $I_\beta(\mu)$ denote the β-dimensional energy of μ; that is,

$$I_\beta(\mu) := \iint \mathrm{K}_\beta(\|x-y\|)\, \mu(\mathrm{d}x)\, \mu(\mathrm{d}y).$$

Here and throughout,

$$\mathrm{K}_\beta(r) := \begin{cases} r^{-\beta} & \text{if } \beta > 0, \\ \log(N_0/r) & \text{if } \beta = 0, \\ 1 & \text{if } \beta < 0, \end{cases} \tag{1.4}$$

where N_0 is a constant whose value will be specified later in the proof of Lemma 4.1.

For all $\beta \in \mathbb{R}$ and Borel sets $F \subset \mathbb{R}^d$, $\mathrm{Cap}_\beta(F)$ denotes the β-dimensional capacity of F; that is,

$$\mathrm{Cap}_\beta(F) := \left[\inf_{\mu \in \mathscr{P}(F)} I_\beta(\mu) \right]^{-1},$$

where $1/\infty := 0$.

Given $\beta \geq 0$, the β-dimensional Hausdorff measure of F is defined by

$$\mathscr{H}_\beta(F) = \lim_{\epsilon \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} (2r_i)^\beta : F \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ \sup_{i \geq 1} r_i \leq \epsilon \right\},$$

where $B(x, r)$ denotes the open (Euclidean) ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. When $\beta < 0$, we define $\mathscr{H}_\beta(F)$ to be infinite.

Let $u(S)$ denote the range of S under the random map $r \mapsto u(r)$, where S is some Borel-measurable subset of $\mathbb{R}_+ \times S^1$.

Theorem 1.1. *Assume hypothesis (1.3). Let $I \subset (0, T]$ and $J \subset [0, 2\pi) \equiv S^1$ be two fixed non-trivial compact intervals. Then for all $T > 0$ and $M > 0$, there exists a finite constant $c_H > 0$ depending on H, M, I and J such that for all compact sets $A \subseteq [-M, M]^d$,*

$$c_H^{-1}\, \mathrm{Cap}_{d-\beta}(A) \leq \mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leq c_H\, \mathscr{H}_{d-\beta}(A).$$

where $\beta := \frac{1}{\alpha} + (\frac{2}{\alpha} \vee \frac{1}{H})$.

Remark 1.2. (a) When B^H is white in time and space, that is, $H = \frac{1}{2}$ and $q_n = 1$ for all n, Theorem 1.1 gives the same hitting probability estimates obtained in [4, Theorem 4.6.].

(b) Because of the inequalities between capacity and Hausdorff measure, the right-hand side of Theorem 1.1 can be replaced by $c\, \mathrm{Cap}_{d-\beta-\eta}(A)$ for all $\eta > 0$ (cf. [9, p. 133]).

We say that a Borel set $A \subseteq \mathbb{R}^d$ is called polar for u if $\mathrm{P}\{u(S) \cap A \neq \varnothing\} = 0$; otherwise, A is called non-polar.

The following results are consequences of Theorem 1.1.

Corollary 1.3. *Assume hypothesis (1.3) and let* $\beta := \frac{1}{\alpha} + (\frac{2}{\alpha} \vee \frac{1}{H})$.

(a) *A (non-random) Borel set* $A \subset \mathbb{R}^d$ *is non-polar for* u *if it has positive* $d - \beta$*-dimensional capacity. On the other hand, if* A *has zero* $d - \beta$*-dimensional Hausdorff measure, then* A *is polar for* u.

(b) *Singletons are polar for* u *if* $d > \beta$ *and are non-polar when* $d < \beta$. *The case* $d = \beta$ *is open.*

(c) *If* $d \geq \beta$, *then*

$$\dim_{\mathrm{H}}(u(\mathbb{R}_+ \times S^1)) = \beta, \quad a.s.$$

Let us consider the same examples as we had for the regularity statements.

- In the case where B^H is white in space, then $\alpha = 2H - \frac{1}{2}$ and $\beta = \frac{6}{4H-1}$.
- In the case where B^H has a covariance function in space given by the Riesz kernel, that is, $Q(x-y) = |x-y|^{-\gamma}$, $0 < \gamma < 1$, then $\alpha = 2H - \frac{\gamma}{2}$ and $\beta = \frac{6}{4H-\gamma}$.
- In the case where B^H is the fractional Brownian noise with Hurst parameter $H > 1/3$ in time and space, then $\alpha = 3H - 1$ and $\beta = \frac{3}{3H-1}$.
- In the case where B^H is the fractional Brownian noise with Hurst parameter H in time and K in space, and $2H + K > 1$, then $\alpha = 2H + K - 1$ and $\beta = \frac{3}{2H+K-1}$.

This paper is organized as follows. In Section 2 we prove the path continuity results of u stated in Section 1 using fractional stochastic calculus. In Section 3 we obtain an upper bound of Gaussian type for the bivariate density of u that will be needed for the proof of Theorem 1.1. Finally, Section 4 is devoted to the proofs of Theorem 1.1 and Corollary 1.3.

Throughout the paper, c_H, C_H will denote universal constants depending on H whose value may change from line to line.

2. Regularity of the solution

We consider the two canonical metrics of u in the space and time parameter, respectively, defined by

$$\delta_t^2(x, y) := \mathrm{E}[\|u(t,x) - u(t,y)\|^2],$$
$$\delta_x^2(s, t) := \mathrm{E}[\|u(t,x) - u(s,x)\|^2],$$

for all $x, y \in S^1$ and $s, t \in \mathbb{R}_+$.

The aim of this section is to obtain upper and lower bounds in terms of the differences $|x - y|$ and $|t - s|$ for the two canonical metrics above. These imply, in particular, the Hölder regularity of u that we have described in detail in the introduction. We begin by introducing some elements of fractional stochastic calculus.

2.1. Elements of fractional stochastic calculus

In this section, we recall, following [12], some elements on stochastic integration with respect to one-dimensional fractional Brownian motion needed for the analysis of the regularity of u in time.

Fix $T > 0$. Let $B^H = (B^H(t), t \in [0, T])$ be a one-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$. That is, B^H is a centered Gaussian process with covariance function given by

$$R(t,s) = \mathrm{E}[B^H(t)B^H(s)] = 2^{-1}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Note that for $H = \frac{1}{2}$, B^H is a standard Brownian motion. Moreover, B^H has the integral representation

$$B^H(t) = \int_0^t K^H(t,s)W(\mathrm{d}s),$$

where $W = (W(t), t \in [0, T])$ is a Wiener process and $K^H(t, s)$ is the kernel defined as

$$K^H(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + s^{\frac{1}{2}-H} F\left(\frac{t}{s}\right), \tag{2.1}$$

where c_H is a positive constant and

$$F(z) = c_H \left(\frac{1}{2} - H\right) \int_0^{z-1} r^{H-\frac{3}{2}} \left(1 - (1+r)^{H-\frac{1}{2}}\right) \mathrm{d}r.$$

From (2.1) we get

$$\frac{\partial K^H}{\partial t}(t,s) = c_H \left(H - \frac{1}{2}\right) (t-s)^{H-\frac{3}{2}} \left(\frac{s}{t}\right)^{\frac{1}{2}-H}. \tag{2.2}$$

It is important to note that $\frac{\partial K^H}{\partial t}$ is positive if $H > 1/2$, but is negative when $H < 1/2$. This negativity causes problems when evaluating the time-canonical metric's lower bound.

We denote by $\mathscr{E}$ the set of step functions on $[0, T]$. Let $\mathscr{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathscr{H}} = R(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \mapsto B_t^H$ can be extended to an isometry between $\mathscr{H}$ and the Gaussian space $\mathscr{H}_1$ associated with B^H. Then $\{B^H(\phi), \phi \in \mathscr{H}\}$ is an isonormal Gaussian process associated with the Hilbert space $\mathscr{H}$. For every element $\phi \in \mathscr{H}$, $B^H(\phi)$ is called the Wiener integral of ϕ with respect to B^H and is denoted

$$\int_0^T \phi(s) B^H(\mathrm{d}s).$$

For every $s < t$, consider the linear operator K^* from $\mathscr{E}$ to $L^2([0, T])$ defined by

$$K_t^*\phi(s) = K^H(t,s)\phi(s) + \int_s^t (\phi(u) - \phi(s)) \frac{\partial K^H}{\partial u}(u,s)\,\mathrm{d}u.$$

When $H > \frac{1}{2}$, since $K^H(t, t) = 0$, this operator has the simpler expression

$$K_t^*\phi(s) = \int_s^t \phi(u) \frac{\partial K^H}{\partial u}(u,s)\,\mathrm{d}u.$$

The operator K^* is an isometry between $\mathscr{E}$ and $L^2([0,T])$ that can be extended to the Hilbert space $\mathscr{H}$. As a consequence, we have the following relationship between the Wiener integral with respect to the fractional Brownian motion B^H and the Wiener integral with respect to the Wiener process W:

$$\int_0^t \phi(s) B^H(\mathrm{d}s) = \int_0^t K_t^* \phi(s) W(\mathrm{d}s),$$

which holds for every $\phi \in \mathscr{H}$, which is true if and only if $K_t^* \phi \in L^2([0,T])$.

Recall also that when $H > \frac{1}{2}$, for any $\phi, \psi \in |\mathscr{H}|$,

$$\mathrm{E}\left[\int_0^t \phi(s) B^H(\mathrm{d}s) \int_0^t \psi(s) B^H(\mathrm{d}s)\right] = H(2H-1) \int_0^t \mathrm{d}s \int_0^t \mathrm{d}u\, \phi(s)\psi(u)|s-u|^{2H-2}. \tag{2.3}$$

Here the notation $|\mathscr{H}|$ designates the set of all functions $\phi \in L^2([0,T])$ such that the quantity on the right-hand side of formula (2.3) is finite for $\psi = \phi$. The reader can also consult the original work [1] for more details.

2.2. Space regularity

The next lemma gives a precise connection between a generic condition of type (1.3) and the Fourier expansion of a canonical metric for a homogeneous Gaussian field on the circle.

Lemma 2.1. *Let Y be a homogeneous, centered and separable Gaussian field on S^1 with canonical metric $\delta(x,y) = \delta(x-y)$ for some univariate function δ. Then, there exists a sequence of non-negative real numbers $\{r_n\}_{n\in\mathbb{N}}$ such that for any $r \in S^1$,*

$$\delta^2(r) = 2\sum_{n=1}^{\infty} r_n (1-\cos nr). \tag{2.4}$$

Moreover, if there exist constants c and C positive, and $\alpha \in (0,1]$, such that for all n large enough,

$$cn^{-2\alpha-1} \le r_n \le Cn^{-2\alpha-1}, \tag{2.5}$$

then for all r close enough to 0,

$$\sqrt{k_\alpha c}\, r^\alpha \le \delta(r) \le \sqrt{K_\alpha C}\, r^\alpha, \tag{2.6}$$

where k_α and K_α are constants depending only on α. More specifically, the upper bound (resp. lower bound) in (2.5) implies the upper bound (resp. lower bound) in (2.6).

Proof. We start proving (2.4). Let $C(x,y)$ denote the covariance function of Y, that is, for any $x, y \in S^1$,

$$\mathrm{E}[Y(x)Y(y)] = C(x,y),$$

where C depends only on the difference $x-y$. Because C is positive definite, it holds that there exists a sequence of non-negative real numbers $\{r_n\}_{n\in\mathbb{N}}$ such that

$$C(x,y) = \sum_{n\in\mathbb{N}} r_n \cos(n(x-y)).$$

Hence, for any $r \in S^1$,

$$\delta^2(r) = \mathrm{E}[(Y(0) - Y(r))^2] = 2\sum_{n=1}^{\infty} r_n (1 - \cos nr).$$

This proves (2.4).

We now prove the second statement of the lemma. We begin by proving the upper bound statement. Assuming that the upper bound of (2.5) holds for all $n > n_0 \geq 1$, we restrict r accordingly: we assume $n_0 \leq [1/r]$, that is, $r \leq 1/n_0$. In this case, we immediately get $r^2 \leq r^{2\alpha}$. We write

$$\begin{aligned}
2^{-1}\delta^2(r) &= \sum_{n=1}^{n_0-1} r_n (1-\cos nr) + \sum_{n=n_0}^{[1/r]} r_n (1-\cos nr) + \sum_{n=[1/r]+1}^{\infty} r_n (1-\cos nr) \\
&\leq \max_{n \leq n_0} \{r_n\} \sum_{n=1}^{n_0-1} (nr)^2 + \sum_{n=1}^{[1/r]} C n^{-2\alpha-1} (nr)^2 + 2 \sum_{n=[1/r]+1}^{\infty} C n^{-2\alpha-1} \\
&\leq n_0^2 \max_{n\leq n_0} \{r_n\} r^2 + C r^2 \sum_{n=1}^{[1/r]} n^{-2\alpha+1} + 2 \sum_{n=[1/r]+1}^{\infty} C n^{-2\alpha-1} \\
&\leq r^{2-2\alpha} n_0^2 \max_{n \leq n_0} \{r_n\} r^{2\alpha} + C C_\alpha r^2 (1/r)^{-2\alpha+2} + 2 C C'_\alpha (1/r)^{-2\alpha} \\
&\leq 2C \left(C_\alpha + 2C'_\alpha\right) r^{2\alpha},
\end{aligned}$$

provided $r \leq r_1 := \min\left\{1/n_0; C\left(C_\alpha + 2C'_\alpha\right)\left[n_0^2 \max_{n\leq n_0}\{r_n\}\right]^{1/(2-2\alpha)}\right\}$, where C_α and C'_α are constants depending only on α. It is elementary to check that C'_α can be taken as $1/(2\alpha)$. If $\alpha \in (0, 1/2)$, then one checks that C_α can be taken as 1; while if $\alpha \in [1/2, 1]$, and we assume moreover that $r < r_2 := (1 - 2\alpha)^{-1/(2\alpha)}$, then C_α can be taken as α^{-1}. In other words, when $\alpha < 1/2$, we obtain the upper bound of (2.6) for all $r \leq r_1$, with $K_\alpha = 4\left(\alpha^{-1} + 1\right)$, while when $\alpha \in [1/2, 1]$, we obtain the upper bound of (2.6) for all $r \leq \min\{r_1; r_2\}$ with $K_\alpha = 8\alpha^{-1}$. In fact, the formula $K_\alpha = 8\alpha^{-1}$ can be used for both cases.

In order to prove the lower bound on $\delta(r)$, we write instead, still assuming $r \leq 1/n_0$, that

$$\begin{aligned}
2^{-1}\delta^2(r) &= \sum_{n=1}^{\infty} r_n (1-\cos nr) \geq c \sum_{n=n_0}^{\infty} n^{-2\alpha-1} (1-\cos nr) \\
&\geq c \sum_{n=[1/r]+1}^{[\pi/(2r)]} n^{-2\alpha-1} (1-\cos nr) \geq c(1-\cos 1) \sum_{n=[1/r]+1}^{[\pi/(2r)]} n^{-2\alpha-1} \\
&\geq c(1-\cos 1)\left(\frac{\pi}{2r}\right)^{-2\alpha-1} \left(\left[\frac{\pi}{2r}\right] - 1 - \left[\frac{1}{r}\right]\right) \\
&\geq r^{2\alpha} c (1-\cos 1)\left(\frac{\pi}{2}\right)^{-2\alpha}\left(\frac{\pi}{2} - 1 - 2r\right).
\end{aligned}$$

Note here that $1 - \cos 1 > 0.459$ and $\pi/2 - 1 > 0.57$. It is now clear that choosing $r \leq r_0 := \min\{0.035; 1/n_0\}$, we get

$$\delta^2(r) \geq r^{2\alpha} c (1-\cos 1)(\pi/2)^{-2\alpha},$$

which proves the lower bound of (2.6) with $k_\alpha = (1 - \cos 1)(\pi/2)^{-2\alpha}$ for all $r \leq r_0$. The proof of the lemma is complete. □

This lemma can be applied immediately, to find sharp bounds on the spatial-canonical metric of u; the almost-sure continuity results also follow.

