Stochastics and Dynamics, Vol. 8, No. 3 (2008) 451–473

LYAPUNOV EXPONENTS FOR STOCHASTIC ANDERSON MODELS WITH NON-GAUSSIAN NOISE

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Received 31 December 2007

The stochastic Anderson model in discrete or continuous space is defined for a class of non-Gaussian spacetime potentials W as solutions u to the multiplicative stochastic heat equation $u(t,x) = 1 + \int_0^t \kappa\Delta u(s,x)ds + \int_0^t \beta W(ds,x)u(s,x)$ with diffusivity κ and inverse-temperature β. The relation with the corresponding polymer model in a random environment is given. The large time exponential behavior of u is studied via its almost sure Lyapunov exponent $\lambda = \lim_{t\to\infty} t^{-1}\log u(t,x)$, which is proved to exist, and is estimated as a function of β and κ for $\beta^2\kappa^{-1}$ bounded below: positivity and nontrivial upper bounds are established, generalizing and improving existing results. In discrete space λ is of order $\beta^2/\log(\beta^2/\kappa)$ and in continuous space it is between $\beta^2(\kappa/\beta^2)^{H/(H+1)}$ and $\beta^2(\kappa/\beta^2)^{H/(1+3H)}$.

Keywords: Anderson model; polymer; random environment; non-Gaussian; Lyapunov exponent; Malliavin derivative; Feynman–Kac formula; strong disorder.

AMS Subject Classification: 22E46, 53C35, 57S20

1. Introduction

1.1. *Model and motivations*

The *stochastic Anderson model* is a stochastic parabolic partial differential equation, namely the stochastic heat equation with linear multiplicative potential: for all $x \in \mathbf{R}^d$ and all $t \geq 0$,

$$u(t,x) = 1 + \int_0^t \kappa \Delta u(s,x) ds + \int_0^t \beta W(ds,x) u(s,x). \tag{1.1}$$

In this paper, κ is a fixed positive diffusion constant, β is a fixed "inverse-temperature" parameter, Δu is the Laplacian of u or its discrete analogue, the potential $W(s,x)$ is a centered random field on $\mathbf{R}_+ \times \mathbf{R}^d$ or $\mathbf{R}_+ \times \mathbf{Z}^d$, which is stationary in the space parameter x, and whose time-derivative has a white-noise behavior in time. In the relatively long history of previous works on the continuous-time stochastic Anderson model [3, 4, 6–9, 11–13, 18–20], authors have only considered the case of a Gaussian field W, special non-Gaussian cases being used only in discrete time (see [10] and references therein). However, one primary original motivation for studying this model was to understand the structure of its *Lyapunov exponents* — large time exponential explosion rates — in analogy to these rates for products of random matrices and other stochastic differential systems, going back to the celebrated multiplicative ergodic theorem of Oseledets, and later extensively developed by Ludwig Arnold and his school in the general cocycle form for random dynamical systems (see Arnold's excellent recent monograph [2]). There is typically no restriction on the type of non-degenerate random elements that can be used to study these finite-dimensional systems' Lyapunov exponents; for instance Oseledets' theorem is valid for wide classes of distributions, just like its additive analogues (e.g. Kingman's sub-additive theorem). This motivates the use of non-Gaussian noise in the infinite-dimensional dynamical system (1.1) here, the objective being to estimate the *almost sure* Lyapunov exponent $\lambda := \lim_{t\to\infty} t^{-1} \log u(t,x)$ when it exists. In order to simplify the presentation, d is taken as 1, but the results herein can be proved using identical techniques for any d.

Beyond extending the study of λ for the Anderson model by considering non-Gaussian potentials, this paper investigates the behavior of λ as a function of *both* the diffusivity parameter κ and the inverse-temperature parameter β. We believe that such a study has never been attempted before. Our results show that λ, which is non-random and x-independent, depends on (κ, β), and is commensurate, in some scales, to the product of a universal factor β^2 and a second factor which is a function of κ/β^2 when this ratio is small, where the function depends on the potential W's spatial regularity. In particular, the dependence on (κ, β) is nontrivial in the sense that no scaling can be performed to reduce the study to $\kappa = 1$ or to $\beta = 1$.

For instance, on $\mathbf{R}_+ \times \mathbf{Z}^d$, when β^2/κ is bounded below, λ is of order $\beta^2/\log(\beta^2/\kappa)$, which has physical interpretations in the sense of fast dynamo and strong disorder, as we allude to briefly at the end of this introduction. In the case of continuous space $\mathbf{R}_+ \times \mathbf{R}^d$, even when restricted to fixed κ or fixed β, our results are sharper than any previously published: we find that, when β^2/κ is bounded below and W is spatially H-Hölder continuous, λ is sandwiched between $\beta^2(\kappa/\beta^2)^{H/(H+1)}$ and $\beta^2(\kappa/\beta^2)^{H/(1+3H)}$, thereby further closing a gap which, in the case of $\beta = 1$, had already been reduced in [12]. These improved results are made possible by borrowing some tools from [12, 18], and the recent preprint [5], using them more efficiently herein, and also introducing new tools.

From the physical standpoint, our results are the hallmarks of an important set of effects known as *strong disorder*. Indeed, consider the random (W-dependent) probability measure P^W defined by

$$dP^W/dP_b = Z_t^{-1} \exp \int_0^t \beta W(ds, b_s),$$

where P_b is the law of a standard Brownian motion b independent of W, and Z_t is the normalizing factor $E_b[\exp \int_0^t \beta W(ds, b_s)]$ needed to make the total mass of P^W equal to 1. This P^W is called the law of the *Brownian polymer* in the *random environment* W (see [18]). It turns out that the law of Z_t is the same as the law of $u(t,x)$ for any fixed x. The polymer measure P^W is interesting if it is significantly different from the Wiener measure, which means that the *random environment's Hamiltonian*: $\int_0^t \beta W(ds, b_s)$ has a nontrivial effect on each path (polymer) b, a property which can be called strong disorder. If W does not depend on the parameter x, then W cannot have any effect on b, and we see that in this case $\lambda = \lim_{t\to\infty} t^{-1} \log Z_t = 0$; thus it is interesting to be able to ensure that $\lambda > 0$. Our lower bound results show that this holds for arbitrarily high temperature β^{-1} as long as the diffusivity is accordingly small; whether this positivity of λ also holds for β^{-1} arbitrarily large with κ fixed is yet an open problem.

Another way to measure the nontriviality of the Hamiltonian's influence on the polymer path b (strong disorder) is to look for a gap between λ (the "quenched" Lyapunov exponent) and its "annealed" analog, the Lyapunov exponent of its average:

$$\lambda_a = \lim_{t\to\infty} t^{-1} \log \mathbf{E}[Z_t].$$

If W had little or no effect on Z_t, one should arguably obtain the same Lyapunov exponent whether or not one averages Z against W's randomness. For instance in the Gaussian case, it is an elementary calculation (see (3.1) below) to prove that $\lambda_a = 2^{-1}\beta^2 Q(0)$ where $Q(0)$ is the common conditional variance of $W(1,x)$. Thus being able to ensure that $\lambda < 2^{-1}\beta^2 Q(0)$ is another sign of strong disorder. Our upper bound results prove that this holds for arbitrarily high temperature as long as κ/β^2 is sufficiently small; in fact the factors $1/\log(\beta^2/\kappa)$ or $(\kappa/\beta^2)^{H/(H+1)}$ can be made arbitrarily small, indicating a very pronounced strong disorder in the corresponding parameter range. Whether $\lambda < \lambda_a$ still holds for arbitrarily small κ/β^2 is also an open question.

Lastly, we mention the issue of stochastic fast dynamo. The Anderson model is a 1-D toy model for the fundamental equation of 3-D magneto-hydrodynamics (MHD) describing the evolution of a magnetic field H, which is a system of three coupled Anderson models with an additional first-order transport term (see [15]). The stochastic fast dynamo conjecture is that if the velocity field is a random field with enough turbulence, the almost-sure Lyapunov exponent of the magnetic energy is positive, and increases dramatically as diffusion is turned on (as κ goes from 0 to being positive, for fixed temperature). Our lower bounds $\beta^2/\log(\beta^2/\kappa)$ or $\beta^2(\kappa/\beta^2)^{H/(H+1)}$ increase indeed very rapidly from 0 as κ increases from 0, showing that fast dynamo can be expected in the 3-D problem as well. This is not a new observation; our results show, however, that this holds for any temperature, and for non-Gaussian noise.

