Time regularity of the evolution solution to the fractional stochastic heat equation

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Abstract

We study the time-regularity of the paths of solutions to stochastic partial differential equations (SPDE) driven by additive infinite-dimensional fractional Brownian noise. Sharp sufficient conditions for almost-sure Hölder continuity, and other, more irregular levels of uniform continuity, are given when the space parameter is fixed. Additionally, a result is included on time-continuity when the solution is understood as a spatially Hölder-continuous-function-valued stochastic process. Tools used for the study include the Brownian representation of fractional Brownian motion, and sharp Gaussian regularity results.

Key words and phrases: Fractional Brownian motion, stochastic PDE, path regularity, Gaussian theory, Banach-space-valued process.

1 Preliminaries

1.1 Introduction

In this article, we study the path regularity of solutions to stochastic partial differential equations (SPDE) which are driven by fractional Brownian noise. Before explaining the precise implications of our results, we begin this article with a quick survey of the subject of path regularity for SPDEs with standard Brownian noise. This theory has existed for many years. The now classical analytic techniques, such as the factorization method made popular by Da Prato and Zabczyk (see [3]), represent an important functional-analytic step in the direction of understanding the local behavior of the solution of a SPDE; the basic premise in this framework is that the equation's solution is a stochastic process of a one-dimensional time parameter, with values in an infinite-dimensional Hilbert space of functions. Path regularity is then given in the time parameter

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only. Such a framework can be traced back further, to the original inception of SPDEs, and questions of existence and uniqueness, such as in [12]. By using the embedding of continuity-defined Banach spaces (e.g. Hölder spaces) inside the natural Hilbert spaces where an SPDE's solution lives, results of joint continuity in time and space can be obtained. The book [3] can again be consulted on this topic.

On the other hand, if a SPDE has a well-understood probability law, the standard technique of using the Kolmogorov lemma (see [7, Theorem I.2.1]) can be invoked to prove joint continuity of the solution in both time and space parameters. This technique was used repeatedly in the 1990s, for various problems including stochastic versions of the Heat and Wave equations. We mention [6] as an example, but a complete list would be quite lengthy. Around the year 2000, the incorporation of fractional Brownian behavior in time for SPDEs' potential, which began with such articles as [4] or [5], still only addressed regularity issues by Kolmogorov-type techniques.

Consider the basic example of an SPDE, the stochastic heat equation for all $x \in \mathbb{R}$ and all $t \geq 0$:

$$X(t,x) = X(0,x) + \int_0^t \Delta_x X(s,x)\, ds + W(t,x). \tag{1}$$

When looking at linear equations, especially those with additive noise as above, much more is known about the solutions, and in particular, since the great majority of SPDEs studied are driven by Gaussian noise, one would be well-advised to try and use the Gaussian property of the solution, if any, to obtain sharper characterizations of the solution's regularity. This program was achieved for spatial regularity of stochastic heat equations (on general compact Lie groups) in [8] and [9]: necessary and sufficient conditions for Hölder-continuity of the solution in the space parameter were given in terms of the regularity of the equation's potential, and in terms of the potential's spatial covariance. These results proved to be a sharpening of those obtained previously by the Kolmogorov lemma.

It was not until the work of [10] and [11] that one realized these types of results could be made even more precise, leaving the scale of Hölder-regularity behind, and working with SPDEs driven by infinite-dimensional fractional Brownian motion (fBm) as opposed to simply infinite-dimensional Brownian motion (BM). Precise information about fBm can be found below in the present section. Loosely speaking, fBm has correlated increments, while standard BM has independent increments. The correlations may be positive (the case $H > 1/2$, in the notation below), in which case fBm is more regular than BM, or negative (the case $H < 1/2$), in which case fBm is more irregular than BM. Infinite-dimensional versions of these processes have spatial regularities that are determined by their spatial covariance functions, and are not necessarily related to the time regularity distinctions just described. It was proved in [10] and [11] that the space regularity of the solution of a stochastic heat equation with linear additive infinite-dimensional fBm-noise W (such as 1) is exactly (up to non-random constants) the same as the spatial regularity of the random field $Y = (I - \Delta)^{-H} W$.