Corollary 2.2. *Let $H \in (0,1)$, $t_0 > 0$ and $t \in [t_0, T]$ be fixed. Assume hypothesis (1.3). Then the canonical metric $\delta_t(x-y)$ for $u(t,\cdot)$ satisfies, for all r enough close to 0,*

$$\sqrt{k_\alpha cc(t_0,T,H)}r^\alpha \le \delta_t(r) \le \sqrt{K_\alpha CC(t_0,T,H)}r^\alpha,$$

where k_α and K_α are constants depending only on α, $c(t_0,T,H)$ and $c(t_0,T,H)$ are constants depending only on t_0, T and H and c, C are the constants in (1.3). In particular, $u(t,\cdot)$ is β-Hölder continuous for any $\beta \in (0,\alpha)$. More specifically, up to a non-random constant, the function $r \mapsto r^\alpha \log^{1/2}(1/r)$ is an almost-sure uniform modulus of continuity for $u(t,\cdot)$.

Proof. Let $(\beta^H(t), t \ge 0)$ be a one-dimensional fractional Brownian motion. Let $t_0 > 0$ and $t \in [t_0, T]$ be fixed. From the proof of Theorems 2 and 3 of [16] we deduce that there exist positive constants $c(t_0,T,H)$ and $C(t_0,T,H)$ such that

$$c(t_0,T,H)\, n^{-4H} \le \mathrm{E}\left[\left(\int_0^t \mathrm{e}^{-n^2(t-s)} \beta_n^H(\mathrm{d}s)\right)^2\right] \le C(t_0,T,H)\, n^{-4H}.$$

Thus, appealing to (1.2), we find that for all n sufficiently large,

$$2c(t_0,T,H)\, n^{-4H} q_n(1-\cos(nr)) \le \delta_t^2(r) \le 2C(t_0,T,H)\, q_n n^{-4H}(1-\cos(nr)).$$

Then hypothesis (1.3) and Lemma 2.1 conclude the first result of the corollary.

The second statement of the corollary, which is a repeat of one of the continuity results described in the introduction, is proved using the arguments described therein as well. In fact, a simple application of Dudley's entropy upper bound theorem is sufficient (see [10, Theorem 2.7.1]). We do not elaborate further on this point. □

2.3. Time regularity

We now concentrate our efforts on finding sharp bounds on the time-canonical metric of u. The bounds we find for $H > 1/2$ were essentially already obtained in [18], although the result and its proof were not stated explicitly therein, an omission which we deal with here. When $H < 1/2$, no results were known, either for upper or lower bounds: we perform these calculations from scratch. This portion of our calculations is very delicate. As in the previous section, our new estimates can be used to also derive almost-sure regularity results.

Proposition 2.3. *Let $H \in (0,1)$. Assume hypothesis (1.3). Let $T > 0$, $t_0 \in (0,1]$ and $s, t \in [t_0, T]$ with $|t-s| \le \frac{t_0}{2}$ be fixed. Then the canonical metric $\delta_x(t-s)$ for $u(\cdot,x)$ satisfies for every $x \in S^1$*

$$c_{t_0,T,H}|t-s|^{\alpha\wedge(2H)} \le \delta_x^2(t-s) \le C_{t_0,T,H}|t-s|^{\alpha\wedge(2H)}, \tag{2.7}$$

where $c_{t_0,T,H}$ and $C_{t_0,T,H}$ are positive constant depending only on t_0, T and H. In particular, $u(\cdot,x)$ is β-Hölder continuous for any $\beta \in (0, \frac{\alpha}{2} \wedge H)$.

In particular, $u(\cdot,x)$ is β-Hölder continuous for any $\beta \in (0, \frac{\alpha}{2} \wedge H)$. More specifically, up to a non-random constant, the function $r \mapsto r^{\frac{\alpha}{2}\wedge H} \log^{1/2}(1/r)$ is an almost-sure uniform modulus of continuity for $u(\cdot,x)$.

Proof. The statement on almost-sure continuity is established using the arguments described in the introduction, or simply by applying Dudley's entropy upper bound theorem (see [10, Theorem 2.7.1]). We detail only the proof of (2.7), separating the cases $H > 1/2$ and $H < 1/2$.

Fix $T > 0$, $t_0 \in (0, 1]$ and $s, t \in [t_0, T]$ such that $|t - s| \leq \frac{t_0}{2}$. We assume without loss of generality that $s \leq t$. Following [18, Section 2.1], it yields that

$$\delta_x^2(s,t) = q_0|t-s|^{2H} + \sum_{n=1}^{+\infty} q_n \,\mathrm{E}\left[\left\{\int_0^s (\mathrm{e}^{-n^2(t-r)} - \mathrm{e}^{-n^2(s-r)})\beta_n^H(\mathrm{d}r) + \int_s^t \mathrm{e}^{-n^2(t-r)}\beta_n^H(\mathrm{d}r)\right\}^2\right], \tag{2.8}$$

where $\{(\beta_n^H(t), t \geq 0)\}_{n\geq 1}$ is a sequence of fractional Brownian motions.

In order to bound the last expectation we consider two different cases:

Case 1: $H \geq \frac{1}{2}$. In [18, (15)] it is proved that $\delta_x^2(s,t)$ is bounded above and below by

$$q_0|t-s|^{2H} + \sum_{n^2(t-s)>1} \frac{c_H q_n}{n^{4H}} + \sum_{n^2(t-s)\leq 1} C_H q_n |t-s|^{2H}.$$

Taking q_n and $\alpha \in (0, 1]$ from hypothesis (1.3), we obtain that $\delta_x^2(s,t)$ is bounded above and below by

$$c_H(|t-s|^{2H} + |t-s|^{\alpha}) = c_H|t-s|^{\alpha}(|t-s|^{2H-\alpha} + 1).$$

Hence, as $2H \geq 1 \geq \alpha > 0$, the upper and lower bounds of (2.7) follow for $H \geq \frac{1}{2}$.

Case 2: $H < \frac{1}{2}$. We prove the upper and lower bounds of (2.7) separately.

The upper bound. In order to prove the upper bound of (2.7), we start estimating the expectation in (2.8). Using the results in Section 2.2, we have that

$$\mathrm{E}\left[\left(\int_0^s \left(\mathrm{e}^{-n^2(t-r)} - \mathrm{e}^{-n^2(s-r)}\right)\beta_n^H(\mathrm{d}r) + \int_s^t \mathrm{e}^{-n^2(t-r)}\beta_n^H(\mathrm{d}r)\right)^2\right] \leq 2I_1 + I_2 + 2I_3, \tag{2.9}$$

where

$$\begin{aligned} I_1 &:= \int_0^s (K_s^* f(r))^2 \mathrm{d}r, \qquad f(r) = \mathrm{e}^{-n^2(t-r)} - \mathrm{e}^{-n^2(s-r)},\\ I_2 &:= \int_s^t (K_t^* g(r))^2 \mathrm{d}r, \qquad g(r) = \mathrm{e}^{-n^2(t-r)},\\ I_3 &:= \int_0^s (K_t^* g(r) - K_s^* g(r))^2 \mathrm{d}r. \end{aligned} \tag{2.10}$$

We next study each of the terms I_1, I_2 and I_3 separately. We will compute the order of each series $\sum_{n=1}^{+\infty} q_n I_i$, for $i = 1, 2, 3$, for q_n as in (1.3). For this, we will separate the sum into two terms: $n^2(t-s) > 1$ (tail) and $n^2(t-s) \leq 1$ (head). We will then prove that the tails of the series are of order $|t-s|^{\alpha}$ for all the terms, and the heads are of order $|t-s|^{\alpha\wedge(2H)}$ for I_1 and I_3, and of order $(|t-s|^{\alpha\wedge(2H)} + |t-s|^{\alpha})$ for I_2.

We start estimating I_1. We write

$$I_1 \le 2\int_0^s (K(s,r)f(r))^2 \mathrm{d}r + 2\int_0^s \left(\int_r^s (f(u)-f(r))\frac{\partial K}{\partial u}(u,r)\mathrm{d}u\right)^2 \mathrm{d}r$$
$$:= 2I_{1,1} + 2I_{1,2}. \tag{2.11}$$

Using Lemma A.1 and the change of variables $2n^2(s-r)=v$, we have

$$I_{1,1} \le c_H \int_0^s (s-r)^{2H-1} r^{2H-1} (\mathrm{e}^{-n^2(t-r)} - \mathrm{e}^{-n^2(s-r)})^2 \,\mathrm{d}r$$
$$= \frac{c_H}{n^{4H}}(1-\mathrm{e}^{-n^2(t-s)})^2 \int_0^{2n^2 s} \left(s - \frac{v}{2n^2}\right)^{2H-1} v^{2H-1}\mathrm{e}^{-v}\,\mathrm{d}v.$$

By Lemma A.2, it yields

$$I_{1,1} \le \frac{c_H}{n^{4H}}(1-\mathrm{e}^{-n^2(t-s)})^2.$$

We now treat $I_{1,2}$. Using Lemma A.1 and the change of variables $s-r=v$, $s-u=v'$, we have

$$I_{1,2} \le c_H(1-\mathrm{e}^{-n^2(t-s)})^2 \int_0^s \mathrm{d}v \left(\int_0^v \mathrm{d}v'(v-v')^{H-\frac{3}{2}}(\mathrm{e}^{-n^2 v'} - \mathrm{e}^{-n^2 v})\right)^2.$$

By the change of variables $v-v'=u$, we find

$$I_{1,2} \le c_H(1-\mathrm{e}^{-n^2(t-s)})^2 \int_0^s \mathrm{d}v\,\mathrm{e}^{-2n^2 v} \left(\int_0^v \mathrm{d}u\, u^{H-\frac{3}{2}}(\mathrm{e}^{n^2 u}-1)\right)^2.$$

Then using [16, Lemma 2] with $a=n^2$ and $A=H-\frac{1}{2}$, we conclude that

$$I_{1,2} \le \frac{c_H}{n^{4H}}(1-\mathrm{e}^{-n^2(t-s)})^2.$$

Writing $I_{1,1}$ and $I_{1,2}$ together, we get

$$I_1 \le \frac{c_H}{n^{4H}}(1-\mathrm{e}^{-n^2(t-s)})^2.$$

We now separate the sum in (2.8) into two terms, as $n^2(t-s) > 1$ (tail) and $n^2(t-s) \le 1$ (head), and take q_n and $\alpha \in (0,1]$ from hypothesis (1.3). Then we obtain for the tail of the series

$$\sum_{n^2(t-s)>1} q_n I_1 \le c_H \sum_{n^2(t-s)>1} n^{-2\alpha-1} \le c_H|t-s|^{\alpha}.$$

For the head of the series, use the inequality $1-\mathrm{e}^{-x} \le x$, valid for all $x \ge 0$, to get

$$\sum_{n^2(t-s)\le 1} q_n I_1 \le \sum_{n^2(t-s)\le 1} q_n \frac{c(t_0,H)}{n^{4H}}(1-\mathrm{e}^{-n^2(t-s)})^{2H}(1-\mathrm{e}^{-n^2(t-s)})^{2-2H}$$
$$\le c_H|t-s|^{2H} \sum_{n^2(t-s)\le 1} n^{4H-2\alpha-1}$$
$$\le c_H|t-s|^{\alpha\wedge(2H)}.$$

We now bound I_2.

$$I_2 \le 2\int_s^t (K(t,r)g(r))^2 \mathrm{d}r + 2\int_s^t \mathrm{d}r \left(\int_r^t \mathrm{d}u\,(g(u)-g(r))\frac{\partial K}{\partial u}(u,r)\right)^2 := 2I_{2,1} + 2I_{2,2}.$$

Using Lemma A.1 and the change of variables $2n^2(t-r)=u$, we have

$$I_{2,1} \le c_H \int_s^t \mathrm{d}r\,(t-r)^{2H-1} r^{2H-1} \mathrm{e}^{-2n^2(t-r)} = \frac{c_H}{n^{4H}} \int_0^{2n^2(t-s)} \mathrm{d}u\,\left(t-\frac{u}{2n^2}\right)^{2H-1} u^{2H-1}\mathrm{e}^{-u}.$$

Using Lemma A.2, we obtain for the tail of the series

$$\sum_{n^2(t-s)>1} q_n I_{2,1} \le c_H \sum_{n^2(t-s)>1} n^{-2\alpha-1} \le c_H|t-s|^{\alpha}.$$

For the head of the series, as $|t-s| \le \frac{t_0}{2}$, we have

$$\sum_{n^2(t-s)\le 1} q_n I_{2,1} \le \sum_{n^2(t-s)\le 1} q_n \frac{c(t_0,H)}{n^{4H}} \left(\frac{t}{2}\right)^{2H-1} \int_0^{2n^2(t-s)} \mathrm{d}u\, u^{2H-1} \le c_H|t-s|^{2H} \sum_{n^2(t-s)\le 1} n^{4H-2\alpha-1}.$$

This proves that $\sum_{n^2(t-s)\le 1} q_n I_{2,1}$ is of the same order as $\sum_{n^2(t-s)\le 1} q_n I_1$ which we calculated above to be of order $|t-s|^{\alpha\wedge(2H)}$.