1.2. *Summary of main results*

We begin with a random field W which, conditionally on a stochastic process q defined on the real line (or on the unit circle in the case of $\mathbf{Z}$), is spatially homogeneous and Brownian in time; W is constructed so that q is the density of the Fourier transform of its random spatial convariance; in particular the conditional variance of $W(1,x)$ is $Q(0) = \int_{\mathbf{R}} q(y)dy$ for any x (or $\int_{[0,2\pi)} q(y)dy$ in the case of $\mathbf{Z}$). See the next section for a precise construction of this non-Gaussian noise W and its relation to the random variable $Q(0)$. In this paper, we prove the following:

(1) the so-called *almost-sure Lyapunov exponent* λ defined by

$$\lambda = \lim_{t\to\infty, t\in\mathbf{N}} \frac{1}{t}\log u(t,x), \tag{1.2}$$

exists, does not depend on x, and is non-random (Theorem 3.1, p. 457);

(2) for $(t,x) \in \mathbf{R}_+ \times \mathbf{Z}$ and c_+, c_1, c_3 constant depending only on the law of q, if $Q(0)$ has a moment of order > 1,

$$c_1\beta^2/\log(\beta^2/\kappa) \le \lambda \le c_3\beta^2/\log(\beta^2/\kappa)$$

hold for $\beta^2/\kappa > c_+$ (Theorem 4.1, p. 462);

(3) for $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$ and c_{++}, c_2, c_4 constant depending only on the law of q, if $Q(0)$ has a moment of order > 1,

$$c_2\beta^2(\kappa/\beta^2)^{H/(H+1)} \le \lambda \le c_4\beta^2(\kappa/\beta^2)^{H/(1+3H)}$$

hold for $\beta^2/\kappa > c_{++}$ (Theorems 5.1 and 5.2, p. 469).

The paper is structured as follows. Section 2 constructs W and gives the Feynman–Kac formula for the solution $u(t,x)$ of (1.1); all proofs in this paper rely on this formula. In Sec. 3, existence of λ is proved by first showing in Sec. 3.1 that $\mathbf{E}[t^{-1}\log u(t,x)]$ converges, and then proving almost sure convergence via a concentration inequality established in Sec. 3.2 by using a Malliavin calculus method. The lower and upper bounds for the Anderson model in discrete space are established in Secs. 4.1 and 4.2 respectively, while the corresponding results in continuous space are dealt with in Secs. 5.1 and 5.2.

2. Preliminaries

To simplify our presentation, we consider mainly the case of $d = 1$, but all our results hold for arbitrary spatial dimension d. Moreover, in this section, we present our model for the parameter space $\mathbf{R}_+ \times \mathbf{R}$, but nearly identical constructions also hold for $\mathbf{R}_+ \times \mathbf{Z}$, a fact which we will not comment on further. Here and throughout, the letter λ is used to denote the Fourier variable, a standard notation; this should not cause any confusion with the use of the letter λ for the Lyapunov exponent.

Let W be a separable centered random field on $\mathbf{R}_+ \times \mathbf{R}$, defined under some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that

$$W(t,x) = \int_{\mathbf{R}_+ \times \mathbf{R}} \mathbf{1}_{[0,t]}(s) e^{i\lambda x} \sqrt{q(\lambda)} M(ds, d\lambda), \tag{2.1}$$

where M is a Gaussian independently scattered white-noise measure on $\mathbf{R}_+ \times \mathbf{R}$, defined on some probability space $(\Omega_M, \mathcal{F}_M, \mathbf{P}_M)$ and q is a non-negative random process on $\mathbf{R}$ defined on another probability space $(\Omega_q, \mathcal{F}_q, \mathbf{P}_q)$ independent of M, satisfying $q \in L^1 = L^1(\Omega_q \times \mathbf{R})$ with respect to $\mathbf{P}_q \times$ Lebesgue's measure, i.e.

$$\mathbf{E}_q \left[\int_{\mathbf{R}} q(\lambda) d\lambda \right] < \infty, \tag{2.2}$$

where $\mathbf{E}_q$ denotes the expectation with respect to $\mathbf{P}_q$. Thus, the probability space where W is defined can be taken as $\Omega = \Omega_M \otimes \Omega_q$, $\mathcal{F} = \mathcal{F}_M \times \mathcal{F}_q$, and $\mathbf{P} = \mathbf{P}_M \times \mathbf{P}_q$. The law of the Gaussian measure M is defined by the following covariance structure: for any square-integrable test functions $f, g : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{C}$, we have

$$\begin{aligned} &\mathbf{E}_M \left[\int_{\mathbf{R}_+ \times \mathbf{R}} f(s,\lambda) M(ds, d\lambda) \int_{\mathbf{R}_+ \times \mathbf{R}} g(s,\lambda) M(ds, d\lambda) \right] \\ &\quad = \int_{\mathbf{R}_+ \times \mathbf{R}} f(s,\lambda) \overline{g(s,\lambda)} ds\, d\lambda, \end{aligned} \tag{2.3}$$

where $\mathbf{E}_M$ denotes the expectation with respect to $\mathbf{P}_M$ and the bar denotes complex conjugation.

Conditionally on the process q, W has a covariance structure similar to the case where q is non-random: for all $s, t \in \mathbf{R}_+$ and all $x, y \in \mathbf{R}$,

$$\mathbf{E}[W(t,x) W(s,y) | \mathcal{F}_q] = \min(s,t) Q(x-y), \tag{2.4}$$

where Q is a homogeneous covariance function that is random, and is $\mathcal{F}_q$-measurable. This fact is obtained using the representation of W in (2.1) and the covariance structure of M in (2.3), in the following elementary way:

$$\begin{aligned} &\mathbf{E}[W(t,x) W(s,y) | \mathcal{F}_q] \\ &\quad = \mathbf{E}_M \left[\int_{\mathbf{R}_+ \times \mathbf{R}} \mathbf{1}_{[0,t]}(r) e^{i\lambda x} \sqrt{q(\lambda)} M(dr, d\lambda) \right. \\ &\qquad \left. \times \int_{\mathbf{R}_+ \times \mathbf{R}} \mathbf{1}_{[0,s]}(r) e^{i\lambda y} \sqrt{q(\lambda)} M(dr, d\lambda) \right] \\ &\quad = \int_{\mathbf{R}_+ \times \mathbf{R}} \mathbf{1}_{[0,t]}(r) \mathbf{1}_{[0,s]}(r) e^{i\lambda(x-y)} q(\lambda) dr\, d\lambda \\ &\quad = \min(s,t) \int_{\mathbf{R}} e^{i\lambda(x-y)} q(\lambda) d\lambda, \end{aligned}$$

which is precisely the claim in (2.4), and proves in addition that

$$Q(x) = \int_{\mathbf{R}} e^{i\lambda x} q(\lambda) d\lambda.$$

The fact that no restriction is placed on q, other than the very weak L^1-integrability in (2.2), means that modulo this integrability, any mixture of homogeneous Gaussian fields can be considered as a potential for the Anderson model, which exhausts a wide range of random fields. We also have $\mathbf{E}[(W(t,x) - W(s,x))^2] = |t-s| \ \mathbf{E}_q[Q(0)]$. Note that Condition (2.2) on q is equivalent to $\mathbf{E}_q[Q(0)] < \infty$, and is thus equivalent to the square-integrability of W. Also note that conditional on q, and for fixed $x \in \mathbf{R}$, the map $t \mapsto W(t,x)$ is a Brownian motion with scale $\sqrt{Q(0)}$. This can allow us to define the stochastic integral $\int_0^t W(ds,x)u(s,x)$ as an Ito integral conditionally on q as long as u is adapted and square integrable given q; there seems to be little hope of defining such integrals without assuming $\mathbf{E}_q[Q(0)] < \infty$.

The solution of the Anderson model equation (1.1) can be represented using the stochastic Feynman–Kac formula. Let b be a Wiener process started at 0 with variance κ defined on some probability space $(\Omega_b, \mathcal{F}_b, \mathbf{P}_b)$ equipped with a filtration $\{\mathcal{F}_t^b : t \geq 0\}$, and assume that b is independent of W. For fixed t and x, we have

$$u(t,x) = \mathbf{E}_b \left[\exp\left(\beta \int_0^t W(ds, b_t - b_s + x) \right) \right],$$

where $\mathbf{E}_b$ denotes the expectation with respect to $\mathbf{P}_b$. This formula can be established using standard techniques such as in [9], by conditioning on q. The proof is omitted. It is also easy to show that W has stationary and independent increments in time. Using this, and the fact that the covariance structure of W given in (2.4) depends only on spatial differences, we have the following non-time-reversed Feynman–Kac formula: for fixed t and x,

$$u(t,x) \underset{\mathbf{P}}{=} \mathbf{E}_b \left[\exp\left(\beta \int_0^t W(ds, b_s + x) \right) \right] \tag{2.5}$$

$$\underset{\mathbf{P}}{=} \mathbf{E}_b \left[\exp\left(\beta \int_0^t W(ds, b_s) \right) \right] \tag{2.6}$$

where the equality holds in distribution under $\mathbf{P} = \mathbf{P}_M \times \mathbf{P}_q$. The expression on the right-hand side of (2.6) also has the interpretation of the partition function Z_t in the *polymer measure* P^W based on the Hamiltonian $-\int_0^t W(ds, b_s)$, as we already mentioned on p. 453.