This random field Y can be understood, in some generalized-function sense, as the fractional spatial antiderivative of order $2H$ of Y.

These works leave entirely open the question of time regularity for fBm-driven SPDEs. This is the topic which we propose to study here. In this article, we restrict our study to the case $H > 1/2$. The case of smaller H will be the subject of a separate article, and will require different tools. One conclusion we can draw, as a consequence of our Theorem 1, is that the solution X to (1) is α-Hölder continuous *in time* as long as the order-$2H$-*spatial* antiderivative $Y = (I - \Delta)^{-H} W$ is 2α-Hölder continuous *in space*, where if $2\alpha > 1$, this regularity is interpreted as Y having a derivative which is $2\alpha-1$-Hölder continuous *in space*. Also note that these statements can only be made if $\alpha \leq H$. We obtain a time-continuity that is twice as rough as the space continuity, since, as mentioned in the previous paragraph, space continuity in this situation would be of the class 2α-Hölder.

In this article we also prove that our time continuity theorems are sharp, in the sense that if Y is not 2β-Hölder continuous in space, then X will never be β-Hölder continuous in time. Our statements are so sharp that precise (up to non-random constants) moduli of continuity can be given for X depending on X's covariance (Theorem 5). In the example in the previous paragraph, $f(r) = r^{\alpha}\sqrt{\log 1/r}$ would be an almost-sure modulus of continuity for X in time. Going even further in the generality of our results, our theorem 5 is not restricted to Hölder-regularity: it actually reaches all regularity scales. This kind of precise and wide-ranging formulation is possibly only by using the full strength of the Gaussian property, via the continuity characterization results of [11].

Nevertheless, the results in our article are perhaps best appreciated when their implications in the Hölder-scale are combined together with what was obtained in [11] for SPDE spatial regularity. Specifically our conclusions are the following.

- If X is the evolution solution of the Stochastic Heat Equation (1) driven by an infinite-dimensional noise term that is the time differential of a random field W which is fBm in time with Hurst parameter H (see below for the definition of the evolution solution), and if W is the order-$2H$-spatial derivative, in the sense of generalized function (Schwartz distributions), of a random field Y which is 2α-Hölder-continuous in *space* for some $\alpha \leq H$, then X is α-Hölder-continuous in *time*, and 2α-Hölder-continuous in *space*.

- Moreover this result is sharp in the sense that the conclusion fails if Y is not 2α-Hölder-continuous in *space*.

- The joint space-time continuity of X under the above hypotheses can only be guaranteed up to Hölder order α. In other words, we lose the knowledge of having twice as much regularity in space as in time, if regularity is considered jointly in space-time.

The last part of this article shows that the techniques of Da Prato and Zabczyk can be employed in our situation in order to give continuity of the solution X when it is to be understood as a stochastic process with values in the Banach space $\mathcal{B}$ of K-Hölder-continuous function, for any fixed $K < H$. Our result (Theorem 7) shows that if Y is spatially $2\alpha + K$-Hölder, then X is, as a $\mathcal{B}$-valued function, Hölder-continuous of order β for any $\beta < \alpha$. Note here that $2\alpha + K$ may be larger than 1, and in that case, we interpret the result as saying that the derivative of Y is $2\alpha + K - 1$-Hölder continuous.

The paper is structured as follows: in the remainder of this section, we give the basic mathematical setup and tools needed for our analysis, including the one-dimensional and infinite dimensional fBm objects. Section 2 deals with the regularity results in time for fixed space parameter. Section 3 presents the time-regularity results when X lives in a Banach space of Hölder-continuous functions.