We now bound $I_{2,2}$. Using Lemma A.1 and the change of variables $t-r=v$, $t-u=v'$, we have

$$I_{2,2} \le c_H \int_0^{t-s} \mathrm{d}v \left(\int_0^v \mathrm{d}v'(v-v')^{H-\frac{3}{2}}(\mathrm{e}^{-n^2v'}-\mathrm{e}^{-n^2v})\right)^2.$$

Using the change of variables $n^2(v-v')=y$ and $2n^2v=x$, we find

$$I_{2,2} \le \frac{c_H}{n^{4H}} \int_0^{2n^2(t-s)} \mathrm{d}x\,\mathrm{e}^{-x} \left(\int_0^{x/2} \mathrm{d}y\, y^{H-\frac{3}{2}}(\mathrm{e}^y-1)\right)^2.$$

Appealing to [16, Lemma 2] with $a=n^2$ and $A=H-\frac{1}{2}$, we obtain for the tail of the series

$$\sum_{n^2(t-s)>1} q_n I_{2,2} \le c_H \sum_{n^2(t-s)>1} n^{-2\alpha-1} \le c_H|t-s|^{\alpha}.$$

For the head of the series, we have

$$\sum_{n^2(t-s)\le 1} q_n I_{2,2} \le \sum_{n^2(t-s)\le 1} q_n \frac{c_H}{n^{4H}} \int_0^{2n^2(t-s)} \mathrm{d}x \left(\int_0^{1/2} \mathrm{d}y y^{H-\frac{3}{2}}(\mathrm{e}^y-1)\right)^2 \le c_H|t-s| \sum_{n^2(t-s)\le 1} n^{-2\alpha+1} \le c_H|t-s|^{\alpha}.$$

We now estimate I_3.

$$I_3 \leq 2\int_0^s (K(t,r)-K(s,r))^2(g(r))^2 \mathrm{d}r + 2\int_0^s \mathrm{d}r \left(\int_s^t \mathrm{d}u\,(g(u)-g(r))\frac{\partial K}{\partial u}(u,r)\right)^2$$
$$:= 2I_{3,1} + 2I_{3,2}.$$

By Lemma A.1, we get, for every $r < s < t$,

$$K(t,r) - K(s,r) = (t-s)\int_0^1 \left|\frac{\partial K}{\partial u}(s+v(t-s),r)\right| \mathrm{d}v$$
$$\leq c_H (t-s)\int_0^1 |s+v(t-s)-r|^{H-3/2}\,\mathrm{d}v \qquad (2.12)$$
$$\leq c_H \left|(s-r)^{H-1/2} - (t-r)^{H-1/2}\right|. \qquad (2.13)$$

We now separate the evaluation of the integral in $I_{3,1}$ depending upon whether r is bigger or smaller than $s-(t-s)/2$. In the first case, we evaluate

$$I_{3,1,1} := \int_{s-(t-s)/2}^s (K(t,r)-K(s,r))^2\, \mathrm{e}^{-n^2(t-r)}\mathrm{d}r.$$

Here, we have $s-r < (t-s)/2$ and $t-r > t-s$; therefore, using (2.13), we have

$$I_{3,1,1} \leq c_H \int_{s-(t-s)/2}^s \left(\left(1+2^{H-1/2}\right)(s-r)^{H-1/2}\right)^2 \mathrm{e}^{-n^2(t-r)}\mathrm{d}r$$
$$\leq c_H \mathrm{e}^{-n^2(t-s)} \int_{s-(t-s)/2}^s (s-r)^{2H-1}\,\mathrm{d}r$$
$$= c_H \mathrm{e}^{-n^2(t-s)}\,(t-s)^{2H}.$$

For the head of the series, we find

$$\sum_{n^2(t-s)\leq 1} q_n I_{3,1,1} \leq c_H (t-s)^{2H} \sum_{n^2(t-s)\leq 1} n^{4H-2\alpha-1},$$

which is bounded above by $c_H|t-s|^{\alpha\wedge(2H)}$ while for the tail of the series we have

$$\sum_{n^2(t-s)>1} q_n I_{3,1,1} \leq c_H (t-s)^{2H} \sum_{n^2(t-s)>1} n^{4H-2\alpha-1}\mathrm{e}^{-n^2(t-s)}$$
$$\leq c_H (t-s)^{2H} \int_{(t-s)^{-1/2}}^\infty \mathrm{e}^{-x^2(t-s)} x^{4H-2\alpha-1}\mathrm{d}x$$
$$= c_H (t-s)^\alpha \int_1^\infty \mathrm{e}^{-y^2} y^{4H-2\alpha-1}\mathrm{d}y = c_H (t-s)^\alpha.$$

Second we evaluate

$$I_{3,1,2} := \int_0^{s-(t-s)/2} (K(t,r)-K(s,r))^2\, \mathrm{e}^{-n^2(t-r)}\mathrm{d}r.$$

Here, we have $s-r > (t-s)/2$; we simply use (2.12) where an upper bound is obtained by replacing $|s+v(t-s)-r|^{H-3/2}$ by $|s-r|^{H-3/2}$; the latter can now be bounded above by $2^{3/2-H}\,|t-s|^{H-3/2}$. Thus

$$I_{3,1,2} \le c_H\, |t-s|^{2+2H-3} \int_0^{s-(t-s)/2} \mathrm{e}^{-n^2(t-r)} \mathrm{d}r.$$
$$\le c_H\, |t-s|^{2H-1}\, n^{-2} \mathrm{e}^{-n^2(t-s)}.$$

This estimate will not help us in the case $n^2\,(t-s) \le 1$. In the other case, we have

$$\sum_{n^2(t-s)>1} q_n I_{3,1,2} \le c_H\, |t-s|^{2H-1} \sum_{n^2(t-s)>1} n^{4H-2\alpha-3} \mathrm{e}^{-n^2(t-s)}$$
$$\le c_H\, |t-s|^{2H-1} \int_{(t-s)^{-1/2}}^{\infty} x^{4H-2\alpha-3} \mathrm{e}^{-x^2(t-s)} \mathrm{d}x$$
$$= c_H\, |t-s|^{2H-1}\, (t-s)^{-2H+\alpha+1} \int_1^{\infty} y^{4H-2\alpha-3} \mathrm{e}^{-y^2} \mathrm{d}y = c_H\, (t-s)^{\alpha}.$$

The third and last step of the estimation of $I_{3,1}$ is the sum for $n^2\,(t-s) < 1$ of $I_{3,1,2}$. In this case, we use (2.13) and obtain an upper bound by bounding $(s-r)^{H-1/2} - (t-r)^{H-1/2}$ above by $c_H\,(t-s)\,(s-r)^{H-3/2}$. Thus

$$I_{3,1,2} \le c_H\,(t-s)^2 \int_0^{s-(t-s)/2} (s-r)^{2H-3}\,\mathrm{d}r \le c_H\,(t-s)^{2H}.$$

This proves that $\sum_{n^2(t-s)\le 1} q_n I_{3,1,2}$ is of the same order as $\sum_{n^2(t-s)\le 1} q_n I_{3,1,1}$ which we calculated above to be of order $|t-s|^{\alpha\wedge(2H)}$.

We now bound $I_{3,2}$. Using Lemma A.1 and the change of variables $s-r=v$, $s-u=v'$, we have

$$I_{3,2} \le c_H \mathrm{e}^{-2n^2(t-s)} \int_0^s \mathrm{d}v \left(\int_{s-t}^0 \mathrm{d}v' (v-v')^{H-\frac{3}{2}} (\mathrm{e}^{-n^2 v'} - \mathrm{e}^{-n^2 v}) \right)^2.$$

Using the change of variables $v - v' = u$, we find

$$I_{3,2} \le c_H \mathrm{e}^{-2n^2(t-s)} \int_0^s \mathrm{d}v\, \mathrm{e}^{-2n^2 v} \left(\int_v^{v+(t-s)} \mathrm{d}u\, u^{H-\frac{3}{2}} (\mathrm{e}^{n^2 u} - 1) \right)^2.$$

Appealing to [16, Lemma 2] with $a = n^2$ and $A = H - \frac{1}{2}$, we obtain for the tail of the series

$$\sum_{n^2(t-s)>1} q_n I_{3,2} \le c_H \sum_{n^2(t-s)>1} n^{-2\alpha-1} \le c_H |t-s|^{\alpha}.$$

In order to evaluate the head of the series, we separate the evaluation of the integral in $I_{3,2}$ depending upon whether v is bigger or smaller than $t-s$, that is,

$$I_{3,2} \le c_H \int_0^s \mathrm{d}v \left(\int_v^{v+(t-s)} \mathrm{d}u\, u^{H-\frac{3}{2}} \right)^2$$
$$= c_H \left\{ \int_0^{t-s} \mathrm{d}v \left(\int_v^{v+(t-s)} \mathrm{d}u\, u^{H-\frac{3}{2}} \right)^2 + \int_{t-s}^s \mathrm{d}v \left(\int_v^{v+(t-s)} \mathrm{d}u\, u^{H-\frac{3}{2}} \right)^2 \right\}$$
$$\le c_H \left\{ \int_0^{t-s} \mathrm{d}v\, v^{2H-1} + \int_{t-s}^s \mathrm{d}v\, v^{2H-3}(t-s)^2 \right\}$$
$$\le c_H (t-s)^{2H}.$$

Therefore, $\sum_{n^2(t-s)\le 1} q_n I_{3,2}$ is of the same order as $\sum_{n^2(t-s)\le 1} q_n I_{3,1}$ which is of order $|t-s|^{\alpha\wedge(2H)}$.

Use all the estimates above, together with (2.8), to conclude that

$$\delta_x^2(s,t) \le c_H(|t-s|^{2H} + |t-s|^{\alpha}) \le c'_H |t-s|^{\alpha\wedge(2H)}.$$

This proves the upper bound of (2.7) when $H < 1/2$.

The lower bound: We now estimate the lower bound of the expectation in the case $H < 1/2$. We write

$$\begin{aligned} \mathrm{E}&\left[\left(\int_0^s \left(\mathrm{e}^{-n^2(t-r)} - \mathrm{e}^{-n^2(s-r)}\right)\beta_n^H(\mathrm{d}r) + \int_s^t \mathrm{e}^{-n^2(t-r)}\beta_n^H(\mathrm{d}r)\right)^2\right] \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{2.14}$$

where I_1, I_2 and I_3 are as in (2.10), and

$$I_4 := \int_0^s (K_s^* f(r))(K_t^* g(r) - K_s^* g(r))\mathrm{d}r. \tag{2.15}$$

We will prove that the series $\sum_{n\in\mathbb{N}} q_n(I_1+I_2+I_3+I_4)$ is of order $(t-s)^{\alpha}$, and that the tails of the series of the first term I_1 are the ones that contribute on that order. First note that $I_1, I_2, I_3 \ge 0$, but $I_4 = I_{4,1}+I_{4,2}+I_{4,3}+I_{4,4}$, where $I_{4,1}, I_{4,2} \ge 0$ and $I_{4,3}, I_{4,4} \le 0$ (see (2.17)). Hence, it suffices to find a lower bound for $\sum_{n\in\mathbb{N}} q_n(I_1+I_{4,3}+I_{4,4})$. In fact, it suffices to find a lower bound for the tail series $\sum_{n\in S_K} q_n(I_1+I_{4,3}+I_{4,4})$, where $S_K := \left\{n\in\mathbb{N} : n^2(t-s) > K\right\}$ for some (large) constant $K \ge 1$ which will be chosen later. We will prove that for some constant $K \ge 1$ sufficiently large, the series $\sum_{n\in S_K} q_n I_1$ is bigger than $2\sum_{n\in S_K} q_n \left|I_{4,3}\right|$ and than $4\sum_{n\in S_K} q_n \left|I_{4,4}\right|$, and is of order $(t-s)^{\alpha}K^{-\alpha}$. This will imply the desired lower bound.

We start by finding a lower bound for I_1. We have $I_1 := I_{1,1}+I_{1,2}+I_{1,3}$, where $I_{1,1}$ and $I_{1,2}$ are as in (2.11), and

$$I_{1,3} = 2\int_0^s \mathrm{d}r\, K(s,r) f(r)\int_r^s \mathrm{d}u\,(f(u)-f(r))\frac{\partial K}{\partial u}(u,r).$$

The change of variables $s-r=v$, $s-u=w$, $v-w=u'$ gives

$$I_1 = (1-\mathrm{e}^{-n^2(t-s)})^2\int_0^s \mathrm{d}v\,\mathrm{e}^{-2n^2v}\left(K(s,s-v) + \int_0^v \mathrm{d}u'\,\frac{\partial K}{\partial u'}(u',0)(\mathrm{e}^{n^2u'}-1)\right)^2.$$

Appealing to Lemma A.1 in the Appendix, and the change of variables $n^2u'=u$, $n^2v=x$, we obtain

$$\begin{aligned} I_1 &\ge \frac{c_H}{n^{4H}}\left(1-\mathrm{e}^{-n^2(t-s)}\right)^2\int_0^{n^2 s}\mathrm{d}x\,\mathrm{e}^{-2x}\left(x^{H-\frac12} - \left(\frac12 - H\right)\int_0^x \mathrm{d}u\,u^{H-\frac32}(\mathrm{e}^u-1)\right)^2 \\ &\ge \frac{c_H}{n^{4H}}(1-\mathrm{e}^{-n^2(t-s)})^2\int_0^{t_0}\mathrm{d}x\,\mathrm{e}^{-2x}\left(x^{H-\frac12} - \left(\frac12 - H\right)\int_0^x \mathrm{d}u\,u^{H-\frac32}(\mathrm{e}^u-1)\right)^2 \\ &= \frac{c_H}{n^{4H}}(1-\mathrm{e}^{-n^2(t-s)})^2, \end{aligned} \tag{2.16}$$

as the last integral is finite and positive.

Next we evaluate I_4. We write $I_4 = I_{4,1} + I_{4,2} + I_{4,3} + I_{4,4}$, where

$$\begin{aligned} I_{4,1} &= \int_0^s \mathrm{d}r K(s,r) f(r)(K(t,r) - K(s,r)) g(r), \\ I_{4,2} &= \int_0^s \mathrm{d}r K(s,r) f(r) \int_s^t \mathrm{d}u (g(u) - g(r)) \frac{\partial K}{\partial u}(u,r), \\ I_{4,3} &= \int_0^s \mathrm{d}r \int_r^s \mathrm{d}u (f(u) - f(r)) \frac{\partial K}{\partial u}(u,r) \int_s^t \mathrm{d}v (g(v) - g(r)) \frac{\partial K}{\partial v}(v,r), \\ I_{4,4} &= \int_0^s \mathrm{d}r (K(t,r) - K(s,r)) g(r) \int_r^s \mathrm{d}u (f(u) - f(r)) \frac{\partial K}{\partial u}(u,r). \end{aligned} \tag{2.17}$$

Now, note that $I_{4,1}, I_{4,2} \geq 0$ but $I_{4,3}, I_{4,4} \leq 0$.

We claim that, for some subset $S_K \subset \mathbb{N}$,

$$\sum_{n \in S_K} q_n I_1 > 2 \sum_{n \in S_K} q_n \left| I_{4,3} \right|, \tag{2.18}$$

$$\sum_{n \in S_K} q_n I_1 > 4 \sum_{n \in S_K} q_n \left| I_{4,4} \right|, \tag{2.19}$$

where q_n and $\alpha \in (0,1]$ are as in hypothesis (1.3) and $S_K := \left\{ n \in \mathbb{N} : n^2 (t-s) > K \right\}$ for some (large) constant $K \geq 1$ which will be chosen later.

Assume (2.18) and (2.19) are proved. We write, using (2.16),

$$\sum_{n \in S_K} q_n I_1 \geq c_H (1 - \mathrm{e}^{-1})^2 \int_{2\sqrt{K}/\sqrt{t-s}}^{\infty} \mathrm{d}x \, x^{-2\alpha-1} := c^1_{\alpha,H} (t-s)^{\alpha} K^{-\alpha}. \tag{2.20}$$

Because $I_2, I_3, I_{4,1}, I_{4,2} \geq 0$ and using (2.18)–(2.20), we find

$$\begin{aligned} \sum_{n \in \mathbb{N}} q_n (I_1 + I_2 + I_3 + I_4) &\geq \sum_{n \in S_K} q_n I_1 - \sum_{n \in S_K} q_n |I_{4,3}| - \sum_{n \in S_K} q_n |I_{4,4}| \\ &\geq \frac{1}{4} \sum_{n \in S_K} q_n I_1 \\ &\geq c_{\alpha,H,K} \, (t-s)^{\alpha}. \end{aligned}$$

Therefore, by (2.8) and (2.14), we conclude that

$$\delta_x^2(s,t) \geq q_0 |t-s|^{2H} + c_H |t-s|^{\alpha} \geq c'_H |t-s|^{\alpha \wedge (2H)}.$$

This proves the lower bound of (2.7) when $H < \frac{1}{2}$.

We finally prove (2.18) and (2.19).