3. Existence of the Almost-Sure Lyapunov Exponent

In this section, we study the existence of the almost-sure Lyapunov exponent λ in (1.2). We will first show that the limit of its expectation exists, i.e.

$$\bar{\lambda}(x) := \lim_{\substack{t \to \infty \\ t \in \mathbf{N}}} \frac{1}{t} \mathbf{E}[\log(u(t,x))].$$

Because of the invariance of the law of W under spatial shifts, we have the equality between (2.5) and (2.6), implying that $\bar{\lambda}(x) \equiv \bar{\lambda}$ does not depend on x. We then make the connection with the above limit and the Lyapunov exponent: we show using Malliavin derivatives that $(\log u(t,x) - \mathbf{E}[\log u(t,x)])/t$ converges to 0 almost surely, thereby proving the existence of λ and that $\lambda = \bar{\lambda}$, which implies our claim that λ is non-random and not dependent on x. In other words, the proof of the next theorem is an immediate consequence of the following two propositions.

Theorem 3.1. *Assume that there exists $k > 1$ such that*

$$\mathbf{E}_q[Q(0)^k] = \mathbf{E}_q\left[\left(\int_{\mathbf{R}} q(\lambda)d\lambda\right)^k\right] < \infty.$$

Then $\mathbf{P}$-almost surely, for every fixed $x \in \mathbf{R}$,

$$\lambda := \lim_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{1}{t}\log(u(t,x))$$

exists, does not depend on x, is finite, and is non-negative.

3.1. *Convergence of the mean*

Proposition 3.1. *Assume $\mathbf{E}q[Q(0)] < \infty$. There exists a constant $\lambda \geq 0$ such that*

$$\lambda = \lim_{t\to\infty} \frac{1}{t}\mathbf{E}[\log(u(t,x))] = \sup_{t\geq 0} \frac{1}{t}\mathbf{E}[\log(u(t,0))].$$

Proof. As we mentioned above, we can replace x with 0. Let $U(t) = \mathbf{E}[\log(u(t,0))]$. We will show that the function U is super-additive, i.e. for $t,h > 0$,

$$U(t+h) \geq U(t) + U(h).$$

Using the Feynman–Kac formula in (2.6) and the independence of increments of b, we have

$$\begin{aligned}
u(t+h,0) &= \mathbf{E}_b[e^{\beta\int_0^{t+h} W(ds,b_s)}] \\
&= \mathbf{E}_b[e^{\beta\int_0^{t} W(ds,b_s)} e^{\beta\int_t^{t+h} W(ds,b_s)}] \\
&= \mathbf{E}_b[\mathbf{E}_b[e^{\beta\int_0^{t} W(ds,b_s)} e^{\beta\int_t^{t+h} W(ds,b_s)}|\mathcal{F}_t^b]] \\
&= \mathbf{E}_b[e^{\beta\int_0^{t} W(ds,b_s)} \mathbf{E}_b[e^{\beta\int_t^{t+h} W(ds,b_s)}|\mathcal{F}_t^b]].
\end{aligned}$$

Let p_t be the heat kernel on $\mathbf{R}$ at time $t \geq 0$, and b' be an independent copy of b. For $t,s \in \mathbf{R}_+$ and $x \in \mathbf{R}$, set $\theta_t W(s,x) = W(s+t,x)$. We then have

$$\begin{aligned}
u(t+h,0) &= \mathbf{E}_b[e^{\beta\int_0^{t} W(ds,b_s)} \mathbf{E}_{b'}[e^{\beta\int_0^{h} (\theta_t W)(ds,b_t+b'_s)}]] \\
&= \int_{\mathbf{R}} \mathbf{E}_b[e^{\beta\int_0^{t} W(ds,b_s)} \mathbf{E}_{b'}[e^{\beta\int_0^{h} (\theta_t W)(ds,y+b'_s)}]|b_t = y]p_t(dy) \\
&= \int_{\mathbf{R}} \mathbf{E}_{b'}[e^{\beta\int_0^{h} (\theta_t W)(ds,y+b'_s)}]\mathbf{E}_b[e^{\beta\int_0^{t} W(ds,b_s)}|b_t = y]p_t(dy) \\
&= \int_{\mathbf{R}} \mathbf{E}_b[e^{\beta\int_0^{h} (\theta_t W)(ds,y+b_s)}]\mathbf{E}_b[e^{\beta\int_0^{t} W(ds,b_s)}|b_t = y]p_t(dy).
\end{aligned}$$

Substituting this into $U(t+h)$ and using Jensen's inequality for the logarithm, we get

$$
\begin{aligned}
U(t+h) &= \mathbf{E}\left[\log \int_{\mathbf{R}} \mathbf{E}_b[e^{\beta \int_0^h (\theta_t W)(ds, y+b_s)}]\mathbf{E}_b[e^{\beta \int_0^t W(ds,b_s)}|b_t = y]p_t(dy)\right] \\
&= \mathbf{E}[\log(u(t,0))] \\
&\quad + \mathbf{E}\left[\log \int_{\mathbf{R}} \mathbf{E}_b[e^{\beta \int_0^h (\theta_t W)(ds, y+b_s)}]\frac{\mathbf{E}_b[e^{\beta \int_0^t W(ds,b_s)}|b_t = y]}{\mathbf{E}_b[e^{\beta \int_0^t W(ds,b_s)}]}p_t(dy)\right] \\
&\geq \mathbf{E}[\log(u(t,0))] \\
&\quad + \mathbf{E}\left[\int_{\mathbf{R}} \log(\mathbf{E}_b[e^{\beta \int_0^h (\theta_t W)(ds, y+b_s)}])\frac{\mathbf{E}_b[e^{\beta \int_0^t W(ds,b_s)}|b_t = y]}{\mathbf{E}_b[e^{\beta \int_0^t W(ds,b_s)}]}p_t(dy)\right].
\end{aligned}
$$

Note that by the invariance of W in law under shifts in space, for any $y \in \mathbf{R}$, we have the equality in law under $\mathbf{P}$

$$
\mathbf{E}_b[e^{\beta \int_0^h (\theta_t W)(ds, y+b_s)}] \underset{\mathbf{P}}{=} \mathbf{E}_b[e^{\beta \int_0^h (\theta_t W)(ds, b_s)}].
$$

Using this and the independence of increments of W in time, which means that $\theta_t W$ is independent of W restricted to $[0,t]$, we have

$$
\begin{aligned}
U(t+h) &\geq \mathbf{E}[\log(u(t,0))] \\
&\quad + \int_{\mathbf{R}} \mathbf{E}[\log(\mathbf{E}_b[e^{\beta \int_0^h (\theta_t W)(ds, y+b_s)}])]\mathbf{E}\left[\frac{\mathbf{E}_b[e^{\beta \int_0^t W(ds,b_s)}|b_t = y]}{\mathbf{E}_b[e^{\beta \int_0^t W(ds,b_s)}]}\right]p_t(dy) \\
&= \mathbf{E}[\log(u(t,0))] \\
&\quad + \mathbf{E}[\log(\mathbf{E}_b[e^{\beta \int_0^h (\theta_t W)(ds, b_s)}])]\mathbf{E}\left[\int_{\mathbf{R}} \frac{\mathbf{E}_b[e^{\beta \int_0^t W(ds,b_s)}|b_t = y]}{\mathbf{E}_b[e^{\beta \int_0^t W(ds,b_s)}]}p_t(dy)\right] \\
&= \mathbf{E}[\log(u(t,0))] + \mathbf{E}[\log(\mathbf{E}_b[e^{\beta \int_0^h W(ds,b_s)}])] \\
&= \mathbf{E}[\log(u(t,0))] + \mathbf{E}[\log(u(h,0))] \\
&= U(t) + U(h).
\end{aligned}
$$

Thus, U is super-additive. It follows that

$$
\lim_{t\to\infty} \frac{1}{t}U(t) = \sup_{t\geq 0} \frac{1}{t}U(t),
$$

and that the limit exists, although it may be infinite.