1.2 The Wiener integral with respect to fractional Brownian motion

Consider $T = [0, \tau]$ a time interval with arbitrary fixed horizon τ and let $(B_t^H)_{t\in T}$ be the one-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$. This means by definition that B^H is a centered Gaussian process with covariance

$$R(t,s) = \mathbb{E}\left(B_s^H B_t^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{2}$$

It is a process starting from zero with stationary increments,

$$\mathbb{E}\left(B_t^H - B_s^H\right)^2 = |t-s|^{2H}, \tag{3}$$

which is self-similar, that is, $B_{\alpha t}^H$ has the same distribution as $\alpha^H B_t^H$. Note that $B^{1/2}$ is standard Brownian motion. B^H has the following Wiener integral representation:

$$B_t^H = \int_0^t K^H(t,s)dW_s$$

where $W = \{W_t : t \in T\}$ is a Wiener process, and K^H is the kernel given by

$$K^H(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + s^{\frac{1}{2}-H} G\left(\frac{t}{s}\right), \tag{4}$$

for $s < t$, c_H being a constant and

$$G(z) = c_H \left(\frac{1}{2} - H\right) \int_0^{z-1} r^{H-\frac{3}{2}} \left(1 - (1+r)^{H-\frac{1}{2}}\right) dr.$$

From (4) we obtain

$$\frac{\partial K^H}{\partial t}(t,s) = c_H \left(H - \frac{1}{2}\right) (t-s)^{H-\frac{3}{2}} \left(\frac{s}{t}\right)^{\frac{1}{2}-H}.$$

Let $\mathcal{E}_H$ be the linear space of step functions on T of the form

$$\varphi(t) = \sum_{i=1}^{n} a_i 1_{(t_i, t_{i+1}]}(t)$$

where $0 = t_1 < t_2 < \cdots < t_n < t_{n+1} = \tau,\ n \in \mathbb{N},\ a_i \in \mathbb{R}$ and let $\mathcal{H}$ be the closure of $\mathcal{E}_H$ with respect to the scalar product

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}} = R(t,s).$$

For $\varphi \in \mathcal{E}_H$ we define its Wiener integral with respect to the fractional Brownian motion as

$$\int_T \varphi_s dB^H(s) = \sum_{i=1}^{n} a_i \left(B^H_{t_{i+1}} - B^H_{t_i} \right).$$

The mapping

$$\varphi = \sum_{i=1}^{n} a_i 1_{(t_i, t_{i+1}]} \to \int_T \varphi_s dB^H(s)$$

is an isometry between $\mathcal{E}_H$ and the linear space $span\{B^H_t, t \in T\}$ viewed as a subspace of $L^2(\Omega)$ and it can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{span}^{L^2(\Omega)}\{B^H_t, t \in T\}$. The image of an element $\Phi \in \mathcal{H}$ by this isometry is called the Wiener integral of Φ with respect to B^H.

For every $s < \tau$, consider the operator K^* in $L^2(T)$

$$(K^*_\tau \varphi)(s) = K(\tau, s)\varphi(s) + \int_s^\tau (\varphi(r) - \varphi(s)) \frac{\partial K}{\partial r}(r,s) dr.$$

When $H > \frac{1}{2}$, the operator K^*_τ has the simpler expression

$$(K^*_\tau \varphi)(s) = \int_s^\tau \varphi(r) \frac{\partial K}{\partial r}(r,s) dr.$$

For any $t \in T$ we can define K^*_t similarly. The fact that K^*_τ is an isometry between $\mathcal{H}$ and $L^2(T)$ is proved in [2]. As a consequence, we have the following relationship between the Wiener integral with respect to the fractional Brownian motion and the Wiener integral with respect to the Wiener process W:

$$\int_T \varphi(s) dB^H(s) = \int_T (K^*_\tau \varphi)(s) dW(s)$$

which holds for every $\varphi \in \mathcal{H}$ if and only if $K^*_\tau \varphi \in L^2(T)$. For any $s, t \in T$, one can check that $K^*_\tau \left(\varphi 1_{[0,t]}\right)(s) = K^*_t(\varphi)(s) 1_{[0,t]}(s)$. Then we can define the stochastic integral $\int_0^t \varphi(s) dB^H(s)$ by $\int_0^\tau \varphi(s) 1_{[0,t]}(s) dB^H(s)$, and obtain

$$\int_0^t \varphi(s) dB^H(s) = \int_0^t (K^*_t \varphi)(s) dW(s)$$

for every $t \in T$ and $\varphi 1_{[0,t]} \in \mathcal{H}$ if and only if $K_t^* \varphi \in L^2(T)$.