Proof of 2.18. Using Lemma A.1 and the change of variables $s - r = r'$, $s - u = u'$, $s - v = v'$, $r' - u' = u''$, $r' - v' = v''$, $n^2 u'' = x$, $n^2 v'' = v$, we find

$$\begin{aligned} \left| I_{4,3} \right| \leq c_H \left(1 - \mathrm{e}^{-n^2(t-s)} \right) \mathrm{e}^{-n^2(t-s)} n^{-4H} \int_0^{n^2 s} \mathrm{d}x \mathrm{e}^{-2x} \left(\int_0^x \mathrm{d}u \, u^{H-3/2} \left(\mathrm{e}^u - 1 \right) \right) \\ \times \left(\int_x^{x+n^2(t-s)} \mathrm{d}v \, v^{H-3/2} \left(\mathrm{e}^v - 1 \right) \right). \end{aligned} \tag{2.21}$$

Note that with the exception of the factor $e^{-n^2(t-s)}$ in $|I_{4,3}|$, the combinations of all the terms in I_1 and $I_{4,3}$ are in fact largely similar, which makes this portion of the proof quite delicate, and in particular, to exploit the factor $e^{-n^2(t-s)}$, we must restrict the values of $n^2(t-s)$ to being relatively large, which explains the choice of S_K above.

Our strategy is to bound the sum over $n \in S_K$ of $q_n |I_{4,3}|$ above as tightly as possible by performing a "Fubini", dragging the sum over n all the way inside the expression for $\sum_{S_K} q_n |I_{4,3}|$, and evaluating it first using some Gaussian estimates. That these Gaussian estimates work has to do with the precise eigenvalue structure of the Laplacian, not with the Gaussian property of the driving noise.

We proved in (2.20) that the contribution of I_1 is bounded below by an expression of the form $c^1_{\alpha,H}(t-s)^\alpha K^{-\alpha}$, where $c^1_{\alpha,H}$ depends only on α and H. We will now show that

$$\sum_{n \in S_K} q_n |I_{4,3}| \le c^2_{\alpha,H}(t-s)^\alpha K^{-\beta} \tag{2.22}$$

for some $\beta > \alpha$, where $c^2_{\alpha,H}$ depends again only on H and α. Even if $c^2_{\alpha,H}$ is much larger than $c^1_{\alpha,H}$, one only needs to choose $K \ge (2c^2_{\alpha,H}/c^1_{H,\alpha})^{1/(\beta-\alpha)}$ to guarantee that the contribution of I_1 exceeds twice the absolute value of the contribution of $I_{4,3}$ as announced in (2.18), which implies that even though the latter is negative, the sum of the two exceeds $(c^1_{\alpha,H}/2)(t-s)^\alpha K^{-\alpha}$, i.e. for some K depending only on H and α.

First, for fixed x, we perform the announced Fubini, which means that, instead of having the integration and summation limits for n and v as $n > \sqrt{K/(t-s)}$ first and $x < v < x+n^2(t-s)$ next, we get instead $x < v < \infty$ and

$$\begin{aligned} n &> \max\left\{\sqrt{K/(t-s)}, \sqrt{(v-x)/t-s}\right\} \\ &= (t-s)^{-1/2}\sqrt{(v-x)\vee K}. \end{aligned}$$

Therefore, bounding $(1-e^{-n^2(t-s)})$ by 1, and $n^2 s$ by ∞, we have

$$\begin{aligned} \sum_{n \in S_K} q_n |I_{4,3}| \le c_H \int_0^\infty \mathrm{d}x\, e^{-2x} \left(\int_0^x \mathrm{d}u\, u^{H-3/2}(e^u - 1)\right) \\ \times \left(\int_x^\infty \mathrm{d}v\, v^{H-3/2}(e^v-1)\, S(K, v-x, t-s)\right), \end{aligned} \tag{2.23}$$

where the term $S(K, v-x, t-s)$ is defined by a series which we compare to a Gaussian integral as follows

$$\begin{aligned} S(K, v-x, t-s) &:= \sum_{n > (t-s)^{-1/2}\sqrt{(v-x)\vee K}} n^{-2\alpha-1} e^{-n^2(t-s)} \\ &\le \int_{y \ge (t-s)^{-1/2}\sqrt{(v-x)\vee K}}^\infty \mathrm{d}y\, y^{-2\alpha-1} e^{-y^2(t-s)}. \end{aligned}$$

Using the change of variable $w^2 = (t-s)y^2$, we have

$$\begin{aligned} S(K, v-x, t-s) &\le (t-s)^\alpha \int_{\sqrt{(v-x)\vee K}}^\infty \mathrm{d}w\, w^{-2\alpha-1} e^{-w^2} \\ &\le (t-s)^\alpha ((v-x)\vee K)^{-\alpha-1/2} \int_{\sqrt{(v-x)\vee K}}^\infty \mathrm{d}w\, e^{-w^2}. \end{aligned}$$

Now, using the classical Gaussian tail estimate $\int_A^\infty \mathrm{d}w\, \mathrm{e}^{-w^2} \leq 2^{-1} A^{-1} \mathrm{e}^{-A^2}$, we get

$$S(K, v-x, t-s) \leq 2^{-1} (t-s)^{\alpha} \left((v-x) \vee K\right)^{-\alpha-1} \mathrm{e}^{-(v-x)\vee K}. \tag{2.24}$$

Combining (2.23) and (2.24) we have immediately

$$\begin{aligned}
\sum_{n \in S_K} q_n \left| I_{4,3} \right| &\leq c_H (t-s)^{\alpha} \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-2x} \left(\int_0^x u^{H-3/2} \left(\mathrm{e}^u - 1\right) \mathrm{d}u \right) \\
&\quad \times \left(\int_x^\infty \mathrm{d}v v^{H-3/2} \left(\mathrm{e}^v - 1\right) \left((v-x) \vee K\right)^{-\alpha-1} \mathrm{e}^{-(v-x)\vee K} \right) \\
&= c_H (t-s)^{\alpha} \mathrm{e}^{-K} K^{-\alpha-1} \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-2x} \left(\int_0^x \mathrm{d}u\, u^{H-3/2} \left(\mathrm{e}^u - 1\right) \right) \qquad (2.25) \\
&\quad \times \left(\int_x^{x+K} \mathrm{d}v\, v^{H-3/2} \left(\mathrm{e}^v - 1\right) \right) \\
&\quad + c_H (t-s)^{\alpha} \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-2x} \left(\int_0^x u^{H-3/2} \mathrm{d}u \left(\mathrm{e}^u - 1\right) \right) \qquad (2.26) \\
&\quad \times \left(\int_{x+K}^\infty \mathrm{d}v\, v^{H-3/2} \left(\mathrm{e}^v - 1\right) (v-x)^{-\alpha-1} \mathrm{e}^{-(v-x)} \right).
\end{aligned}$$

We separate the last expression into various terms. We will calculate first the term in line (2.25) by separating the x-integration over $x \in [0, K]$ and $x \in (K, \infty)$, which we denote by $J_{4,3,1}$ and $J_{4,3,2}$, respectively. The term in line (2.26), which we denote by $J_{4,3,2}$, can be dealt with more directly. We now perform these evaluations.

Term $J_{4,3,1}$. We write

$$\begin{aligned}
J_{4,3,1} &:= c_H (t-s)^{\alpha} \mathrm{e}^{-K} K^{-\alpha-1} \int_0^K \mathrm{d}x\, \mathrm{e}^{-2x} \left(\int_0^x \mathrm{d}u\, u^{H-3/2} \left(\mathrm{e}^u - 1\right) \right) \\
&\quad \times \left(\int_x^{x+K} \mathrm{d}v\, v^{H-3/2} \left(\mathrm{e}^v - 1\right) \right) \\
&\leq c_H (t-s)^{\alpha} \mathrm{e}^{-K} K^{-\alpha-1} \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-2x} \left(\int_0^x \mathrm{d}u\, u^{H-3/2} \left(\mathrm{e}^u - 1\right) \right) \\
&\quad \times \left(c_H + \int_1^{2K} \mathrm{d}v\, v^{H-3/2} \left(\mathrm{e}^v - 1\right) \right).
\end{aligned}$$

Now, integrating by parts, we get

$$\int_1^{2K} \mathrm{d}v\, v^{H-3/2} \left(\mathrm{e}^v - 1\right) \leq c_H \mathrm{e}^K K^{H+1/2}.$$

The last two estimates imply immediately that

$$J_{4,3,1} \leq c_H (t-s)^{\alpha} K^{-\alpha+H-1/2},$$

which proves the contribution of $J_{4,3,1}$ in (2.22).

Term $J_{4,3,2}$. We write

$$J_{4,3,2} := c_H (t-s)^{\alpha} \mathrm{e}^{-K} K^{-\alpha-1} \int_K^\infty \mathrm{d}x\, \mathrm{e}^{-2x} \left(\int_0^x \mathrm{d}u\, u^{H-3/2} \left(\mathrm{e}^u - 1\right) \right)$$

$$
\begin{aligned}
&\times \left(\int_x^{x+K} \mathrm{d}v\, v^{H-3/2}\left(\mathrm{e}^v - 1\right)\right)\\
\le\ & c_H\,(t-s)^\alpha\,\mathrm{e}^{-K}K^{-\alpha-1}\int_K^\infty \mathrm{d}x\,\mathrm{e}^{-2x}\left(\int_0^x \mathrm{d}u\,u^{H-3/2}\left(\mathrm{e}^u-1\right)\right)\\
&\times x^{H-3/2}\left(\mathrm{e}^{x+K}-\mathrm{e}^x\right)\\
\le\ & c_H\,(t-s)^\alpha\,K^{-\alpha-1}\int_K^\infty \mathrm{d}x\,\mathrm{e}^{-x}x^{H-3/2}\left(\int_0^x \mathrm{d}u\,u^{H-3/2}\left(\mathrm{e}^u-1\right)\right)\\
\le\ & c_H\,(t-s)^\alpha\,K^{-\alpha-1}\int_K^\infty \mathrm{d}x\,\mathrm{e}^{-x}x^{H-3/2}\left(c_H+\mathrm{e}^x\int_1^x \mathrm{d}u\,u^{H-3/2}\right)\\
\le\ & c_H\,(t-s)^\alpha\,K^{-\alpha-1}\left(K^{H-3/2}\mathrm{e}^{-K}+K^{H-1/2}\right)\\
\le\ & c_H\,(t-s)^\alpha\,K^{-\alpha+H-3/2}
\end{aligned}
$$

which proves the contribution of $J_{4,3,2}$ in (2.22).

Term $J_{4,3,3}$. The last part of the estimation is that of

$$
\begin{aligned}
J_{4,3,3} &:= c_H\,(t-s)^\alpha\int_0^\infty \mathrm{d}x\,\mathrm{e}^{-2x}\left(\int_0^x u^{H-3/2}\left(\mathrm{e}^u-1\right)\mathrm{d}u\right)\\
&\quad\times\left(\int_{x+K}^\infty \mathrm{d}v\,v^{H-3/2}\left(\mathrm{e}^v-1\right)(v-x)^{-\alpha-1}\,\mathrm{e}^{-(v-x)}\right)\\
&\le c_H\,(t-s)^\alpha\,K^{H-3/2}\int_0^\infty \mathrm{d}x\,\mathrm{e}^{-x}\cdot\left(\int_0^x \mathrm{d}u\,u^{H-3/2}\left(\mathrm{e}^u-1\right)\right)\\
&\quad\times\left(\int_{x+K}^\infty \mathrm{d}v\ (v-x)^{-\alpha-1}\right)\\
&= c_{\alpha,H}\,(t-s)^\alpha\,K^{-\alpha+H-3/2}\int_0^\infty \mathrm{d}u\,u^{H-3/2}\left(\mathrm{e}^u-1\right)\left(\int_u^\infty \mathrm{d}x\,\mathrm{e}^{-x}\right)\\
&= c_\alpha\,(t-s)^\alpha\,K^{-\alpha+H-3/2}\int_0^\infty u^{H-3/2}\left(\mathrm{e}^u-1\right)\mathrm{e}^{-u}\mathrm{d}u\\
&= c_\alpha\,(t-s)^\alpha\,K^{-\alpha+H-3/2}\left[c_H+\int_1^\infty u^{H-3/2}\mathrm{d}u\right]\\
&= c_{\alpha,H}\,(t-s)^\alpha\,K^{-\alpha+H-3/2}.
\end{aligned}
$$

Therefore, (2.22) holds taking $\beta=\alpha+1/2-H$ which is greater than α as $H<1/2$.

The proof of (2.18) is now finished. □

Proof of 2.19. By (2.12) and Lemma A.1, we have

$$
|I_{4,4}| \le c_H(t-s)\int_0^s \mathrm{d}r\,(s-r)^{H-\frac{3}{2}}g(r)\int_r^s \mathrm{d}u\,(u-r)^{H-\frac{3}{2}}(f(u)-f(r)).
$$

Using the change of variables $s-r=r'$, $s-u=u'$, $r'-u'=v$, $n^2v=u$, $n^2r'=x$, we get

$$
|I_{4,4}| \le \frac{c_H}{n^{4H-2}}(t-s)\mathrm{e}^{-n^2(t-s)}(1-\mathrm{e}^{-n^2(t-s)})\int_0^{n^2 s}\mathrm{d}x\,x^{H-\frac{3}{2}}\mathrm{e}^{-2x}\int_0^x \mathrm{d}u\,u^{H-\frac{3}{2}}(\mathrm{e}^u-1).
$$

Bounding $(1 - e^{-n^2(t-s)})$ by 1 and $n^2 s$ by ∞, we get

$$|I_{4,4}| \le \frac{c_H}{n^{4H-2}}(t-s)e^{-n^2(t-s)}.$$

We will now proceed as in the proof of (2.18); that is we will prove that there exists a constant c_H^3 depending only on H such that

$$\sum_{n \in S_K} q_n |I_{4,4}| \le c_H^3 \, (t-s)^{\alpha} \, K^{-\beta}, \tag{2.27}$$

for some $\beta > \alpha$. It then suffices to choose $K \ge (4c_H^3/c_{H,\alpha}^1)^{1/(\beta-\alpha)}$ to get (2.19).

We now prove (2.27). We write

$$\begin{aligned}\sum_{n \in S_K} q_n |I_{4,4}| &\le c_H (t-s) \int_{\sqrt{K/(t-s)}}^{\infty} dx \, x^{-2\alpha+1} e^{-x^2(t-s)} \\ &= c_H (t-s)^{\alpha} \int_{\sqrt{K}}^{\infty} dy \, y^{-2\alpha+1} e^{-y^2} \\ &\le c_H (t-s)^{\alpha} K^{-\alpha} 2^{-1} \int_{\sqrt{K}}^{\infty} dy \, 2y e^{-y^2} \\ &\le c_H (t-s)^{\alpha} K^{-(\alpha+1)},\end{aligned}$$

which proves (2.27) taking $\beta = \alpha + 1$ and concludes the proof of (2.19). □

This finishes the proof of the entire proposition. □

3. Gaussian upper bound for the bivariate density

We denote by $p_{t,x;s,y}(\cdot,\cdot)$ the (Gaussian) probability density function of the random vector $(u(t,x), u(s,y))$ for all $s, t > 0$ and $x, y \in S^1$ such that $(t,x) \ne (s,y)$.

For every fixed real number $0 < \alpha \le 1$ we consider the metric

$$\mathbf{\Delta}((t,x);(s,y)) = |x-y|^{2\alpha} + |t-s|^{\alpha \wedge (2H)}. \tag{3.1}$$

In this section we establish an upper bound of Gaussian type for the bivariate density $p_{t,x;s,y}(\cdot,\cdot)$ in terms of the metric (3.1). This will be one of the key results in order to show the lower bound of Theorem 1.1. The estimates obtained in the previous section to prove space and time regularity are nearly sufficient to obtain the results in this section. The following further improvement is needed, which deals with precise joint regularity (see [4, (4.11)] for the space–time white noise case).