To show that the limit is finite, we only need to show that $U(t)/t$ is bounded for all t. Indeed, using Jensen's inequality, Fubini's lemma, and the covariance structure in (2.4), we have

$$\begin{aligned}
\frac{1}{t}U(t) &= \frac{1}{t}\mathbf{E}[\log(\mathbf{E}_b[e^{\beta\int_0^t W(ds,b_s)}])] = \frac{1}{t}\mathbf{E}_q\mathbf{E}_M[\log(\mathbf{E}_b[e^{\beta\int_0^t W(ds,b_s)}])] \\
&\le \frac{1}{t}\mathbf{E}_q[\log(\mathbf{E}_b\mathbf{E}_M[e^{\beta\int_0^t W(ds,b_s)}])] \\
&= \frac{1}{t}\mathbf{E}_q\left[\log\left(\mathbf{E}_b\left[\exp\left(\frac{\beta^2}{2}\mathbf{E}_M\left[\left(\int_0^t W(ds,b_s)\right)^2\right]\right)\right]\right)\right] \\
&= \frac{1}{t}\mathbf{E}_q\left[\log\left(\mathbf{E}_b\left[\exp\left(\frac{t\beta^2}{2}Q(0)\right)\right]\right)\right] = \frac{\beta^2}{2}\mathbf{E}_q[Q(0)]. \qquad (3.1)
\end{aligned}$$

This is finite by our assumption on $Q(0)$. To show that $\lambda \ge 0$, we again use Jensen's inequality, but in the other direction, to get that

$$U(t) \ge \mathbf{E}\mathbf{E}_b[\log e^{\beta\int_0^t W(ds,b_s)}] = \beta\mathbf{E}_b\mathbf{E}_q\mathbf{E}_M\left[\int_0^t W(ds,b_s)\right] = 0. \qquad \square$$

3.2. *Almost-sure convergence*

We first start with some notation on Malliavin calculus for the Gaussian measure M, that will be used throughout the paper. Let F be a random variable in the space $L^2(\Omega_M, \mathcal{F}_M, \mathbf{P}_M)$ generated by M. Its Malliavin derivative DF with respect to M, when it exists, is a random field on the parameter space $\mathbf{R}_+ \times \mathbf{R}$ (see [16, 21] for more details). Thus, the Malliavin derivative here is defined only in terms of the randomness in M. For this paper, it is sufficient to note two important properties of D:

(1) Let $(\mathcal{F}_t^M)_{t\ge 0}$ be the filtration of M. If G has a Malliavin derivative and G is $\mathcal{F}_t^M$-measurable for some $t \ge 0$, then for all $\lambda \in \mathbf{R}$ and all $s > t$, $D_{s,\lambda}G = 0$.
(2) Let f be a square integrable function from $\mathbf{R}_+ \times \mathbf{R}$ to $\mathbf{C}$ (non-random with respect to the randomness in M, but possibly dependent on q), let $F = \int_{\mathbf{R}_+\times\mathbf{R}} f(s,\lambda)M(ds,d\lambda)$, and g be a function in $C^1(\mathbf{R})$. Then, conditional on q, the random variable $G = g(F)$ has a Malliavin derivative (w.r.t. M) given by,

$$D_{s,\lambda}G = g'(F)f(s,\lambda),$$

for all $s \ge 0$ and all $\lambda \in \mathbf{R}$, as long as $g'(F)$ is in $L^2(\Omega_M)$. Note that $D_{s,\lambda}F = f(s,\lambda)$.

For $t \ge 0$ and for any bounded measurable function $f : C([0,t];\mathbf{R}) \to \mathbf{R}$, we set

$$\langle f\rangle_t = \mathbf{E}_b[f(b)e^{\beta\int_0^t W(ds,b_s)}]/u(t,0),$$

where b is a Brownian motion. This notation is borrowed from the mathematical physics theory of Gibbs measures: it is the expectation of $f(\cdot)$ with respect to the polymer measure P^W we described in the introduction (up to time t). Note that

P^W is a random probability measure, since it depends on the randomness in W, i.e. in both M and q.

Proposition 3.2. *Assume that there exists $k > 1$ such that*

$$\mathbf{E}_q[Q(0)^k] = \mathbf{E}_q\left[\left(\int_{\mathbf{R}} q(\lambda)d\lambda\right)^k\right] < \infty.$$

Then $\mathbf{P}$-almost surely, for any fixed $x \in \mathbf{R}$,

$$\lim_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{1}{t}\log u(t,x) = \lim_{\substack{t\to\infty \\ t\in\mathbf{N}}} \frac{1}{t}\mathbf{E}[\log(u(t,x))].$$

Proof. Let us first compute the Malliavin derivative of $\log u(t,0)$ (where we replace x again by 0 due to the spatial homogeneity of W) conditional on the stochastic process q. All the computations below hold given $\mathcal{F}_q$, i.e. conditional on q, for all $s \le t$.

$$\begin{aligned}
D_{s,\lambda}\log u(t,0) &= \frac{1}{u(t,0)}D_{s,\lambda}u(t,0) \\
&= \frac{1}{u(t,0)}\mathbf{E}_b\left[D_{s,\lambda}\exp\left(\beta\int_{[0,t]\times\mathbf{R}} e^{i\lambda b_s}\sqrt{q(\lambda)}M(ds,d\lambda)\right)\right] \\
&= \frac{1}{u(t,0)}\mathbf{E}_b\left[\beta e^{i\lambda b_s}\sqrt{q(\lambda)}\exp\left(\beta\int_{[0,t]\times\mathbf{R}} e^{i\lambda b_s}\sqrt{q(\lambda)}M(ds,d\lambda)\right)\right] \\
&= \beta\sqrt{q(\lambda)}\langle e^{i\lambda b_s}\rangle_t.
\end{aligned}$$

Computing the norm of the Malliavin derivative, we have

$$\begin{aligned}
\|D\log u(t,0)\|^2 &= \beta^2\int_0^t\int_{\mathbf{R}} q(\lambda)\langle e^{i\lambda b_s}\rangle_t\langle e^{-i\lambda b_s}\rangle_t d\lambda\, ds \\
&= \beta^2\int_0^t\left\langle\int_{\mathbf{R}} q(\lambda)e^{i\lambda(b_s^1-b_s^2)}d\lambda\right\rangle_t ds \\
&\le t\beta^2 Q(0),
\end{aligned}$$

where b_s^1 and b_s^2 are two independent Brownian motion. Using Theorem 9.2.3(iii) in Üstünel's textbook [21], we have that for $k \in \mathbf{N}$,

$$\begin{aligned}
&\mathbf{E}\left[\left(\frac{1}{t}\log u(t,0) - \frac{1}{t}\mathbf{E}[\log u(t,0)]\right)^{2k}\right] \\
&\quad = \mathbf{E}_q\mathbf{E}_M\left[\left(\frac{1}{t}\log u(t,0) - \frac{1}{t}\mathbf{E}[\log u(t,0)]\right)^{2k}\right] \\
&\quad \le c_k t^{-2k}\mathbf{E}_q\mathbf{E}_M[\|D\log u(t,0)\|^{2k}] \\
&\quad \le C_{k,Q}t^{-k},
\end{aligned}$$

where c_k is a constant depending on k and $C_{k,Q} = c_k \beta^{2k} \mathbf{E}_q[Q(0)^k]$. By Chebyshev's inequality, for any constant $C(t)$,

$$\mathbf{P}\left[\left|\frac{1}{t}\log u(t,0) - \frac{1}{t}\mathbf{E}[\log u(t,0)]\right| > C(t)\right] \leq \frac{C_{k,Q}}{t^k (C(t))^{2k}}.$$

To complete the proof, we apply the Borel–Cantelli lemma: by choosing $C(t) = t^{-\varepsilon/(2k)}$ with some positive $\varepsilon < k-1$, we get that the last expression above is summable in $t \in \mathbf{N}$, showing that almost surely, for t large enough, $\left|\frac{1}{t}\log u(t,0) - \frac{1}{t}\mathbf{E}[\log u(t,0)]\right| < t^{-\varepsilon/(2k)}$. □

We can perform a finer analysis of the speed of concentration of $t^{-1}\log u(t,0)$ around its mean, by considering various integrability hypotheses on $Q(0)$. Such results are physically related to the question of evaluating the so-called *fluctuation exponent*. The latter is defined as the exponent α of t in the asymptotics of the standard deviation $\sqrt{\mathrm{Var}[\log u(t,0)]}$. It is conjectured by physicists that $\mathrm{Var}[\log u(t,0)] \asymp t^{2\alpha}$, with fluctuation exponent $\alpha < 1/2$. This is a difficult and long-standing mathematical conjecture. The tools herein allow us to prove that $\alpha \leq 1/2$, as the reader can easily check. However, when looking at almost-sure convergence rather than mean-square convergence, we obtain a different notion of fluctuation speed, as a trivial consequence of the proof above, which, presumably unlike α, is sensitive to how many moments $Q(0)$ has, or alternately to whether our noise W is (sub)-Gaussian or not.