We also recall that when $H > \frac{1}{2}$

$$\mathbb{E}\left[\int_0^\tau f(u)dB^H(u) \cdot \int_0^\tau g(u)dB^H(u)\right] = \alpha_H \int_{T^2} f(s)g(t)|t-s|^{2H-2}dsdt \quad (5)$$

where $\alpha_H = H(2H-1)$.

1.3 The stochastic heat equation with infinite-dimensional fractional Brownian motion

We consider the stochastic heat equation on the unit circle S^1 driven by an infinite-dimensional fractional Brownian motion (fBm):

$$\frac{\partial X}{\partial t}(t,x) = \Delta_x X(t,x) + \frac{\partial B^H}{\partial t}(t,x) \quad (6)$$

where $x \in S^1$, $t \in [0,1]$, $X(0,x) = 0$, $H \in (0,1)$, Δ is the standard Laplacian on S^1, B^H is a Gaussian field on $[0,1] \times S^1$ whose behavior in time is fBm with Hurst parameter H, and whose behavior in space is homogeneous (*i.e.* for every t, $B^H(t,\cdot)$ is Gaussian and its covariance depends only on differences between points). Note that while the theorems proved in this article are relative to $H > 1/2$, the existence of the solution to (6) can be established for all $H \in (0,1)$, as proved in [10].

The set of functions

$$\{\cos nx,\ \sin nx : n \in \mathbb{N}\} \quad (7)$$

is not only an orthogonal basis for $L^2(S^1, dx)$ where dx is the normalized Lebesgue measure on $[-\pi, \pi)$, but also is exactly the set of eigenfunctions of Δ. We assume, as was done in [10], that the random field B^H is given by a random Fourier series:

$$B^H(t,x) = \sqrt{q_0}\beta_0^H(t) + \sum_{n=1}^{\infty} \sqrt{q_n}\left(\beta_n^H(t)\cos nx + \widetilde{\beta}_n^H(t)\sin nx\right) \quad (8)$$

where $\{\beta_n^H\}_n$ and $\{\widetilde{\beta}_n^H\}_n$ are IID fBm's with common $H \in (0,1)$, and $\{q_n\}_{n=0}^{\infty}$ is a sequence of non-negative terms. The reader can refer to [10] for a detailed treatment of the Gaussian field B^H.

As done in theory of stochastic PDEs, let us write (6) in its weaker evolution form:

$$X(t,x) = \int_0^t P_{t-s}[B(ds,\cdot)](x) \quad (9)$$

where $t \in [0,1]$, $x \in S^1$ and $(P_t)_{t \geq 0}$ is the semigroup of operators generated by the Laplacian on S^1 whose action on $L^2(S^1)$ is characterized by

$$P_t[e^{in\cdot}](x) = \exp(-n^2 t)e^{inx},$$

which can be translated into a characterization using the trigonometric functions (7).

Existence and uniqueness of the solution X to (9) is given in [10]. Moreover, the following random Fourier representation holds for X:

$$X(t,x) = \sum_{n=0}^{\infty} \sqrt{q_n} \cos(nx) \int_0^t e^{-n^2(t-s)} \beta_n^H(ds) + \sum_{n=1}^{\infty} \sqrt{q_n} \sin(nx) \int_0^t e^{-n^2(t-s)} \tilde{\beta}_n^H(ds), \tag{10}$$

under a necessary and sufficient condition for existence

$$\sum_{n=1}^{\infty} \frac{q_n}{n^{4H}} < \infty. \tag{11}$$

2 Pointwise regularity of the evolution solution

A detailed study of the spatial regularity of (10) is provided in [11] for fixed time parameter. Throughout the entire remainder of the article, we fix $H > 1/2$. While in the next section, we will study the Hölder continuity in time when the solution is considered as a function-valued process, this section is devoted to the regularity of (10) in the time variable when the space variable is fixed. Therefore, in this chapter, $x \in S^1$ is fixed. We begin with a precise calculation of the solution's canonical metric; then we apply some simple estimates to obtain a result of Hölder-continuity; lastly, we show this result in the Hölder scale is sharp by formulating a more general time-continuity theory, which also applies to other scales.