Lemma 3.1. *Assume hypothesis (1.3). Fix $t_0, T > 0$. Then there exists $c_H > 0$ such that for any $s, t \in [t_0, T]$, $x, y \in S^1$, with (t,x) is sufficiently near (s,y), and $i = 1, \ldots, d$,*

$$\begin{aligned} c_H^{-1} \mathbf{\Delta}((t,x);(s,y)) &\le \mathrm{E}\left[(u_i(t,x) - u_i(s,y))^2\right] \\ &\le c_H \mathbf{\Delta}((t,x);(s,y)). \end{aligned} \tag{3.2}$$

Proof. The upper bound in (3.2) is a consequence of the upper bounds of Corollary 2.2 and Proposition 2.3, and the following inequality

$$\mathrm{E}\left[(u_i(t,x) - u_i(s,y))^2\right] \le 2\left\{\mathrm{E}\left[(u_i(t,x) - u_i(s,x))^2\right] + \mathrm{E}\left[(u_i(s,x) - u_i(s,y))^2\right]\right\}.$$

We now proceed to the proof of the lower bound in (3.2). By Corollary 2.2, there exist $c_1, c_2 > 0$ such that for all $t \in [t_0, T]$, $x, y \in S^1$, with x is sufficiently near y, and $i = 1, \dots, d$,

$$c_1|x-y|^{2\alpha} \leq \mathrm{E}\left[(u_i(t,x)-u_i(t,y))^2\right] \leq c_2|x-y|^{2\alpha}. \tag{3.3}$$

Moreover, Proposition 2.3 ensures the existence of $c_3, c_4 > 0$ such that that for any $s, t \in [t_0, T]$, $x \in S^1$, with t is sufficiently near s, and $i = 1, \dots, d$,

$$c_3\,|t-s|^{\alpha\wedge(2H)} \leq \mathrm{E}\left[(u_i(t,x)-u_i(t,y))^2\right] \leq c_4\,|t-s|^{\alpha\wedge(2H)}. \tag{3.4}$$

Let us now consider two different cases.

Case 1: $|t-s|^{\alpha\wedge(2H)} < \frac{c_1}{4c_4}|x-y|^{2\alpha}$. Appealing to the lower bound in (3.3) and the upper bound in (3.4),

$$\begin{aligned}\mathrm{E}\left[(u_i(t,x)-u_i(s,y))^2\right] &= \mathrm{E}\left[(u_i(t,x)-u_i(t,y)+u_i(t,y)-u_i(s,y))^2\right]\\ &\geq \frac{1}{2}\mathrm{E}\left[(u_i(t,x)-u_i(t,y))^2\right]-\mathrm{E}\left[(u_i(t,y)-u_i(s,y))^2\right]\\ &\geq \frac{1}{2}c_1|x-y|^{2\alpha}-c_4|t-s|^{\alpha\wedge(2H)}.\end{aligned}$$

Because of the inequality that defines this Case 1, this is bounded below by

$$\begin{aligned}\frac{c_1}{2}|x-y|^{2\alpha}-\frac{c_1}{4}|x-y|^{2\alpha} &= \frac{c_1}{4}|x-y|^{2\alpha}\\ &\geq \frac{c_1}{8}|x-y|^{2\alpha}+\frac{c_1}{8}\frac{4c_4}{c_1}|t-s|^{\alpha\wedge(2H)}\\ &\geq \min\left(\frac{c_1}{8},\frac{c_4}{2}\right)\mathbf{\Delta}((t,x);(s,y)).\end{aligned}$$

This completes the proof of the lower bound in (3.2) in Case 1.

Case 2: $|t-s|^{\alpha\wedge(2H)} > \frac{4c_2}{c_3}|x-y|^{2\alpha}$. The proof of this portion is identical to Case 1, by using the upper bound in (3.3) and the lower bound in (3.4), and writing

$$\begin{aligned}\mathrm{E}\left[(u_i(t,x)-u_i(s,y))^2\right] &= \mathrm{E}\left[(u_i(t,x)-u_i(t,y)+u_i(t,y)-u_i(s,y))^2\right]\\ &\geq \frac{1}{2}\mathrm{E}\left[(u_i\,(t,x)-u_i\,(s,x))^2\right]-\mathrm{E}\left[(u_i\,(s,x)-u_i\,(s,y))^2\right]\end{aligned}$$

which yields the lower bound $\min\left(\frac{c_3}{8},\frac{c_2}{2}\right)\mathbf{\Delta}((t,x);(s,y))$. This completes the proof of Case 2.

Case 3: $\frac{4c_2}{c_3}|x-y|^{2\alpha} \geq |t-s|^{\alpha\wedge(2H)} \geq \frac{c_1}{4c_4}|x-y|^{2\alpha}$. Note that it suffices to prove that,

$$\mathrm{E}\left[(u_i(t,x)-u_i(s,y))^2\right] \geq c|t-s|^{\alpha\wedge(2H)}. \tag{3.5}$$

Indeed, because of the lower bound inequality that defines this Case 3, this is bounded below by

$$\frac{c}{2}|t-s|^{\alpha\wedge(2H)}+\frac{c}{2}\frac{c_1}{4c_4}|x-y|^{2\alpha} \geq c'\mathbf{\Delta}((t,x);(s,y)),$$

which proves the lower bound in (3.2) in this Case 1, provided that (3.5) is proved.

Proof of 3.5. We write

$$\mathrm{E}\left[(u_i(t,x)-u_i(s,y))^2\right] = q_0|t-s|^{2H} + \sum_{n=1}^{\infty} q_n\{\mathscr{W}_1+\mathscr{W}_2\},$$

where

$$\mathscr{W}_1 = \mathrm{E}\left[\left\{\int_0^s (\cos(nx)\,\mathrm{e}^{-n^2(t-r)} - \cos(ny)\,\mathrm{e}^{-n^2(s-r)})\,\beta_n(\mathrm{d}r) + \int_s^t \cos(nx)\,\mathrm{e}^{-n^2(t-r)}\,\beta_n(\mathrm{d}r)\right\}^2\right],$$

$$\mathscr{W}_2 = \mathrm{E}\left[\left\{\int_0^s (\sin(nx)\,\mathrm{e}^{-n^2(t-r)} - \sin(ny)\,\mathrm{e}^{-n^2(s-r)})\,\beta'_n(\mathrm{d}r) + \int_s^t \sin(nx)\,\mathrm{e}^{-n^2(t-r)}\,\beta'_n(\mathrm{d}r)\right\}^2\right],$$

where $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\beta'_n\}_{n\in\mathbb{N}}$ are independent standard fractional Brownian motions.

Now, because the further calculations use fractional stochastic calculus we need to consider the two different cases, namely $H<\frac{1}{2}$ and $H\geq\frac{1}{2}$.

Case $H\geq\frac{1}{2}$. In this case $\alpha\wedge(2H)=\alpha$ and (3.5) is proved in [18] for the case $x=y$.

Straightforward computations using (2.3) give

$$\begin{aligned}
&\mathrm{E}\left[(u_i(t,x)-u_i(s,y))^2\right]\\
&= q_0|t-s|^{2H} + \sum_{n=1}^{\infty} q_n\left\{\left(\mathrm{e}^{-2n^2t}+\mathrm{e}^{-2n^2s}-2\cos(n|x-y|)\mathrm{e}^{-n^2(t+s)}\right)I_1\right.\\
&\quad \left.+\ \mathrm{e}^{-n^2I_2}I_2 + 2\mathrm{e}^{-n^2t}\left(\mathrm{e}^{-n^2t}-\cos(n|x-y|)\mathrm{e}^{-n^2s}\right)I_3\right\}\\
&\geq q_0|t-s|^{2H} + \sum_{n=1}^{\infty} q_n\left\{(\mathrm{e}^{-n^2t}-\mathrm{e}^{-n^2s})^2I_1 + \mathrm{e}^{-n^2I_2}I_2 + 2\mathrm{e}^{-n^2t}(\mathrm{e}^{-n^2t}-\mathrm{e}^{-n^2s})I_3\right\},
\end{aligned}$$

where

$$I_1 = \int_0^s \mathrm{d}w \int_0^s \mathrm{d}v\,\mathrm{e}^{n^2(w+v)}|w-v|^{2H-2},$$

$$I_2 = \int_s^t \mathrm{d}w \int_s^t \mathrm{d}v\,\mathrm{e}^{n^2(w+v)}|w-v|^{2H-2},$$

$$I_3 = \int_0^s \mathrm{d}w \int_s^t \mathrm{d}v\,\mathrm{e}^{n^2(w+v)}|w-v|^{2H-2}.$$

Hence, using the results of [18, Section 2.1 and (17)] and (1.3), it follows that

$$\begin{aligned}
\mathrm{E}\left[(u_i(t,x)-u_i(s,y))^2\right] &\geq q_0|t-s|^{2H} + c_H(t-s)^{2H}\sum_{n^2(t-s)\leq 1} q_n\\
&\geq c_H((t-s)^{2H}+(t-s)^{\alpha}) \geq c_H(t-s)^{\alpha}.
\end{aligned}$$

This proves (3.5) when $H\geq\frac{1}{2}$.

Case $H < \frac{1}{2}$. It is elementary to see that by (2.14), $\mathscr{W}_1 \geq \tilde{I}_1 + \tilde{I}_4$, where $\tilde{I}_1$ and $\tilde{I}_4$ are defined, respectively, as I_1 and I_4 in the previous section (see (2.10) and (2.15)), but replacing f and g by

$$\tilde{f}(r) = \cos(nx)\,\mathrm{e}^{-n^2(t-r)} - \cos(ny)\,\mathrm{e}^{-n^2(s-r)}, \qquad \tilde{g}(r) = \cos(nx)\,\mathrm{e}^{-n^2(t-r)}.$$

Similarly, $\mathscr{W}_2 \geq \bar{I}_1 + \bar{I}_4$, where $\bar{I}_1$ and $\bar{I}_4$ are defined, respectively, as I_1 and I_4 but replacing f and g by

$$\bar{f}(r) = \sin(nx)\,\mathrm{e}^{-n^2(t-r)} - \sin(ny)\,\mathrm{e}^{-n^2(s-r)}, \qquad \bar{g}(r) = \sin(nx)\,\mathrm{e}^{-n^2(t-r)}.$$

Therefore, the proof of (3.5) when $H < \frac{1}{2}$ is similar to the control of I_1 from below by $|I_4|$ in the previous section; yet it is less delicate, because the hardest estimates we will need to use are ones which were already obtained therein. Indeed, proceeding as in (2.16), we find

$$\begin{aligned}\tilde{I}_1 + \bar{I}_1 &\geq \frac{c_H}{n^{4H}}\{(\cos(nx)\,\mathrm{e}^{-n^2(t-s)} - \cos(ny))^2 + (\sin(nx)\,\mathrm{e}^{-n^2(t-s)} - \sin(ny))^2\} \\ &= \frac{c_H}{n^{4H}}\{\mathrm{e}^{-2n^2(t-s)} + 1 - 2\cos(n|x-y|)\,\mathrm{e}^{-n^2(t-s)}\} \\ &\geq \frac{c_H}{n^{4H}}(1 - \mathrm{e}^{-n^2(t-s)})^2. \end{aligned} \tag{3.6}$$

Here we see that the case where $x = y$ is the worst case, in the sense that the lower bound (2.16) obtained for I_1 is a lower bound for all $\tilde{I}_1 + \bar{I}_1$ uniformly in t, x, s, y.

Moreover, simple calculations yield very similar formulas for the four terms in $\tilde{I}_4 + \bar{I}_4$ as we had found for I_4 itself in (2.17); namely we have

$$\begin{aligned}\tilde{I}_{4,1} + \bar{I}_{4,1} &= \int_0^s \mathrm{d}r\, K(s,r)h(r)(K(t,r) - K(s,r))g(r), \\ \tilde{I}_{4,2} + \bar{I}_{4,2} &= \int_0^s \mathrm{d}r\, K(s,r)h(r)\int_s^t \mathrm{d}u(g(u) - g(r))\frac{\partial K}{\partial u}(u,r), \\ \tilde{I}_{4,3} + \bar{I}_{4,3} &= \int_0^s \mathrm{d}r \int_r^s \mathrm{d}u(h(u) - h(r))\frac{\partial K}{\partial u}(u,r)\int_s^t \mathrm{d}v(g(v) - g(r))\frac{\partial K}{\partial v}(v,r), \\ \tilde{I}_{4,4} + \bar{I}_{4,4} &= \int_0^s \mathrm{d}r(K(t,r) - K(s,r))g(r)\int_r^s \mathrm{d}u(h(u) - h(r))\frac{\partial K}{\partial u}(u,r)\end{aligned}$$

where

$$\begin{aligned}h(r) &= \mathrm{e}^{-n^2(t-r)} - \cos(n|x-y|)\,\mathrm{e}^{-n^2(s-r)} \\ &= \mathrm{e}^{-n^2(s-r)}\left(\mathrm{e}^{-n^2(t-s)} - \cos(n|x-y|)\right) \\ &=: \mathrm{e}^{-n^2(s-r)} h_{s,t,x,y}.\end{aligned} \tag{3.7}$$

In other words, for each $j = 1, 2, 3, 4$, the formula for $\tilde{I}_{4,j} + \bar{I}_{4,j}$ is identical to that of $I_{4,j}$, with f replaced by h. Also recall that

$$f(r) = \mathrm{e}^{-n^2(s-r)}\left(\mathrm{e}^{-n^2(t-s)} - 1\right) = \mathrm{e}^{-n^2(s-r)} h_{s,t,x,x}.$$

We see here that f is always negative, while it is much more difficult to control the sign of h. Luckily, for any r, the sign of $h(r)$ is the sign of the fixed coefficient $h_{s,t,x,y}$ defined in (3.7).

When $h_{s,t,x,y}$ is negative, we will be able to use calculations from the previous section directly. When $h_{s,t,x,y}$ is non-negative, we will instead compare $\tilde{I}_1+\bar{I}_1$ with $\left|\tilde{I}_{4,1}+\bar{I}_{4,1}\right|$ and $\left|\tilde{I}_{4,2}+\bar{I}_{4,2}\right|$.