Corollary 3.1. *With the same hypothesis as in Proposition* 3.2, *for any* $\beta > 1$, $\mathbf{P}$*-almost surely for* t *large enough*,

$$|\log u(t,0) - \mathbf{E}[\log u(t,0)]| \leq t^{\frac{1}{2}+\frac{\beta}{2k}}.$$

The value $\frac{1}{2} + \frac{\beta}{2k}$ could be called an upper bound on an "almost-sure fluctuation exponent" $\bar{\alpha}$, where $\bar{\alpha}$ is a value as small as possible such that $|\log u(t,0) - \mathbf{E}[\log u(t,0)]|$ tends to 0 as fast as $t^{\bar{\alpha}}$. In particular, if $Q(0)$ has moments of all orders, $|\log u(t,0) - \mathbf{E}[\log u(t,0)]| \leq t^{\gamma}$ for all $\gamma > 1/2$, i.e. we can take $\bar{\alpha} = 1/2 + \varepsilon$ for any $\varepsilon > 0$. One should expect $\bar{\alpha}$ to always exceed α, since the former can be regarded as an almost-sure statement while the latter is for mean-square convergence. Still, one may reach lower than the threshold $\bar{\alpha} = 1/2 + \varepsilon$, by making stronger integrability hypotheses on $Q(0)$, e.g. assuming that $Q(0)$ is a bounded random variable, which corresponds to saying that W is sub-Gaussian, or assuming that $Q(0)$ is sub-Gaussian, which implies that W has sub-exponential tails. The study of these and other generalizations are left to the reader.

4. Estimation of the Lyapunov Exponent: Discrete Space

In this section, we consider the Anderson model on $\mathbf{R}_+ \times \mathbf{Z}$. The theorem and propositions in the previous section, proved for $\mathbf{R}_+ \times \mathbf{R}$, also hold for $\mathbf{R}_+ \times \mathbf{Z}$

using identical proofs; as we alluded to in the Introduction, we will not comment on these proofs further.

The next theorem is an immediate consequence of the next two propositions combined with Theorem 3.1.

Theorem 4.1. *For the Anderson model on* $\mathbf{R}_+ \times \mathbf{Z}$, *assume that there exists* $k > 1$ *such that*

$$\mathbf{E}_q[Q(0)^k] = \mathbf{E}_q\left[\left(\int_{\mathbf{R}} q(\lambda)d\lambda\right)^k\right] < \infty.$$

Then $\mathbf{P}$*-a.s.*, $\lambda := \lim_{t\to\infty} t^{-1} \log u(t,x)$ *exists, is non-random, does not depend on* x, *and is bounded as*

$$c_1 \frac{\beta^2}{\log(\beta^2/\kappa)} \le \lambda \le c_3 \frac{\beta^2}{\log(\beta^2/\kappa)}$$

as soon as $\beta^2/\kappa > c_+$, *where* c_1, c_3 *and* c_+ *depend only on the law of* q, *and are given more explicitly in Propositions* 4.1 *and* 4.2 *below.*

4.1. *Lower bound result*

Proposition 4.1. *Assume that* $\mathbf{E}_q[\sqrt{Q(0) - Q(2)}] = c > 0$. *Let* $c_1 := c^2/(9\pi)$. *Under this very weak non-degeneracy hypothesis, there exists a constant* c_+ *depending only on* c *such that for* $\beta^2/\kappa \ge c_+$, *we have*

$$\lambda \ge c_1 \frac{\beta^2}{\log(\beta^2/\kappa)}.$$

In fact, we can take $c_+ = e^e \vee x$ *where* x *is the solution of the equation* $x^{-1}\log x = c^2/(18\pi)$.

Proof. The Feynman–Kac formula (2.6) is now to be understood with b replaced by a simple symmetric random walk on $\mathbf{Z}$ in continuous time, with speed parameter κ. In other words, b jumps at the jump times t_i of a Poisson process N_t with parameter $2\kappa t$, and the positions followed by b are those of a discrete-time simple symmetric random walk. Bounding the formula in (2.6) below by throwing away all paths b that do not jump exactly once in the interval $[0,t]$, we have

$$\begin{aligned} u(t,0) &\ge P_b[N_t = 1]\frac{1}{2}\int_0^t \frac{ds}{t}(e^{\beta W(s,0)+\beta W([s,t],+1)} + e^{\beta W(s,0)+\beta W([s,t],-1)}) \\ &= \kappa t e^{-2\kappa t}\int_0^t \frac{ds}{t}(e^{\beta W(s,0)+\beta W([s,t],+1)} + e^{\beta W(s,0)+\beta W([s,t],-1)}), \end{aligned}$$

where $W([s,t],x) := W(t,x) - W(s,x)$. Using Jensen, the fact that W is mean-zero, and then choosing the maximum of the two increments of W, we have

$$\frac{1}{t}\mathbf{E}[\log u(t,0)] \geq \frac{\log \kappa t}{t} - 2\kappa + \int_0^t \frac{ds}{t^2}\mathbf{E}[\beta W(s,0) + \log(e^{\beta W([s,t],+1)} + e^{\beta W([s,t],-1)})]$$
$$\geq \frac{\log \kappa t}{t} - 2\kappa + \beta \int_0^t \frac{ds}{t^2}\mathbf{E}[\max(W([s,t],+1); W([s,t],-1))].$$

Conditional on q, $(W([s,t],+1); W([s,t],-1))$ is a jointly Gaussian vector with covariance matrix

$$\sqrt{t-s}\begin{bmatrix} Q(0) & Q(2) \\ Q(2) & Q(0) \end{bmatrix}.$$

Therefore

$$\mathbf{E}[\max(W([s,t],+1); W([s,t],-1))] = \mathbf{E}_q[\mathbf{E}_M[\max(W([s,t],+1); W([s,t],-1))]]$$
$$= \frac{1}{\sqrt{\pi}}\sqrt{t-s}\,\mathbf{E}_q[\sqrt{Q(0)-Q(2)}].$$

Hence by our non-degeneracy hypothesis

$$\frac{1}{t}\mathbf{E}[\log u(t,0)] \geq \frac{\log \kappa t}{t} - 2\kappa + \beta t^{-2} c \pi^{-1/2} \int_0^t s^{1/2} ds$$
$$= \frac{\log \kappa t}{t} - 2\kappa + \beta t^{-1/2} c_0, \tag{4.1}$$

where $c_0 := 2c/(3\sqrt{\pi})$.

To conclude the proof of the proposition, it is sufficient to find a single value t depending on β and κ such that the last expression above exceeds a positive fraction of the last term in (4.1). We choose

$$t = c'\beta^{-2}\log^2(\beta^2/\kappa),$$

where the constant c' is determined below. Plugging this value into the expression (4.1) we get

$$\frac{1}{t}\mathbf{E}[\log u(t,0)]$$
$$\geq \beta^2\left(\frac{c_0(c')^{-1/2} - (c')^{-1}}{\log(\beta^2/\kappa)} + 2\frac{\log\log(\beta^2/\kappa)}{c'\log^2(\beta^2/\kappa)} + \frac{\log c'}{c'\log^2(\beta^2/\kappa)}\right) - 2\kappa.$$

We may now choose our constant c'. In order to get a weak restriction on β and κ, we simply choose

$$c' = 4/c_0^2.$$

Thence

$$\frac{1}{t}\mathbf{E}[\log u(t,0)] \geq \beta^2\left(\frac{c_0^2/2}{\log(\beta^2/\kappa)} + 2\frac{\log\log(\beta^2/\kappa)}{c'\log^2(\beta^2/\kappa)} + \frac{\log c'}{c'\log^2(\beta^2/\kappa)}\right) - 2\kappa.$$

By reducing c to a smaller constant if necessary, we may assume that $c' > 1$. If $x := \beta^2/\kappa \geq e^e$, we obtain

$$\frac{1}{t}\mathbf{E}[\log u(t,0)] \geq \frac{c_0^2}{2}\frac{\beta^2}{\log(\beta^2/\kappa)} - 2\kappa.$$

Now we only need to check that the term 2κ is negligible. More precisely, let us require that

$$\frac{c_0^2}{2}\frac{\beta^2}{\log(\beta^2/\kappa)} \geq 4\kappa.$$

With our notation $x = \beta^2/\kappa$, this translates as

$$x^{-1}\log x \leq c_0^2/8 = c^2/(18\pi),$$

which is satisfied for a sufficiently large x since the function $f(x) = x^{-1}\log x$ is decreasing for $x > e$. We have thus proved that, with $x := \beta^2/\kappa$, if $x^{-1}\log x \leq c^2/(18\pi)$ and $x > e^e$, then

$$\frac{1}{t}\mathbf{E}[\log u(t,0)] \geq \frac{c_0^2}{4}\frac{\beta^2}{\log(\beta^2/\kappa)}.$$

This finishes the proof of the proposition. □

4.2. *Upper bound result*

The next proposition is valid for all $\beta^2 > \kappa > 0$, but is only useful when β^2 is not too close to κ; indeed, only then can we be in the strong disorder regime, i.e. λ strictly less than the quantity $\frac{1}{2}\beta^2\mathbf{E}_q[Q(0)]$, which is the annealed Lyapunov exponent mentioned on p. 453, as proved in (3.1).