2.1 The canonical metric of X

In this subsection we evaluate the so-called *canonical metric* of X in the time parameter, namely, for this fixed x, for all $t_1, t_2 \in [0,1]$, the quantity

$$\delta^2(t_1,t_2) := \mathbb{E}\left\{ (X(t_2,x) - X(t_1,x))^2 \right\}. \tag{12}$$

The significance of this pseudo-metric is as follows. If one can prove that for some increasing function ℓ defined on a neighborhood of 0 in $\mathbb{R}_+$, such that $\lim_{r\to 0} \ell(r) = 0$, for all t_1, t_2 in the same neighborhood,

$$\delta(t_1,t_2) \le \ell(|t_1 - t_2|), \tag{13}$$

then, since X is a centered Gaussian process, the theory of Gaussian regularity (see for example [1] or [11]) implies that the function η defined by $\eta(r) = \ell(r) \log^{1/2}(1/r)$ is almost surely a uniform modulus of continuity for the stochastic process $X(\cdot, x)$ defined on $[0,1]$ (we must necessarily have $\lim_0 \eta$

for this statement to be non-vacuous). The work in [11] shows that a converse to this result exists, and therefore it is desirable to have a lower bound on δ^2 of the same form as (13). Such bounds, and their implications, are found in the following subsections. Here we simply calculate δ^2.

Using the fact that $\{\beta_n^H\}_n$ and $\{\widetilde{\beta}_n^H\}_n$ are IID fBm's, we obtain

$$\begin{aligned}\delta^2(t_1,t_2) = \; & q_0 \; \mathbb{E}\left\{\int_0^{t_2} \beta_n^H(ds) - \int_0^{t_1} \beta_n^H(ds)\right\}^2 \\ & + \sum_{n=1}^{\infty} q_n \cos^2 nx \; \mathbb{E}\left\{\int_0^{t_2} e^{-n^2(t_2-s)}\beta_n^H(ds) - \int_0^{t_1} e^{-n^2(t_1-s)}\beta_n^H(ds)\right\}^2 \\ & + \sum_{n=1}^{\infty} q_n \sin^2 nx \; \mathbb{E}\left\{\int_0^{t_2} e^{-n^2(t_2-s)}\widetilde{\beta}_n^H(ds) - \int_0^{t_1} e^{-n^2(t_1-s)}\widetilde{\beta}_n^H(ds)\right\}^2 .\end{aligned}$$

Now using (3) and the fact that the expectations in the last two terms are the same, we have

$$\begin{aligned}\delta^2(t_1,t_2) = \; & q_0 \; |t_2-t_1|^{2H} \\ & + \sum_{n=1}^{\infty} q_n \mathbb{E}\left\{\int_0^{t_2} e^{-n^2(t_2-s)}\beta_n^H(ds) - \int_0^{t_1} e^{-n^2(t_1-s)}\beta_n^H(ds)\right\}^2 .\end{aligned}$$