Case $h_{s,t,x,y}<0$. Note that, in this case, $\tilde{I}_{4,1}+\bar{I}_{4,1}>0$ and $\tilde{I}_{4,2}+\bar{I}_{4,2}>0$, while the other two sums are negative. Therefore, identically to the proof of lower bound in the previous section, we only need to show that for sufficiently large K, still using $S_K=\{n:n^2|t-s|\geq K\}$,

$$\sum_{n\in S_K} q_n(\tilde{I}_1+\bar{I}_1)>2\sum_{n\in S_K} q_n\left|\tilde{I}_{4,3}+\bar{I}_{4,3}\right|, \tag{3.8}$$

$$\sum_{n\in S_K} q_n(\tilde{I}_1+\bar{I}_1)>4\sum_{n\in S_K} q_n\left|\tilde{I}_{4,4}+\bar{I}_{4,4}\right|. \tag{3.9}$$

This is not difficult. Indeed, we have that both f and h are decreasing, and for all $u\in[r,s]$,

$$\begin{aligned}|h(u)-h(r)|&=\left(\mathrm{e}^{-n^2(s-u)}-\mathrm{e}^{-n^2(s-r)}\right)\left|h_{s,t,x,y}\right|\\&\leq\left(\mathrm{e}^{-n^2(s-u)}-\mathrm{e}^{-n^2(s-r)}\right)\left|h_{s,t,x,x}\right|=|f(u)-f(r)|.\end{aligned}$$

Hence, exploiting the fact that all the terms in the products defining the $I_{4,3}$ as well as $\tilde{I}_{4,3}+\bar{I}_{4,3}$ have constant signs, we can write

$$\begin{aligned}\left|\tilde{I}_{4,3}+\bar{I}_{4,3}\right|&=\int_0^s\mathrm{d}r\int_r^s\mathrm{d}u\,|h(u)-h(r)|\left|\frac{\partial K}{\partial u}(u,r)\right|\int_s^t\mathrm{d}v(g(v)-g(r))\left|\frac{\partial K}{\partial v}(v,r)\right|\\&\leq\int_0^s\mathrm{d}r\int_r^s\mathrm{d}u\,|f(u)-f(r)|\left|\frac{\partial K}{\partial u}(u,r)\right|\int_s^t\mathrm{d}v(g(v)-g(r))\left|\frac{\partial K}{\partial v}(v,r)\right|\\&=\left|I_{4,3}\right|,\end{aligned}$$

and similarly we get $\left|\tilde{I}_{4,4}+\bar{I}_{4,4}\right|\leq\left|I_{4,4}\right|$. Since the lower bound on $\tilde{I}_1+\bar{I}_1$ in (3.6) is as large as the lower bound (2.16) on I_1, the proof of the lower bound in the previous section implies both (3.8) and (3.9), which finishes the proof of (3.5) when $h_{s,t,x,y}<0$.

Case $h_{s,t,x,y}\geq 0$. Here we cannot rely on previous calculations. Indeed, in this case, $\tilde{I}_{4,3}+\bar{I}_{4,3}\geq 0$ and $\tilde{I}_{4,4}+\bar{I}_{4,4}\geq 0$, while $\tilde{I}_{4,1}+\bar{I}_{4,1}$ and $\tilde{I}_{4,2}+\bar{I}_{4,2}$ are negative, and we must therefore control their absolute values. As in the previous case, we only need to prove that for K large enough,

$$\sum_{n\in S_K} q_n(\tilde{I}_1+\bar{I}_1)>2\sum_{n\in S_K} q_n\left|\tilde{I}_{4,1}+\bar{I}_{4,1}\right|, \tag{3.10}$$

$$\sum_{n\in S_K} q_n(\tilde{I}_1+\bar{I}_1)>4\sum_{n\in S_K} q_n\left|\tilde{I}_{4,2}+\bar{I}_{4,2}\right|. \tag{3.11}$$

Unlike the last section where the full sum had to be invoked to obtain the required lower bounds, here it is possible to prove that the above two inequalities hold without the sums, i.e. for any fixed $n\in S_K$. These fact are established in Appendix A.4.

This proves (3.5) when $H<\frac{1}{2}$. The proof of the lemma is thus complete. □

Proposition 3.2. *Assume hypothesis (1.3). Then for all* $t_0, T, M > 0$*, there exists a finite constant* $c_H > 0$ *such that for all* $s, t \in [t_0, T]$*,* $x, y \in S^1$ *and* $z_1, z_2 \in [-M, M]^d$*,*

$$p_{t,x;s,y}(z_1, z_2) \le c_H\, (\mathbf{\Delta}((t,x);(s,y)))^{-d/2} \exp\left(-\frac{\|z_1 - z_2\|^2}{c_H \mathbf{\Delta}((t,x);(s,y))}\right).$$

Proof. Let $p^i_{t,x;s,y}(\cdot,\cdot)$ denote the bivariate density of the random vector $(u_i(t,x), u_i(s,y))$. Note that $p^i_{t,x;s,y}(\cdot,\cdot)$ does not depend on i.

We follow [3,4]. As in [3, (3.8)] and [4, (4.10)], we have that

$$p^i_{t,x;s,y}(z_1, z_2) \le \frac{1}{2\pi\sigma_{s,y}\tau} \exp\left(-\frac{|z_1 - z_2|^2}{4\tau^2}\right) \times \exp\left(\frac{|z_2|^2\,|1-m|^2}{4\tau^2}\right) \exp\left(-\frac{|z_2|^2}{2\sigma^2_{s,y}}\right), \tag{3.12}$$

where

$$\tau^2 := \sigma^2_{t,x}\left(1 - \rho^2_{t,x;s,y}\right), \qquad \rho_{t,x;s,y} = \frac{\sigma_{t,x;s,y}}{\sigma_{t,x}\sigma_{s,y}}, \qquad \sigma^2_{t,x} = \mathrm{E}[(u_i(t,x))^2]$$

$$m := \frac{\sigma_{t,x;s,y}}{\sigma^2_{s,y}}, \qquad \sigma_{t,x;s,y} = \mathrm{Cov}\,(u_i(t,x), u_i(s,y))\,.$$

We now show the analogues of (4.12) and Lemma 4.3 in [4] in the case of the fractional heat equation. Fix $s, t \in [t_0, T]$, $x, y \in S^1$. We claim that the following estimates hold:

$$|\sigma_{t,x} - \sigma_{s,y}| \le c_H |t-s|^{2\alpha}. \tag{3.13}$$

$$c_H^{-1}\mathbf{\Delta}((t,x);(s,y)) \le \sigma^2_{t,x}\sigma^2_{s,y} - \sigma^2_{t,x;s,y} \le c_H\,\mathbf{\Delta}((t,x);(s,y)), \tag{3.14}$$

$$|\sigma^2_{t,x} - \sigma_{t,x;s,y}| \le c_H\,[\mathbf{\Delta}((t,x);(s,y))]^{1/2}. \tag{3.15}$$

Indeed, in the proof of Proposition 2.3 we have proved that

$$\mathrm{E}\left[\left(\int_0^t \mathrm{e}^{-n^2(t-s)}\beta_n^H\,(\mathrm{d}s)\right)^2\right] \le c_H|t-s|^{2\alpha}.$$

Therefore, using [4, (4.31)], we have

$$|\sigma_{t,x} - \sigma_{s,y}| \le c_H\,|\sigma^2_{t,x} - \sigma^2_{s,y}| \le c_H|t-s|^{2\alpha}$$

where c_H does not depend on $t \in [t_0, T]$. This proves (3.13).

We now prove (3.14). Let $\gamma^2_{t,x;s,y} := \mathrm{E}[(u_i(t,x) - u_i(s,y))^2]$. Then using [4, (4.42)],

$$\sigma^2_{t,x}\sigma^2_{s,y} - \sigma^2_{t,x;s,y} = \frac{1}{4}\left(\gamma^2_{t,x;s,y} - (\sigma_{t,x} - \sigma_{s,y})^2\right)\left((\sigma_{t,x} + \sigma_{s,y})^2 - \gamma^2_{t,x;s,y}\right). \tag{3.16}$$

By Lemma 3.1, $\gamma^2_{t,x,s,y} \le c\mathbf{\Delta}((t,x);(s,y))$. Therefore, the second factor of (3.16) is bounded below by a positive constant when (t,x) is near (s,y). Furthermore, Lemma 3.1 and (3.13) yield

$$\gamma^2_{t,x,s,y} - (\sigma_{t,x} - \sigma_{s,y})^2 \ge c_H\mathbf{\Delta}((t,x);(s,y)).$$

This proves the lower bound of (3.14) provided (t,x) is sufficiently near (s,y).

In order to extend this inequality to all (t, x) and (s, y) in $[t_0, T] \times S^1$, note that by the continuity of the function $(t, x, s, y) \mapsto \sigma_{t,x}^2 \sigma_{s,y}^2 - \sigma_{t,x;s,y}^2$, it suffices to show that

$$\sigma_{t,x}^2 \sigma_{s,y}^2 - \sigma_{t,x;s,y}^2 > 0 \quad \text{if } (t, x) \neq (s, y).$$

If this last function was equal to zero there would be $\lambda \in \mathbb{R}$ such that $u_i(t, x) = \lambda u_i(s, y)$ a.s., which is a contradiction to the lower bound in (3.2) and the fact that $\mathbf{\Delta}((t, x); (s, y))$ is zero only if $(t, x) = (s, y)$. This completes the proof of the lower bound of (3.14).

In order to prove the upper bound of (3.14), use Lemma 3.1 to see that the first factor in (3.16) is bounded above by $c_H \mathbf{\Delta}((t, x); (s, y))$. As the second factor in (3.16) is bounded above by a constant c_H, the desired upper bound follows.

It remains to prove (3.15). Use [4, (4.47)] to find

$$\begin{aligned} |\sigma_{t,x}^2 - \sigma_{t,x;s,y}| &= \left|\gamma_{t,x;s,y}^2 + \mathrm{Cov}\,(u_i(t, x) - u_i(s, y), u_i(s, y))\right| \\ &\leq \gamma_{t,x;s,y}^2 + \gamma_{t,x;s,y}\sigma_{s,y} \\ &\leq c_H \,[\mathbf{\Delta}((t, x); (s, y))]^{1/2}, \end{aligned}$$

where we have used Lemma 3.1 twice in the last inequality. This implies the desired bound.

Finally, introducing inequalities (3.14) and (3.15) into (3.12) and using the independence of the components $u_1, \ldots, u_d$, the proposition follows. □

4. Proof of Theorem 1.1 and Corollary 1.3

In order to prove Theorem 1.1 we will follow the approach developed in [4] extended to our situation. For this we shall state and prove the versions of Theorem 2.1(1), Lemma 2.2(1), Theorem 3.1(1) and Lemma 4.5 in [4] needed in our situation.

The first result is an extension of [4, Lemma 2.2(1)] (take $\alpha = 1/2$, $H = 1/2$ and $d = \beta$).

Lemma 4.1. *Let I and J two intervals as in Theorem 1.1. Then for all $N > 0$, there exists a finite and positive constant $C = C(I, J, N)$ such that for all $a \in [0, N]$,*

$$\int_I \mathrm{d}t \int_I \mathrm{d}s \int_J \mathrm{d}x \int_J \mathrm{d}y \, \frac{\mathrm{e}^{-a^2/\mathbf{\Delta}((t,x);(s,y))}}{\mathbf{\Delta}^{d/2}((t, x); (s, y))} \leq C\, \mathrm{K}_{d-(\frac{1}{\alpha}+\frac{2}{\alpha\wedge(2H)})}(a), \tag{4.1}$$

where $\mathbf{\Delta}((t, x)\,; (s, y))$ is the metric defined in (3.1).

Proof. Write $\alpha_1 := 2\alpha$ and $\alpha_2 := \alpha \wedge (2H)$. Using the change of variables $\tilde{u} = t - s$ (t fixed), $\tilde{v} = x - y$ (x fixed) we have that the integral in (4.1) is bounded above by

$$4|I|\,|J| \int_0^{|I|} \mathrm{d}\tilde{u} \int_0^{|J|} \mathrm{d}\tilde{v}\, (\tilde{u}^{\alpha_1} + \tilde{v}^{\alpha_2})^{-d/2} \exp\left(-\frac{a^2}{\tilde{u}^{\alpha_1} + \tilde{v}^{\alpha_2}}\right). \tag{4.2}$$

A change of variables $[\tilde{u}^{\alpha_1} = a^2 u, \tilde{v}^{\alpha_2} = a^2 v]$ implies that this is equal to

$$C a^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-d} \int_0^{|I|^{\alpha_1} a^{-2}} \mathrm{d}u \int_0^{|J|^{\alpha_2} a^{-2}} \mathrm{d}v\, \frac{u^{\frac{1}{\alpha_1}-1} v^{\frac{1}{\alpha_2}-1}}{(u + v)^{d/2}} \exp\left(-\frac{1}{u + v}\right). \tag{4.3}$$

Observe that the last integral is bounded above by

$$\int_0^{|I|^{\alpha_1} a^{-2}} \mathrm{d}u \int_0^{|J|^{\alpha_2} a^{-2}} \mathrm{d}v\, (u + v)^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}-2-\frac{d}{2}} \exp\left(-\frac{1}{u + v}\right).$$

Pass to polar coordinates to deduce that the preceding is bounded above by $I_1 + I_2(a)$, where

$$I_1 := \int_0^{KN^{-2}} \mathrm{d}\rho\, \rho^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}-1-\frac{d}{2}} \exp(-c/\rho),$$

$$I_2(a) := \int_{KN^{-2}}^{Ka^{-2}} \mathrm{d}\rho\, \rho^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}-1-\frac{d}{2}},$$

where $K = |I|^{\alpha_1} \vee |J|^{\alpha_2}$. Clearly, $I_1 \leq C < \infty$, and if $d \neq \frac{2}{\alpha_1} + \frac{2}{\alpha_2}$, then

$$I_2(a) = K^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}-\frac{d}{2}} \frac{a^{d-\frac{2}{\alpha_1}-\frac{2}{\alpha_2}} - N^{d-\frac{2}{\alpha_1}-\frac{2}{\alpha_2}}}{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}-\frac{d}{2}}.$$

If $d > \frac{2}{\alpha_1} + \frac{2}{\alpha_2}$, then $I_2(a) \leq C$ for all $a \in [0, N]$. If $d < \frac{2}{\alpha_1} + \frac{2}{\alpha_2}$, then $I_2(a) \leq C a^{d-(\frac{2}{\alpha_1}+\frac{2}{\alpha_2})}$. Finally, if $d = \frac{2}{\alpha_1} + \frac{2}{\alpha_2}$, then

$$I_2(a) = 2\left[\ln\left(\frac{1}{a}\right) + \ln(N)\right].$$

Hence, we deduce that for all $a \in [0, N]$, the expression in (4.3) is bounded above by $C\,\mathrm{K}_{d-(\frac{2}{\alpha_1}+\frac{2}{\alpha_2})}(a)$, provided that N_0 in (1.4) is sufficiently large. This proves the lemma. □

The next result uses the proof of [4, Theorem 2.1(1)] applied to our situation and establishes the lower bound of Theorem 1.1.

Theorem 4.2. *Assume hypothesis (1.3). Let $I \subset (0, T]$ and $J \subset [0, 2\pi)$ be two fixed non-trivial compact intervals. Then for all $T > 0$ and $M > 0$, there exists a finite constant $c_H > 0$ such that for all compact sets $A \subseteq [-M, M]^d$,*

$$c_H\, \mathrm{Cap}_{d-\beta}(A) \leq \mathrm{P}\{u(I \times J) \cap A \neq \emptyset\},$$

where $\beta := \frac{1}{\alpha} + (\frac{2}{\alpha} \vee \frac{1}{H})$.

Proof. The proof of this result follows exactly the same lines as the proof of [4, Theorem 2.1(1)], therefore we will only sketch the steps that differ. It suffices to replace their $\beta - 6$ by our $d - \beta$ with $\beta := \frac{1}{\alpha} + (\frac{2}{\alpha} \vee \frac{1}{H})$. Moreover, if $p_{t,x}(y)$ denotes the density of $u(t, x)$ solution of (1.1), then we have that for all $y \in [-M, M]^d$ and $(t, x) \in I \times J$,

$$p_{t,x}(y) = (2\pi\sigma_{t,x}^2)^{-d/2} \mathrm{e}^{-\|y\|^2/(2\sigma_{t,x}^2)} \geq c_H, \tag{4.4}$$

which proves hypothesis **A1** of [4, Theorem 2.1(1)]. On the other hand, our Proposition 3.2 proves hypothesis **A2** with $\mathbf{\Delta}((t, x)\,; (s, y))$ defined as in (3.1).