Proposition 4.2. *Assume that $\mathbf{E}_q[Q(0)] < \infty$. Then there is a non-random constant c_3 depending only on the law of $Q(0)$ such that for all $\beta^2 > \kappa > 0$,*

$$\lambda \leq \left(c_3\frac{\beta^2}{\log(\beta^2/\kappa)}\right) \wedge \left(\frac{1}{2}\beta^2\mathbf{E}_q[Q(0)]\right).$$

In fact, we can take $c_3 = K_u\mathbf{E}_q[\max(\sqrt{2}, 6\sqrt{2}K_u\sqrt{Q(0)})\sqrt{Q(0)}]$ where K_u is the universal constant in the so-called Dudley entropy upper bound for Gaussian expected suprema.

Proof. From (3.1), we have that

$$t^{-1}\mathbf{E}[\log u(t,0)] \leq \frac{1}{2}\beta^2\mathbf{E}_q[Q(0)],$$

which explains the corresponding upper bound in the statement of the proposition. We thus only need to prove $\lambda \leq c_3\frac{\beta^2}{\log(\beta^2/\kappa)}$.

We begin by recalling notation and a technical result which can be traced back to [6] in the case where $Q(0)$ is non-random, and was expressed more quantitatively in [12] for continuous space. Here it can be proved directly by conditioning on q. The details are left to the reader. Let α be any fixed positive number. Let I_α be the set of all paths of the random walk b on $[0,t]$ which have at most αt jumps. Let $Y_\alpha = \sup_{b\in I_\alpha} \int_0^t \beta W(ds, b_s)$. The following holds:

$$\mathbf{E}[Y_\alpha | \mathcal{F}_q] = \mathbf{E}_M[Y_\alpha] \leq K_u \sqrt{Q(0)}\beta t \sqrt{\alpha}, \tag{4.2}$$

where K_u is the universal constant in the so-called Dudley Entropy upper bound (see [1] or [14]).

Let us now decompose $u(t,0)$ according to the number of jumps of the random walk b. With N_t the number of jumps of the path b before time t, we have:

$$\begin{aligned} u(t,0) &= P_b[N_t \leq \alpha t] E_b\left[\exp\left(\int_0^t \beta W(ds,b_s)\right)\middle| N_t \leq \alpha t\right] \\ &\quad + \sum_{n=1}^{\infty} P_b[n\alpha t < N_t \leq (n+1)\alpha t] \\ &\quad \times E_b\left[\exp\left(\int_0^t \beta W(ds,b_s)\right)\middle| n\alpha t < N_t \leq (n+1)\alpha t\right]. \\ &\leq P_b[N_t \leq \alpha t]\exp(Y_\alpha) + \sum_{n=1}^{\infty} P_b[n\alpha t < N_t]\exp(Y_{(n+1)\alpha}) \\ &\leq \exp(Y_\alpha) + \sum_{n=1}^{\infty} P_b[n\alpha t < N_t]\exp(Y_{(n+1)\alpha}). \end{aligned} \tag{4.3}$$

We will need to use the tail of N_t, which is a Poisson process with parameter κ. We simply use the well-known bound, valid for all $a > \kappa$ and t large enough,

$$P_b[at < N_t] \leq \exp(-at\log(a/\kappa)). \tag{4.4}$$

With the shorthand notation

$$p_n = p_n(t) = \exp(-n\alpha t\log(n\alpha/\kappa)) \tag{4.5}$$

and $p_0 = 1$, the above upper bound on $u(t,0)$ becomes

$$u(t,0) \leq \sum_{n=0}^{\infty} p_n(t)\exp(Y_{(n+1)\alpha}).$$

Now using the fact that for $A, B > 0$ and $t > 1$, $(A+B)^{1/\sqrt{t}} \leq A^{1/\sqrt{t}} + B^{1/\sqrt{t}}$, we get

$$\frac{1}{\sqrt{t}}\log u(t,0) \leq \log\left(\sum_{n=0}^{\infty}(p_n)^{1/\sqrt{t}}\exp\left(\frac{1}{\sqrt{t}}Y_{(n+1)\alpha}\right)\right).$$

To evaluate the expectation of the above, we first evaluate the expectation conditional on q, i.e. the operator $\mathbf{E}_M$. Hence by Jensen's inequality

$$\mathbf{E}_M\left[\frac{1}{\sqrt{t}}\log u(t,0)\right] \le \log\sum_{n=0}^{\infty}(p_n)^{1/\sqrt{t}}\mathbf{E}_M\left[\exp\left(\frac{1}{\sqrt{t}}Y_{(n+1)\alpha}\right)\right]. \tag{4.6}$$

By standard calculations in Gaussian analysis (see for instance applications of the Borell–Sudakov inequality in [9]), using the fact that the conditional variance of $\int_0^t W(ds,b_s)$ is bounded above by $Q(0)t$ for any b, we have

$$\begin{aligned}\mathbf{E}_M\left[\exp\left(\frac{1}{\sqrt{t}}Y_{n\alpha}\right)\right] &\le \exp(\mathbf{E}_M[Y_{n\alpha}]/\sqrt{t})\exp(\beta^2 Q(0))\\ &\le \exp(K_u\sqrt{Q(0)}\beta\sqrt{n\alpha t}+\beta^2 Q(0)),\end{aligned} \tag{4.7}$$

where we used (4.2) in the last inequality.

Combining (4.6) and (4.7), we get

$$\begin{aligned}&\mathbf{E}_M[t^{-1}\log u(t,0)]\\ &\quad\le \frac{1}{\sqrt{t}}\log\left(\sum_{n=0}^{\infty}(p_n)^{1/\sqrt{t}}\exp(K_u\sqrt{Q(0)}\beta\sqrt{(n+1)\alpha t}+\beta^2Q(0))\right)\\ &\quad= \beta^2 Q(0)/\sqrt{t}+\frac{1}{\sqrt{t}}\log\left(\sum_{n=0}^{\infty}(p_n)^{1/\sqrt{t}}\exp(K_u\sqrt{Q(0)}\beta\sqrt{(n+1)\alpha t})\right).\end{aligned} \tag{4.8}$$

We immediately get that

$$\begin{aligned}&\lim_{t\to\infty}\mathbf{E}[t^{-1}\log u(t,0)]\\ &\qquad\le \limsup_{t\to\infty}\frac{1}{\sqrt{t}}\mathbf{E}_q\left[\log\left(\sum_{n=0}^{\infty}(p_n)^{1/\sqrt{t}}\exp(K_u\sqrt{Q(0)}\beta\sqrt{(n+1)\alpha t})\right)\right].\end{aligned} \tag{4.9}$$

We need to transform the right-hand side of the above using $\log(A+B)\le\log_+ A+\log_+ B+\log 2$ valid for $A,B>0$ where $\log_+ = 0\vee\log$:

$$\begin{aligned}&\frac{1}{\sqrt{t}}\mathbf{E}_q\left[\log\left(\sum_{n=0}^{\infty}(p_n)^{1/\sqrt{t}}\exp(K_u\sqrt{Q(0)}\beta\sqrt{(n+1)\alpha t})\right)\right]\\ &\qquad\le \frac{1}{\sqrt{t}}\mathbf{E}_q[\log(\exp(K_u\sqrt{Q(0)}\beta\sqrt{\alpha t}))]\\ &\qquad\quad+\frac{1}{\sqrt{t}}\mathbf{E}_q\left[\log_+\left(\sum_{n=1}^{\infty}(p_n)^{1/\sqrt{t}}\exp(K_u\sqrt{Q(0)}\beta\sqrt{(n+1)\alpha t})\right)\right]+\frac{\log 2}{\sqrt{t}}.\end{aligned}$$

In view of (4.9), and using Fatou's lemma, and the expression (4.5) for p_n, we now have

$$\begin{aligned}&\lim_{t\to\infty}\mathbf{E}[t^{-1}\log u(t,0)] \le \beta K_u\mathbf{E}_q[\sqrt{Q(0)}\sqrt{\alpha}]\\ &\qquad+\mathbf{E}_q\left[\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\log_+\left(\sum_{n=1}^{\infty}e^{-n\alpha\sqrt{t}\log(n\alpha/\kappa)+K_u\sqrt{Q(0)}\beta\sqrt{(n+1)\alpha t}}\right)\right].\end{aligned} \tag{4.10}$$

We may now choose the coefficient α in order to minimize the last expression above; α may depend on q. We choose

$$\alpha = c\beta^2 / \log^2(\beta^2/\kappa),$$

where the constant c will be chosen below as a function of $Q(0)$. The fact that the lim sup in (4.10) is inside the expectation means that we can choose t arbitrarily large and possibly dependent on c. With $x = \beta^2/\kappa$, for any $n \geq 1$, one readily checks that the exponent in (4.10) will be smaller than $-2^{-1}n\alpha\sqrt{t}\log(n\alpha/\kappa)$ as soon as

$$c\frac{n}{\log^2 x}(\log(ncx) - 2\log\log x) > 2K_u\sqrt{c}\sqrt{Q(0)}\frac{\sqrt{n+1}}{\log x}. \tag{4.11}$$

We also impose $c \geq 2$ (see footnote[a]). In this case, it is easy to check that $\log(ncx)$ always exceeds $3\log\log x$ for all $x > 1$, which implies that $\log(ncx) - 2\log\log x > 3^{-1}\log(ncx)$. This in turn implies that Condition (4.11) is true as soon as

$$\begin{gathered} \frac{cn}{3} \geq 2K_u\sqrt{c}\sqrt{Q(0)}\sqrt{n+1} \\ \Longleftrightarrow \\ c \geq 36K_u^2 Q(0)\frac{n+1}{n^2} \\ \Longleftarrow \\ c \geq 72K_u^2 Q(0), \end{gathered}$$

where the last implication holds because $n \geq 1$.