It remains to estimate the expectation in the above term using (5) for calculating expectations. Assume $t_1 < t_2$, then

$$\begin{aligned}& \frac{1}{\alpha_H}\mathbb{E}\left\{\int_0^{t_2} e^{-n^2(t_2-s)}\beta_n^H(ds) - \int_0^{t_1} e^{-n^2(t_1-s)}\beta_n^H(ds)\right\}^2 \\ =& \frac{1}{\alpha_H}\mathbb{E}\left\{\int_0^{t_1} \left(e^{-n^2(t_2-s)} - e^{-n^2(t_1-s)}\right)\beta_n^H(ds) + \int_{t_1}^{t_2} e^{-n^2(t_2-s)}\beta_n^H(ds)\right\}^2 \\ =& \int_0^{t_1}\int_0^{t_1} \left(e^{-n^2(t_2-s)} - e^{-n^2(t_1-s)}\right)\left(e^{-n^2(t_2-s')} - e^{-n^2(t_1-s')}\right)|s-s'|^{2H-2}ds ds' \\ & + \int_{t_1}^{t_2}\int_{t_1}^{t_2} \left(e^{-n^2(t_2-s)}\right)\left(e^{-n^2(t_2-s')}\right)|s-s'|^{2H-2}ds ds' \\ & + 2\int_{t_1}^{t_2}\int_0^{t_1} \left(e^{-n^2(t_2-s)} - e^{-n^2(t_1-s)}\right)\left(e^{-n^2(t_2-s')}\right)|s-s'|^{2H-2}ds ds' \\ =& \left(e^{-n^2t_2} - e^{-n^2t_1}\right)^2 \int_0^{t_1}\int_0^{t_1} e^{n^2s}e^{n^2s'}|s-s'|^{2H-2}ds' ds \\ & + e^{-2n^2t_2}\int_{t_1}^{t_2}\int_{t_1}^{t_2} e^{n^2s}e^{n^2s'}|s-s'|^{2H-2}ds' ds \\ & + 2\left(e^{-n^2t_2} - e^{-n^2t_1}\right)e^{-n^2t_2}\int_0^{t_1}\int_{t_1}^{t_2} e^{n^2s}e^{n^2s'}|s-s'|^{2H-2}ds' ds \\ =& \left(e^{-n^2t_2} - e^{-n^2t_1}\right)^2 I_1 + e^{-2n^2t_2}I_2 + 2\left(e^{-n^2t_2} - e^{-n^2t_1}\right)e^{-n^2t_2}I_3,\end{aligned}$$

where

$$(I_1) = \int_0^{t_1}\int_0^{t_1} e^{n^2 s} e^{n^2 s'} |s-s'|^{2H-2} ds' ds$$
$$(I_2) = \int_{t_1}^{t_2}\int_{t_1}^{t_2} e^{n^2 s} e^{n^2 s'} |s-s'|^{2H-2} ds' ds$$
$$(I_3) = \int_0^{t_1}\int_{t_1}^{t_2} e^{n^2 s} e^{n^2 s'} |s-s'|^{2H-2} ds' ds.$$

Letting

$$(I_4) = \int_0^{t_2}\int_0^{t_2} e^{n^2 s} e^{n^2 s'} |s-s'|^{2H-2} ds' ds,$$

we observe that $2(I_3) = (I_4) - (I_1) - (I_2)$ holds by symmetry.

Now let us calculate (I_1) using change of variables $u = s - s'$, $v = n^2 u$ and finally applying Fubini's theorem:

$$\begin{aligned}
(I_1) &= 2\int_0^{t_1}\int_0^{s} e^{n^2 s} e^{n^2 s'} |s-s'|^{2H-2} ds' ds \\
&= 2\int_0^{t_1}\int_0^{s} e^{n^2 s} e^{n^2 (s-u)} u^{2H-2} du ds \\
&= 2\int_0^{t_1}\int_0^{s} e^{2n^2 s} e^{-n^2 u} u^{2H-2} du ds \\
&= \frac{2}{n^{4H-2}}\int_0^{t_1}\int_0^{n^2 s} e^{2n^2 s} e^{-v} v^{2H-2} dv ds \\
&= \frac{2}{n^{4H-2}}\int_0^{n^2 t_1}\left(\int_{v/n^2}^{t_1} e^{2n^2 s} ds\right) e^{-v} v^{2H-2} dv \\
&= \frac{1}{n^{4H}}\int_0^{n^2 t_1}\left(e^{2n^2 t_1} - e^{2v}\right) e^{-v} v^{2H-2} dv \\
&= \frac{e^{2n^2 t_1}}{n^{4H}}\int_0^{n^2 t_1}\left(1 - e^{2v-2n^2 t_1}\right) e^{-v} v^{2H-2} dv.
\end{aligned}$$