We then follow the proof of [4, Theorem 2.1(1)]. Define, for all $z \in \mathbb{R}^d$ and $\epsilon > 0$, $\tilde{B}(z, \epsilon) := \{y \in \mathbb{R}^d : |y - z| < \epsilon\}$, where $|z| := \max_{1\leq j\leq d} |z_j|$, and

$$J_\epsilon(z) = \frac{1}{(2\epsilon)^d} \int_I \mathrm{d}t \int_J \mathrm{d}x\, \mathbf{1}_{\tilde{B}(z,\epsilon)}(u(t, x)).$$

In the case $d < \beta$, following as in [4, (2.30)] and using Proposition 3.2, we find that for all $z \in A \subseteq [-M, M]^d$ and $\epsilon > 0$,

$$\mathrm{E}\left[(J_\epsilon(z))^2\right] \leq c_H \int_I \mathrm{d}t \int_I \mathrm{d}s \int_J \mathrm{d}x \int_J \mathrm{d}y\, \mathbf{\Delta}^{-d/2}((t, x); (s, y)).$$

Then instead of [4, (2.31)], we get following as in (4.2) and using [4, Lemma 2.3], that for all $z \in A \subseteq [-M, M]^d$ and $\epsilon > 0$,

$$\mathrm{E}\left[(J_\epsilon(z))^2\right] \leq c_H \int_0^{|I|} \mathrm{d}u \int_0^{|J|} \mathrm{d}v\, (u^{2\alpha} + v^{\alpha\wedge(2H)})^{-d/2}$$
$$\leq c_H \int_0^{|I|} \mathrm{d}u\ \Psi_{|J|,d(\frac{\alpha}{2}\wedge H)}(u^{d\alpha})$$
$$\leq c_H \int_0^{|I|} \mathrm{d}u\, \mathrm{K}_{1-(\frac{2}{\alpha}\vee\frac{1}{H})/d}(u^{d\alpha}).$$

We will then consider the different cases: $d < \frac{2}{\alpha}\vee\frac{1}{H}$, $\frac{2}{\alpha}\vee\frac{1}{H} < d < \frac{1}{\alpha}+(\frac{2}{\alpha}\vee\frac{1}{H})$ and $d = \frac{2}{\alpha}\vee\frac{1}{H}$. This will prove the case $d < \beta$.

The case $d \geq \beta$ is proved exactly along the same lines as the proof of [4, Theorem 2.1(1)], appealing to (4.4), Proposition 3.2 and Lemma 4.1. □

The following result is an extension of [4, Lemma 4.5].

Lemma 4.3. *Assume hypothesis (1.3). For all $p \geq 1$, there exists $C_{p,H} > 0$ such that for all $\epsilon > 0$ and all (t, x) fixed,*

$$\mathrm{E}\left[\sup_{[\mathbf{\Delta}((t,x)\,;(s,y))]^{1/2}\leq\epsilon} \|u(t,x) - u(s,y)\|^p\right] \leq C_{p,H}\epsilon^p, \tag{4.5}$$

where $\mathbf{\Delta}((t, x)\,; (s, y))$ is defined as in (3.1).

Proof. It suffices to prove (4.5) for each coordinate u_i, $i = 1, \ldots, d$. We proceed as in [4, Lemma 4.5], that is, we will use [4, Proposition A.1] with $S := S_\epsilon = \{(s, y) : [\mathbf{\Delta}((t, x); (s, y))]^{1/2} < \epsilon\}$, $\rho((t, x), (s, y)) := [\mathbf{\Delta}((t, x); (s, y))]^{1/2}$, $\mu(\mathrm{d}t\mathrm{d}x) := \mathrm{d}t\mathrm{d}x$, $\Psi(x) := \mathrm{e}^{|x|} - 1$, $p(x) := x$, and $f := u_i$.

Moreover, by (3.2), the random variable $\mathscr{C}$ defined in [4, Proposition A.1] satisfies

$$\mathrm{E}\,[\mathscr{C}] \leq \mathrm{E}\left[\int_{S_\epsilon} \mathrm{d}t\,\mathrm{d}x \int_{S_\epsilon} \mathrm{d}s\,\mathrm{d}y\ \exp\left(\frac{|u_i(t,x) - u_i(s,y)|}{[\mathbf{\Delta}((t,x);(s,y))]^{1/2}}\right)\right] \leq c_H\epsilon^\beta,$$

where $\beta = \frac{2}{\alpha} + (\frac{4}{\alpha}\vee\frac{2}{H})$.

The rest of the proof follows exactly as in [4, (4.51)] and is therefore omitted. □

The next result uses the proof of [4, Theorem 3.1(1)] applied to our situation and establishes the upper bound of Theorem 1.1.

Theorem 4.4. *Assume hypothesis (1.3). Let $I \subset (0, T]$ and $J \subset [0, 2\pi)$ be two fixed non-trivial compact intervals. Then for all $T > 0$ and $M > 0$, there exists a finite constant $c_H > 0$ such that for all Borel sets $A \subseteq [-M, M]^d$,*

$$\mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leq c_H\, \mathscr{H}_{d-\beta}(A),$$

where $\beta := \frac{1}{\alpha} + (\frac{2}{\alpha}\vee\frac{1}{H})$.

Proof. When $d < \beta$, there is nothing to prove, so we assume that $d \geq \beta$. For all positive integers n, set $t_k^n := k2^{-n/\alpha}$, $x_\ell^n := \ell 2^{-(2n/\alpha)\vee(n/H)}$, and

$$I_k^n = [t_k^n, t_{k+1}^n], \qquad J_\ell^n = [x_\ell^n, x_{\ell+1}^n], \qquad R_{k,\ell}^n = I_k^n \times J_\ell^n.$$

Then for all $R^n_{k,\ell} \subset I \times J$, there exists a constant $c_H > 0$ such that the following estimate for hitting a small ball holds for all $z \in \mathbb{R}^d$ and $\epsilon > 0$,

$$\mathrm{P}\{u(R^n_{k,\ell}) \cap B(z,\epsilon) \neq \varnothing\} \leq c_H \epsilon^d. \tag{4.6}$$

Indeed, as $\{u_i(t,x)\}_{i=1,..,d}$ are independent, centered, Gaussian random variables, with variance bounded above and below by positive constants, and such that the upper bound in (3.2) and Lemma 4.3 hold, the proof of (4.6) follows exactly along the same lines as the proof of [4, Proposition 4.4]. Note also that because $\{u_i(t,x), u_i(t^n_k, x^n_\ell)\}$ is a two-dimensional centered Gaussian vector, the random variables $Y^n_{k,\ell}$ and $Z^n_{k,\ell}$ defined in [4, (4.58)] are independent.

Finally, the result follows directly from [4, Theorem 3.1(1)] replacing their β by d and the 6 by our β. □

Proof of Theorem 1.1. Theorems 4.2 and 4.4 prove the lower and upper bounds of Theorem 1.1, respectively. □

Proof of Corollary 1.3. (a) This is an immediate consequence of Theorem 1.1.

(b) Let $z \in \mathbb{R}^d$. If $d < \beta$, then $\mathrm{Cap}_{d-\beta}(\{z\}) = 1$. Hence, the lower bound of Theorem 1.1 implies that $\{z\}$ is not polar. On the other hand, if $d > \beta$, then $\mathscr{H}_{d-\beta}(\{z\}) = 0$ and the upper bound of Theorem 1.1 implies that $\{z\}$ is polar.

(c) Theorem 1.1 implies that for $d \geq 1$: $\mathrm{codim}(u(\mathbb{R}_+ \times S^1)) = (d-\beta)^+$; where $\mathrm{codim}(E)$ with E a random set is defined in [4, (5.12)]. Then, when $d > \beta$, [4, (5.13)] implies the desired result.

The case $d = \beta$ follows using exactly the same argument that led to the result in [4, Corollary 5.3(a)] for $d = 6$, and is therefore omitted. □

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Appendix

A.1. Riesz-kernel example

We consider the example of the Riesz kernel. There, we assume that $Q(x) = |x|^{-\gamma}$ for some $\gamma \in (0,1)$. We then first need to show that this is a *bonafide* homogeneous spatial covariance function on the circle (that this is such a function in Euclidean space is well-known, but here we are restricted to the circle). In other words, we need to show that

$$Q(x) = \sum_{n=0}^{\infty} q_n \cos nx,$$

where $\{q_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers. Since Q is integrable, we simply calculate the values q_n by (inverse) Fourier transform: using the symmetry of Q, and some scaling, we obtain

$$\begin{aligned} q_n &= \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}nx} |x|^{-\gamma} \mathrm{d}x = 2\int_0^{\pi} \cos(nx)\, x^{-\gamma} \mathrm{d}x \\ &= 2n^{\gamma-1} \int_0^{n\pi} \cos(x)\, x^{-\gamma} \mathrm{d}x \end{aligned}$$

$$= n^{\gamma-1} \sum_{k=0}^{n-1} r(k),$$

where $r(k) = 2\int_{k\pi}^{(k+1)\pi} \cos(x)\, x^{-\gamma} \mathrm{d}x$. We can calculate this $r(k)$ a bit further: using an integration by parts, we get

$$r(k) = 2\gamma \int_{k\pi}^{(k+1)\pi} x^{-\gamma-1} \sin(x)\,\mathrm{d}x$$
$$= 2\gamma(-1)^k \int_{k\pi}^{(k+1)\pi} x^{-\gamma-1} |\sin(x)|\,\mathrm{d}x.$$

Hence we do indeed have, as announced in the Riesz-kernel example, that $q_n = n^{\gamma-1}c(n)$ where $c(n)$ is the partial sum of the alternating sequence with general term $2r(k)$. Also as announced, we clearly see that $r(0) > 0$, and it is trivial to prove that $|r(k+1)| < |r(k)|$, by simply using the change of variable $x' = x - \pi$, and the fact that $\sin(x' + \pi) = -\sin(x')$. The partial sums of such an alternating series are always positive since the first term is positive. All the claims in the Riesz-kernel example are justified.

A.2. Fractional Brownian example

In the fractional Brownian noise class of examples, with $H < 1/2$ and where $q_n = c(n)n^{1-2H}$ for some function c which is bounded above and below by positive constants, the Fourier series representation $Q(x) = \sum_{n=0}^{\infty} c(n)n^{1-2H}\cos(nx)$ is only formal because this series diverges even as an alternating series. Yet we can interpret B^H as the spatial derivative of a process similar to a space–time fractional Brownian sheet. Indeed, consider the centered Gaussian field $Y(t,x)$ which is fractional Brownian in time with parameter H, and has spatial covariance equal to $R(x,y) = R(0) - |x-y|^{2H}$. This field, which is spatially homogeneous on the circle for fixed t, is not the usual fractional Brownian sheet on the circle since the latter is not spatially homogeneous. However, the reader will immediately check that B^H and the standard fractional Brownian sheet share the same canonical metric (standard deviation of their increments), which means that their increments have the same regularity and scaling properties. Using exactly the same calculations as in the Riesz-kernel case above, but this time with $\gamma = -2H$, we can still invoke the fact that $x^{-\gamma-1}$ is decreasing, since $2H - 1 < 0$, and thus $R(x,y)$ can be written as $R(0) + \sum_{n=1}^{\infty} c(n)\, n^{-2H-1}\cos(nx)$ where, as in the previous example, $c(n)$ is the partial sum of a positive alternating series. It is then easy to see that Y can be represented as

$$Y(t,x) = \sqrt{R(0)}B_{0,H}(t) + \sum_{n=1}^{\infty} \sqrt{c(n)}n^{-H-1/2}\cos(nx)\, B_{n,H}(t)$$
$$+ \sum_{n=1}^{\infty} \sqrt{c(n)}n^{-H-1/2}\sin(nx)\, \tilde{B}_{n,H}(t),$$

where $\{B_{n,H}\}_{n\in\mathbb{N}}$ and $\{\tilde{B}_{n,H}\}_{n\in\mathbb{N}}$ are independent sequences of IID standard fractional Brownian motions. If one then defines the noise in the heat equation formally (i.e. in the sense of distributions) by

$$B_H(t,x) = \frac{\partial}{\partial x} Y(t,x),$$

a factor n comes out in the Fourier representation, and one gets that B_H can be written, in the sense of distributions, as

$$B_H(t,x) = \sum_{n=1}^{\infty} \sqrt{c(n)} n^{-H+1/2} \cos(nx) B_{n,H}(t) + \sum_{n=1}^{\infty} \sqrt{c(n)} n^{-H+1/2} \sin(nx) \tilde{B}_{n,H}(t),$$

from which the formula $q_n = c(n) n^{1-2H}$ follows, i.e. the formal expansion $Q(x) = \sum_{n=0}^{\infty} c(n) n^{-2H+1} \cos(nx)$ follows immediately. This justifies using the scale n^{1-2H} as the covariance's Fourier coefficient in order to construct a space–time fractional Brownian noise. Note that this justification also works when $H > 1/2$.

It is instructive to note that one can also formally write

$$Q(x-y) = \mathrm{E}\left[\frac{\partial}{\partial x} Y(1,x) \frac{\partial}{\partial y} Y(1,y)\right] = (\partial^2/\partial x \partial y) |x-y|^{2H} = 2H(2H-1)|x-y|^{2H-2},$$

which is not integrable at the origin ($x = y$) when $H < 1/2$, which explains why one cannot use the pointwise Fourier and/or the Riesz-kernel representation in this case.

A.3. Estimates of the kernel K^H

We have the following estimates on the kernel K^H.

Lemma A.1. *Let $t_0, T \geq 0$ be fixed. Then for any $H < \frac{1}{2}$ and $s, t \in [t_0, T]$ with $s \leq t$, there exist positive constants $c(t_0, T, H)$ and $C(t_0, T, H)$ such that*

$$c(t_0,T,H)^{-1}(t-s)^{H-\frac{1}{2}} \leq K^H(t,s) \leq c(t_0,T,H)(t-s)^{H-\frac{1}{2}} s^{H-\frac{1}{2}},$$
$$C(t_0,T,H)^{-1}\left(H-\frac{1}{2}\right)(t-s)^{H-\frac{3}{2}} \leq \frac{\partial K^H}{\partial t}(t,s) \leq C(t_0,T,H)\left(H-\frac{1}{2}\right)(t-s)^{H-\frac{3}{2}}.$$

Proof. Theses estimates follow immediately from (2.1), (2.2) and [6, Theorem 3.2]. □

The following is a two real variable technical result that is used several times in this paper.