Summarizing, what we have proved is that if

$$\begin{aligned} 1 &< \beta^2/\kappa \\ c &= \max(2, 72K_u^2 Q(0)), \qquad (4.12) \\ \alpha &= c\beta^2/\log^2(\beta^2/\kappa), \qquad (4.13) \end{aligned}$$

then

$$\begin{aligned} \lim_{t\to\infty} \mathbf{E}[t^{-1}\log u(t,0)] &\leq \beta K_u \mathbf{E}_q\left[\sqrt{Q(0)}\sqrt{\alpha}\right] \\ &\quad + \mathbf{E}_q\left[\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\log_+\left(\sum_{n=1}^{\infty}\exp(-2^{-1}n\alpha\sqrt{t}\log(n\alpha/\kappa))\right)\right]. \end{aligned} \tag{4.14}$$

[a] This is done for convenience; in cases where the random variable $Q(0)$ is bounded above by a (very small) non-random constant, an improvement on this lower bound on c is possible, but it creates problems such as requiring a random lower bound on β.

To evaluate the last series above, we brutally ignore the term n inside the logarithm, yielding an upper bound

$$\sum_{n=1}^{\infty} \exp(-2^{-1} n\alpha\sqrt{t}\log(n\alpha/\kappa)) \leq \sum_{n=1}^{\infty} \exp(-2^{-1} n\alpha\sqrt{t}\log(\alpha/\kappa))$$
$$= \frac{\exp(-2^{-1}\alpha\sqrt{t}\log(\alpha/\kappa))}{1 - \exp(-2^{-1}\alpha\sqrt{t}\log(\alpha/\kappa))} \leq 1,$$

where the last inequality holds as soon as $t > (2\log 2)^2/(\alpha\log(\alpha/\kappa))^2$. Although α is random because c is random, since c is greater than 2 by definition, this restriction on t is met as soon as $t > (2\log 2)^2/(\alpha'\log(\alpha'/\kappa))^2$ where α' is the same as α in (4.13) but with c replaced by 2; therefore our lower bound on t is non-random. Hence from (4.14), and the expressions (4.12) and (4.13) we get

$$\lim_{t\to\infty} \mathbf{E}[t^{-1}\log u(t,0)] \leq \beta K_u \mathbf{E}_q\left[\sqrt{Q(0)}\sqrt{\alpha}\right]$$
$$= \beta^2 K_u \mathbf{E}_q\left[\sqrt{cQ(0)}\right] / \log(\beta^2/\kappa)$$
$$= \frac{\beta^2}{\log(\beta^2/\kappa)} K_u \mathbf{E}_q\left[\max(\sqrt{2}, 6\sqrt{2}K_u\sqrt{Q(0)})\sqrt{Q(0)}\right].$$

However, in (4.4) we used the fact that $a = \alpha > \kappa$, which by (4.13), with $x = \beta^2/\kappa$, means $cx/\log^2(x) > 1$, i.e.

$$\max(2, 72K_u^2 Q(0)) > \frac{\log^2 x}{x}.$$

Our restriction on c being greater than 2, which leads to the $\max(2,\cdot)$ above, is also convenient here because it means it is not necessary to impose a random lower bound on x; indeed for all $x > 1$, $x^{-1}\log^2 x < 1$, which means that α always exceeds κ. This finishes the proof of the proposition. □

5. Estimation of the Lyapunov Exponent: Continuous Space

In this section, we consider the Anderson model on $\mathbf{R}_+ \times \mathbf{R}$. The Lyapunov exponent λ, which still exists (is non-random, and does not depend on x) thanks to Theorem 3.1, now satisfies bounds that scale as powers, rather than containing a logarithmic term as in the discrete case. For the bounds below, we assume that there exists $k > 1$ such that

$$\mathbf{E}_q[Q(0)^k] = \mathbf{E}_q\left[\left(\int_{\mathbf{R}} q(\lambda)d\lambda\right)^k\right] < \infty. \tag{5.1}$$

We also assume some regularity (resp. irregularity) of Hölder-continuity type on the spatial behavior of W in order to prove an upper bound (resp. lower bound) on λ. This is condition (5.3) (resp. condition (5.2)) below.

5.1. *Lower bound result*

Theorem 5.1. *Assume* (5.1) *and that for some* $H \in (0,1)$, *for all* $|x| \leq r_1$,

$$\mathbf{E}_q[Q(0) - Q(x)] \geq c_1 x^{2H}. \tag{5.2}$$

Then there exists a constant c_{++} *depending on* r_1 *and* c_1, *and a constant* c_2 *depending only on the law of* q, *such that for* $\beta^2/\kappa > c_{++}$, *we have*

$$\lambda \geq c_2 \beta^2 (\beta^2/\kappa)^{-F(H)},$$

where $F(H) = (1/3) \wedge (H/(H+1))$.

Proof. The proof of this theorem is very similar to the proof of the corresponding result in [5]: one only needs to replace t by κt and check all of the details. We omit them. □

5.2. *Upper bound result*

The theorem below for the upper bound is an improvement on the corresponding results in [5] and [12], and indeed on all previous upper bound results for the stochastic Anderson model's Lyapunov exponent in the case of a space-time potential which is white in time. The proof is also more streamlined and efficient.

Theorem 5.2. *Assume* (5.1) *and that for some* $H \in (0,1)$, *for all* $|x| \leq r_2$,

$$\mathbf{E}_q[Q(0) - Q(x)] \leq c_3 x^{2H}. \tag{5.3}$$

Then there exists a constant c_{++} *depending on* r_2 *and* c_3, *and a constant* c_4 *depending only on the law of* q, *such that for* $\beta^2/\kappa > c_{++}$, *we have*

$$\lambda \leq (c_4 \beta^2 (\beta^2/\kappa)^{-G(H)}) \wedge \left(\frac{1}{2}\beta^2 \mathbf{E}_q[Q(0)]\right),$$

where $G(H) = H/(1+3H)$.

In fact, we may take $c_{++} = 1 \vee (\frac{4\sqrt{2}K_u}{c_3 r_2^{1+3H}})^2$ *and* $c_4 = 4\sqrt{2}(1 \vee c_3) K_u^{4/3} (1 + 2\mathbf{E}_q[Q(0)^{2/3}])$. *Here* K_u *is still the universal constant in the Dudley entropy upper bound for Gaussian expected suprema.*

Proof. The proof starts off similarly to the proof of Proposition 3.1 in [5], up to Step 3. In that proof, a discretization was constructed, where b is replaced by a process $\tilde{b}$ in discrete space $\varepsilon\mathbf{Z}$, which jumps to the position of b at a distance ε from the previous visited site in $\varepsilon\mathbf{Z}$, the first time that this new site in $\varepsilon\mathbf{Z}$ is reached by b. We call N_t the total number of jumps of $\tilde{b}$ before time t. Using the same notation Y_α as in the proof of Proposition 4.2, we still have that $u(t,0)$ is bounded above as follows:

$$u(t,0) \leq \exp(Y_\alpha) + \sum_{n=1}^{\infty} P_b[n\alpha t < N_t] \exp(Y_{(n+1)\alpha}).$$

Evidently, while Y_α is the same as in our discrete space proofs, on the other hand, N_t is not a Poisson process, and $\tilde{b}$ is not a Markov process, but a useful estimate was still obtained in [5] and [12]. Specifically, estimate (22) in [5] was

$$\mathbf{P}_b[N_t > n\alpha t] \leq \exp\left(-\frac{t}{2}(\alpha n\varepsilon)^2 + t\alpha n\right). \tag{5.4}$$

Here we need to modify this estimate to account for the diffusion parameter κ. To go from $\kappa = 1$ to $\kappa \neq 1$, under $\mathbf{P}_b$, we simply need to multiply b by $\sqrt{\kappa}$. By the definition of the jumps of $\tilde{b}$ as hitting times of $\varepsilon\mathbf{Z}$, this modification is equivalent to replacing ε by $\varepsilon/\sqrt{\kappa}$. Therefore (5.4) becomes

$$\mathbf{P}_b[N_t > n\alpha t] \leq \exp\left(-\frac{t}{2\kappa}(\alpha n\varepsilon)^2 + t\alpha n\right).$$