Lemma A.2. *Let $t_0 > 0$ be fixed. Then for any $s \geq t_0$, there exists a positive constant $c(t_0, H)$ such that*

$$\int_0^{2n^2 s} \left(s - \frac{v}{2n^2}\right)^{2H-1} v^{2H-1} \mathrm{e}^{-v} \, \mathrm{d}v \leq c(t_0, H).$$

Proof. We write, following [16, eq. (25)],

$$\int_0^{2n^2 s} \left(s - \frac{v}{2n^2}\right)^{2H-1} v^{2H-1} \mathrm{e}^{-v} \, \mathrm{d}v$$
$$\leq \left(\frac{s}{2}\right)^{2H-1} \int_0^{\infty} v^{2H-1} \mathrm{e}^{-v} \, \mathrm{d}v + (n^2 s)^{2H-1} \int_{n^2 s}^{2n^2 s} \left(s - \frac{v}{2n^2}\right)^{2H-1} \mathrm{e}^{-v} \, \mathrm{d}v$$

$$
\begin{aligned}
&\leq c_H {t_0}^{2H-1} + (n^2 s)^{2H-1} \int_0^{n^2 s} \left(\frac{v'}{2n^2} \right)^{2H-1} \mathrm{e}^{-(2n^2 s - v')} \, \mathrm{d}v' \\
&\leq C(t_0, H) + c_H {t_0}^{2H-1} \mathrm{e}^{-n^2 s} (n^2 s)^{2H} \\
&\leq C(t_0, H) + C(t_0, H) \sup_{x \geq s} |\mathrm{e}^{-x} x^{2H}| \\
&\leq C(t_0, H). \quad \square
\end{aligned}
$$

A.4. *Further covariance calculations*

Proof of 3.10. With the notations of the proof of Lemma 3.1, we will show that for K large enough and for all n such that $n^2(t-s) \geq K$, when $h_{t,s,x,y} \geq 0$,

$$
\tilde{I}_1 + \bar{I}_1 > 2 \left| \tilde{I}_{4,1} + \bar{I}_{4,1} \right|. \tag{A.1}
$$

This will prove (3.10).

Using Lemma A.1, and the trivial bound $h_{t,s,x,y} \leq 2$ applied to (3.7), we have

$$
\begin{aligned}
\left| \tilde{I}_{4,1} + \bar{I}_{4,1} \right| &= \int_0^s \mathrm{d}r \, K(s,r) h(r) \left(\int_s^t \left| \frac{\partial K}{\partial u}(u,r) \right| \mathrm{d}u \right) g(r) \\
&\leq c_H \int_0^s \mathrm{d}r \, (s-r)^{H-1/2} \mathrm{e}^{-n^2(t+s-2r)} \left((s-r)^{H-1/2} - (t-r)^{H-1/2} \right) \\
&= c_H \mathrm{e}^{-n^2(t-s)} \int_0^s \mathrm{d}r \, r^{H-1/2} \left(r^{H-1/2} - (r+t-s)^{H-1/2} \right) \mathrm{e}^{-2n^2 r}.
\end{aligned}
$$

We evaluate the integral above by splitting it up according to whether r exceeds n^{-2}. We also assume that $n^2(t-s) \geq 1$, i.e. we restrict $K \geq 1$. Hence

$$
\begin{aligned}
&\int_0^{n^{-2}} \mathrm{d}r \, r^{H-1/2} \left(r^{H-1/2} - (r+t-s)^{H-1/2} \right) \mathrm{e}^{-2n^2 r} \\
&\leq \int_0^{n^{-2}} \mathrm{d}r \, r^{H-1/2} \left(r^{H-1/2} - (2t-2s)^{H-1/2} \right) \\
&= \int_0^{n^{-2}} \mathrm{d}r \, \left(r^{2H-1} - r^{H-1/2} (2t-2s)^{H-1/2} \right) \\
&\leq c_H n^{-4H}.
\end{aligned}
$$

The other piece is

$$
\begin{aligned}
&\int_{n^{-2}}^s \mathrm{d}r \, r^{H-1/2} \left(r^{H-1/2} - (r+t-s)^{H-1/2} \right) \mathrm{e}^{-2n^2 r} \\
&\leq c_H \int_{n^{-2}}^s \mathrm{d}r \, r^{H-1/2} (t-s) \, r^{H-3/2} \mathrm{e}^{-2n^2 r} = c_H (t-s) \int_{n^{-2}}^s \mathrm{d}r \, r^{2H-2} \mathrm{e}^{-2n^2 r} \\
&= c_H n^{-2} n^{4-4H} (t-s) \int_1^{n^2 s} \mathrm{d}x \, x^{2H-2} \mathrm{e}^{-2x} \\
&\leq c_H n^{-4H} n^2 (t-s) \int_1^{\infty} \mathrm{d}x \, x^{2H-2} \mathrm{e}^{-2x} \\
&= c_H n^{-4H} n^2 (t-s).
\end{aligned}
$$

In conclusion, we get

$$\left|\tilde{I}_{4,1}+\bar{I}_{4,1}\right| \le c_H^1 n^{-4H}\left(1+n^2(t-s)\right)e^{-n^2(t-s)}.$$

Since the function $x \mapsto (1+x)e^{-x}$ decreases to 0 as x increases to ∞, we only need to choose K sufficiently large such that for all n with $n^2(t-s) \ge K$, $\left|\tilde{I}_{4,1}+\bar{I}_{4,1}\right| \le 2^{-1}c_H^1 n^{-4H}(1-e^{-n^2(t-s)})^2 \le \tilde{I}_1+\bar{I}_1$, where c_H^1 is the constant in (3.6). This completes the proof of (A.1). □

Proof of 3.11. We now show that for K large enough and for all n such that $n^2(t-s) \ge K$, when $h_{t,s,x,y} \ge 0$,

$$\tilde{I}_1+\bar{I}_1 > 2\left|\tilde{I}_{4,2}+\bar{I}_{4,2}\right|. \tag{A.2}$$

This will prove (3.11).

Again using Lemma A.1, and the bound $h_{t,s,x,y} \le 2$ applied to (3.7), we have

$$\begin{aligned}\left|\tilde{I}_{4,2}+\bar{I}_{4,2}\right| &= h_{t,s,x,y}\int_0^s \mathrm{d}r\,K(s,r)e^{-n^2(s-r)}\int_s^t \mathrm{d}u(g(u)-g(r))\left|\frac{\partial K}{\partial u}(u,r)\right|\\ &\le c_H\int_0^s \mathrm{d}r(s-r)^{H-1/2}e^{-n^2(s+t-r)}\int_s^t \mathrm{d}u\left(e^{n^2u}-e^{n^2r}\right)(u-r)^{H-3/2}.\end{aligned}$$

We cut this integral into three pieces. First calculate the piece for $u > s+n^{-2}$:

$$\begin{aligned}&\int_0^s \mathrm{d}r(s-r)^{H-1/2}e^{-n^2(s+t-r)}\int_{s+n^{-2}}^t \mathrm{d}u\left(e^{n^2u}-e^{n^2r}\right)(u-r)^{H-3/2}\\ &\le \int_0^s \mathrm{d}r(s-r)^{H-1/2}e^{-n^2(s+t-2r)}\int_{s+n^{-2}}^t \mathrm{d}u e^{n^2(u-r)}(u-r)^{H-3/2}\\ &= n^{-2H+1}\int_0^s \mathrm{d}r(s-r)^{H-1/2}e^{-n^2(s+t-2r)}\int_{(s-r)n^2+1}^{(t-r)n^2} e^x x^{H-3/2}\mathrm{d}x\\ &= n^{-4H}\int_0^{sn^2}\mathrm{d}y\, y^{H-1/2}e^{-y}e^{-n^2(t-s)}\int_{y+1}^{y+n^2(t-s)} e^x x^{H-3/2}\mathrm{d}x.\end{aligned}$$

Now, for any fixed constants $y_0(H)$ and $y_1(H)$ such that $y_1 > y_0+1$, the above term with the y-integral restricted to $y \le y_0$ can be written as follows:

$$\begin{aligned}&n^{-4H}\int_0^{y_0}\mathrm{d}y\, y^{H-1/2}e^{-y}e^{-n^2(t-s)}\left(\int_{y+1}^{y_1} e^x x^{H-3/2}\mathrm{d}x+\int_{y_1}^{y+n^2(t-s)} e^x x^{H-3/2}\mathrm{d}x\right)\\ &\le n^{-4H}\int_0^{y_0}\mathrm{d}y\, y^{H-1/2}\left(e^{-n^2(t-s)}c(H,y_1)+y_1^{H-3/2}e^{y_0}\right).\end{aligned}$$

We now choose y_1 and K large enough such that for all n with $n^2(t-s) \ge K$ and for any choice of y_0, the above equation is smaller than $c_H n^{-4H}$ with $c_H \le 2^{-1}c_H^1(1-e^{-n^2(t-s)})^2$, where c_H^1 is the constant in (3.6).

For the other part of the integral in y we get

$$n^{-4H}\int_{y_0}^{sn^2}\mathrm{d}y\, y^{H-1/2}e^{-y}e^{-n^2(t-s)}\int_{y+1}^{y+n^2(t-s)} e^x x^{H-3/2}\mathrm{d}x$$

$$
\begin{aligned}
&\leq n^{-4H} \int_{y_0}^{sn^2} \mathrm{d}y\; y^{2H-2} \mathrm{e}^{-y} \mathrm{e}^{-n^2(t-s)} \int_{y+1}^{y+n^2(t-s)} \mathrm{e}^{x} \mathrm{d}x \\
&\leq n^{-4H} \int_{y_0}^{sn^2} \mathrm{d}y\; y^{2H-2} \\
&\leq c_H n^{-4H} y_0^{2H-1},
\end{aligned}
$$

and it is sufficient to take y_0 large enough to ensure that this last expression is smaller than $c_H n^{-4H}$ with $c_H \leq 2^{-1} c_H^1 (1 - \mathrm{e}^{-n^2(t-s)})^2$.

Now we calculate the piece for $u \in [s, s+n^{-2}]$ and $r \in [s-n^{-2}, s]$. This yields a piece bounded above by

$$
\begin{aligned}
&c_H \int_{s-n^{-2}}^{s} \mathrm{d}r (s-r)^{H-1/2} \mathrm{e}^{-n^2 t} \int_{s}^{s+n^{-2}} \mathrm{d}u \left(\mathrm{e}^{n^2 s+1} - \mathrm{e}^{n^2 s-1}\right) (u-r)^{H-3/2} \\
&\leq c_H \mathrm{e}^{-n^2(t-s)} \int_{s-n^{-2}}^{s} \mathrm{d}r (s-r)^{H-1/2} \left((s-r)^{H-1/2} - (s-r+n^{-2})^{H-1/2}\right) \\
&= c_H \mathrm{e}^{-n^2(t-s)} n^{-4H} \int_0^1 x^{H-1/2} \left(x^{H-1/2} - (x+1)^{H-1/2}\right) \mathrm{d}x \\
&= c_H \mathrm{e}^{-n^2(t-s)} n^{-4H}
\end{aligned}
$$

which can obviously be made smaller than $2^{-1} c_H^1 (1 - \mathrm{e}^{-n^2(t-s)})^2$, for all n such that $n^2\,(t-s) \geq K$, provided that K is large enough.

The last piece to deal with is

$$
\begin{aligned}
&c_H \int_0^{s-n^{-2}} \mathrm{d}r (s-r)^{H-1/2} \mathrm{e}^{-n^2(s+t-r)} \int_s^{s+n^{-2}} \mathrm{d}u \left(\mathrm{e}^{n^2 u} - \mathrm{e}^{n^2 r}\right) (u-r)^{H-3/2} \\
&\leq c_H \int_0^{s-n^{-2}} \mathrm{d}r (s-r)^{H-1/2} \mathrm{e}^{-n^2(s+t-r)} \int_s^{s+n^{-2}} \mathrm{d}u\; \mathrm{e}^{n^2 u} (u-r)^{H-3/2} \\
&\leq c_H e \int_0^{s-n^{-2}} \mathrm{d}r (s-r)^{H-1/2} \mathrm{e}^{-n^2(t-r)} \int_s^{s+n^{-2}} \mathrm{d}u\; (u-r)^{H-3/2} \\
&= c_H \mathrm{e}^{-n^2(t-s)} \int_0^{s-n^{-2}} \mathrm{d}r (s-r)^{H-1/2} \left((s-r)^{H-1/2} - (s-r+n^{-2})^{H-1/2}\right) \\
&\leq c_H \mathrm{e}^{-n^2(t-s)} n^{-4H} \int_1^{\infty} x^{H-1/2} \left(x^{H-1/2} - (x+1)^{H-1/2}\right) \mathrm{d}x \\
&\leq c_H \mathrm{e}^{-n^2(t-s)} n^{-4H} \int_1^{\infty} x^{H-3/2} \mathrm{d}x \\
&= c_H \mathrm{e}^{-n^2(t-s)} n^{-4H},
\end{aligned}
$$

and the conclusion is the same as before. This finishes the proof of (A.2). □

References

[1] E. Alós, O. Mazet, D. Nualart, Stochastic calculus with respect to Gaussian processes, Ann. Probab. 29 (2) (1999) 766–801.

[2] R. Balan, C. Tudor, The stochastic heat equation with a fractional colored noise: Existence of the solution, Preprint. Available at: http://arxiv.org/abs/math/0703088, 2008.

[3] R.C. Dalang, E. Nualart, Potential theory for hyperbolic SPDEs, Ann. Probab. 32 (2004) 2099–2148.

[4] R.C. Dalang, D. Khoshnevisan, E. Nualart, Hitting probabilities for systems of non-linear stochastic heat equations with additive noise, Latin Amer. J. Probab. Math. Stat. 3 (2007) 231–271.

[5] R.C. Dalang, D. Khoshnevisan, E. Nualart, Hitting probabilities for systems of non-linear stochastic heat equation with multiplicative noise, Probab. Theory Related Fields (2008) (in press). Published online. http://www.springerlink.com/content/100451/.

[6] L. Decreusefond, A.-S. Ustunel, Stochastic analysis of the fractional Brownian motion, Potential Anal. 10 (1997) 177–214.

[7] T.E. Duncan, B. Maslowski, B. Pasik-Duncan, Fractional Brownian motion and stochastic equations in Hilbert spaces, Stoch. Dynam. 2 (2002) 225–250.

[8] T.E. Duncan, B. Maslowski, B. Pasik-Duncan, Some solutions of semilinear stochastic equations in a hilbert space with a fractional brownian motion, in: 45th IEEE Conference on Decision and Control, 2006, pp. 3077–3082.

[9] J.-P. Kahane, Some Random Series of Functions, Cambridge University Press, 1985.

[10] D. Khoshnevisan, Multiparameter Processes, Springer-Verlag, New York, 2002.

[11] B. Maslowski, D. Nualart, Evolution equations driven by fractional Brownian motion, J. Funct. Anal. 202 (2003) 277–305.

[12] D. Nualart, The Malliavin Calculus and Related Topics, second ed., Springer-Verlag, 2006.

[13] L. Quer-Sardanyons, S. Tindel, The 1-d stochastic wave equation driven by a fractional Brownian sheet, Stoch. Processes Appl. 117 (2007) 1448–1472.

[14] M. Sanz-Solé, M. Sarrà, Path properties of a class of Gaussian processes with applications to SPDE's, Canadian Mathematical Society Conference Proceedings 28 (2000) 303–316.

[15] M. Sanz-Solé, M. Sarrà, Hölder continuity for the stochastic heat equation with spatially correlated noise, in: Seminar on Stochastic Analysis, Random Fields ans Applications, III (Ascona, 1999), Progr. Probab. 52 (2002) 259–268.

[16] S. Tindel, C.A. Tudor, F. Viens, Stochastic evolution equations with fractional Brownian motions, Probab. Theory Related Fields 127 (2003) 186–204.

[17] S. Tindel, C.A. Tudor, F. Viens, Sharp Gaussian regularity on the circle, and applications to the fractional stochastic heat equation, J. Funct. Anal. 217 (2004) 280–313.

[18] Y. Sarol, F. Viens, Time regularity of the evolution solution to the fractional heat equation, Discrete Contin. Dyn. Syst. B 6 (2006) 895–910.