For Y_α, we still have from (4.2),

$$\mathbf{E}_M[Y_\alpha] \leq \beta K_u\sqrt{Q(0)}t\sqrt{\alpha}. \tag{5.5}$$

At this point, borrowing calculations from the proof of Proposition 3.1 in [5] up to Step 3, and using the hypothesis of the theorem, the discretization method amounts to introducing an error of order $\varepsilon^{\,2H}$, or more precisely,

$$\mathbf{E}[t^{-1}\log u(t,0)] \leq c_3\beta^2\varepsilon^{2H} + \frac{1}{2}p_t^\varepsilon(\beta), \tag{5.6}$$

where $p_t^\varepsilon(\beta)$ is bounded as

$$tp_t^\varepsilon(\beta) \leq 2\beta K_u t\sqrt{\alpha}\mathbf{E}_q[\sqrt{Q(0)}] + \mathbf{E}[\log_+ B] + \log 2,$$

where $B = \sum_{n=1}^\infty \mathbf{P}_b[n\alpha t < N_t]\exp(2\beta Y_{\alpha(n+1)})$. The proof of Proposition 3.1 in [5] is valid in our case if

$$\varepsilon \leq r_2. \tag{5.7}$$

We will see below how this effects our parameters once we have chosen ε.

We are now able to use the two estimates (5.4) and (5.5), proceeding as in the proof of Proposition 4.2, to get, as in (4.10),

$$\begin{aligned}
&\limsup_{t\to\infty} p_t^\varepsilon(\beta) \\
&\quad\leq 2\beta K_u\sqrt{\alpha}\mathbf{E}_q[\sqrt{Q(0)}] \\
&\qquad + \mathbf{E}_q\left[\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\log_+\left(\sum_{n=1}^\infty e^{-\frac{\sqrt{t}}{2\kappa}(\alpha n\varepsilon)^2 + \sqrt{t}\alpha n + 2\beta K_u\sqrt{Q(0)}\sqrt{(n+1)\alpha t}}\right)\right] \\
&\quad =: A_2 + A_3.
\end{aligned} \tag{5.8}$$

In other words, we have proved

$$\mathbf{E}[t^{-1}\log u(t,0)] \leq A_1 + A_2 + A_3$$

with $A_1 = c_3\beta^2\varepsilon^{2H}$, and A_2 and A_3 given in (5.8) above.

The remainder of the proof is more complex than the upper bound in the discrete case herein. In order to motivate our choices for the free parameters α and ε, let us imagine for the moment that the term $\sqrt{n+1}$ in (5.8) is not present. This decouples the problem of choosing α and ε as functions of β, and the problem of choosing an optimal relation between α and ε. Hence let us first impose that the negative term $\frac{\sqrt{t}}{2\kappa}(\alpha n\varepsilon)^2$ be four times as large as $\sqrt{t}\alpha n$: this means $\alpha \geq n^{-1}8\kappa\varepsilon^{-2}$. If we simply choose

$$\alpha = 8\kappa\varepsilon^{-2}, \tag{5.9}$$

the last inequality becomes true for all $n \geq 1$. On the other hand, let us now check to see in what situation the negative term is also four times as large as the term with the square root:

$$\frac{1}{2\kappa}(\alpha n\varepsilon)^2 \geq 8\beta K_u\sqrt{Q(0)}\sqrt{(n+1)\alpha}.$$

This means that we must consider low values of n separately. Thus let n_0 be the first integer such that the above inequality is true: n_0 is the smallest n such that

$$\frac{n^2}{\sqrt{n+1}} \geq K_u\sqrt{2^{-1}Q_0}\beta\kappa^{-1/2}\varepsilon.$$

Therefore

$$n_0 \leq (K_u\sqrt{Q_0}\beta\kappa^{-1/2}\varepsilon)^{2/3}. \tag{5.10}$$

Before we can see the effect of n_0, and indeed of the entire series term A_3 in (5.8), we must choose ε by comparing the first term $A_2 = 2\beta K_u\mathbf{E}_q[\sqrt{Q(0)}\sqrt{\alpha}]$ in (5.8), with the discretization error term $A_1 = c_3\beta^2\varepsilon^{2H}$ in (5.6). By choosing to make these two terms equal, except for the factor $\sqrt{Q(0)}$ to avoid having to make ε random, we impose $2\beta K_u\sqrt{\alpha} = c_3\beta^2\varepsilon^{2H}$; in other words, with our choice of α above in (5.9), we have

$$\varepsilon = (4\sqrt{2}K_u c_3^{-1}\sqrt{\kappa}/\beta)^{1/(1+3H)}.$$

We record now that this choice of ε and the condition (5.7) mean that we are restricting the parameters β and κ as announced in the statement of the theorem with the constant c_{++}:

$$\frac{\beta}{\sqrt{\kappa}} \geq \frac{4\sqrt{2}K_u}{c_3 r_2^{1+3H}} =: \sqrt{c_{++}}.$$

We can now evaluate the term A_3 in (5.8). We have

$$\begin{aligned} A_3 &= \mathbf{E}_q\left[\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\log_+\left(\left(\sum_{n=1}^{n_0-1}+\sum_{n=n_0}^{\infty}\right)e^{-\frac{\sqrt{t}}{2\kappa}(\alpha n\varepsilon)^2+\sqrt{t}\alpha n+2\beta K_u\sqrt{Q(0)}\sqrt{(n+1)\alpha t}}\right)\right] \\ &\leq \mathbf{E}_q\left[\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\log n_0\right] \end{aligned} \tag{5.11}$$

$$+\,\mathbf{E}_q\left[2\beta K_u\sqrt{Q(0)}\sqrt{n_0\alpha}\right] \tag{5.12}$$

$$+\,\mathbf{E}_q\left[\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\log_+\left(\sum_{n=1}^{\infty}\exp\left(-\frac{\sqrt{t}}{4\kappa}(\alpha n\varepsilon)^2\right)\right)\right]. \tag{5.13}$$

The term in line (5.11) is zero. The series $\sum_{n=1}^{\infty}\exp(-\frac{\sqrt{t}}{4\kappa}(\alpha n\varepsilon)^2)$ can be made less than twice its first term by choosing t large enough, and therefore the term in line (5.13) is also zero. With the estimate on n_0 in line (5.10), we have that the term in line (5.12) is

$$\begin{aligned}
&\mathbf{E}_q\left[2\beta K_u\sqrt{Q(0)}\sqrt{n_0\alpha}\right]\\
&\quad= \mathbf{E}_q\left[2\beta K_u\sqrt{8}\kappa^{1/2}\varepsilon^{-1}\sqrt{n_0}\sqrt{Q(0)}\right]\\
&\quad\le \mathbf{E}_q\left[2\beta K_u\sqrt{8}\kappa^{1/2}\varepsilon^{-1}(K_u\sqrt{Q(0)}\beta\kappa^{-1/2}\varepsilon)^{1/3}\sqrt{Q(0)}\right]\\
&\quad= 2\sqrt{8}K_u^{4/3}\mathbf{E}_q[Q(0)^{2/3}]\beta^2(\sqrt{\kappa}/\beta)^{2/3}.
\end{aligned}$$

We have proved

$$A_3 \le 2\sqrt{8}K_u^{4/3}\mathbf{E}_q[Q_0^{2/3}]\beta^2(\sqrt{\kappa}/\beta)^{2H/(1+3H)}.$$

Returning to the evaluation of A_1 and A_2, with our choice of ε, we have

$$\begin{aligned}
A_1 + A_2 &= c_3\beta^2\varepsilon^{2H}(1+\mathbf{E}_q[\sqrt{Q(0)}])\\
&= c_3(4\sqrt{2}K_u c_3^{-1})^{2H/(1+3H)}(1+\mathbf{E}_q[\sqrt{Q(0)}])\beta^2(\sqrt{\kappa}/\beta)^{2H/(1+3H)}.
\end{aligned}$$

Since we assumed in the hypothesis of the theorem that $\beta^2 \ge \kappa$, and since the power of $\sqrt{\kappa}/\beta$ in A_3 is greater than in $A_1 + A_2$ ($2/3$ is greater than $2H/(1+3H)$ since $H < 1$), we can summarize our estimates by

$$\mathbf{E}[t^{-1}\log u(t,0)] \le 4\sqrt{2}(1\vee c_3)K_u^{4/3}(1+2\mathbf{E}_q[Q(0)^{2/3}])\beta^2(\sqrt{\kappa}/\beta)^{2H/(1+3H)},$$

which, together with Theorem 3.1, proves all statements in the theorem. □

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