Similarly,

$$(I_2) = \frac{e^{2n^2 t_2}}{n^{4H}}\int_0^{n^2(t_2-t_1)}\left(1 - e^{2v-2n^2(t_2-t_1)}\right) e^{-v} v^{2H-2} dv$$

and

$$(I_4) = \frac{e^{2n^2 t_2}}{n^{4H}}\int_0^{n^2 t_2}\left(1 - e^{2v-2n^2 t_2}\right) e^{-v} v^{2H-2} dv.$$

Keeping in mind that $2(I_3) = (I_4) - (I_1) - (I_2)$ we are ready to write (12) explicitly:

$$\delta^2(t_1,t_2) = \; q_0 \; |t_2 - t_1|^{2H} + \sum_{n=1}^{\infty} q_n c_H \Big\{ \left(e^{-n^2 t_1} - e^{-n^2 t_2}\right) \left(e^{-n^2 t_1}\right) (I_1) \\ + \left(e^{-n^2 t_1}\right) \left(e^{-n^2 t_2}\right) (I_2) - \left(e^{-n^2 t_1} - e^{-n^2 t_2}\right) \left(e^{-n^2 t_2}\right) (I_4) \Big\}.$$

We can rewrite this using the following function:

$$F(z) := \int_0^z \left(1 - e^{2v-2z}\right) e^{-v} v^{2H-2} dv. \tag{14}$$

We finally have

$$\delta^2(t_1,t_2) = \; q_0 \; |t_2 - t_1|^{2H} + \sum_{n=1}^{\infty} \frac{q_n c_H}{n^{4H}} \Big\{ \left(1 - e^{-n^2(t_2-t_1)}\right) F\left(n^2 t_1\right) \\ + \left(e^{n^2(t_2-t_1)}\right) F\left(n^2(t_2-t_1)\right) - \left(e^{n^2(t_2-t_1)} - 1\right) F\left(n^2 t_2\right) \Big\}. \tag{15}$$

2.2 Hölder-continuity of the trajectories

For the sake of our presentation's clarity, this subsection deals only with the Hölder continuity of (10) in the time variable. The basic estimates introduced here will be used to formulate a more general theory in the next subsection. We use the general notation $f \asymp g$ for two positive function whose ratio is bounded above and from below by two positive constants.

Theorem 1 *Let $H > \frac{1}{2}$ and $\alpha \leq H$ be such that*

$$\sum_{n=1}^{\infty} \frac{q_n}{n^{4H-4\alpha}} < \infty.$$

Then for any $x \in S^1$, $X(\cdot,x)$ has β-Hölder continuous trajectories $\forall \beta \in (0,\alpha)$. More precisely, the function $r \mapsto r^{\alpha} \log^{1/2}(1/r)$ is almost surely a uniform modulus of continuity for $X(\cdot,x)$.

Corollary 2 *If $\sum q_n < \infty$, or a fortiori if there exists a sequence a_n such that $\lim_{n\to\infty} a_n = +\infty$ and $\sum q_n a_n < \infty$, we can guarantee that $X(\cdot,x)$ admits the function $k(r) = r^H \log^{1/2}(1/r)$ as an almost sure uniform modulus of continuity. But this regularity cannot be improved no matter how fast a_n tends to $+\infty$. Specifically, if $q_0 \neq 0$, no function $\tilde{k}$ such that $\tilde{k}(r) = o(k(r))$ is a uniform modulus of continuity for $X(\cdot,x)$.*

In order to prove these result, we will use the following estimates.

Lemma 3 *F is increasing and $F(z) \leq \left(1 - e^{-2z}\right)\Gamma(2H-1)$.*

Proof. Trivial. □

Lemma 4 *For all $z \in [0,1]$, $F(z) \asymp z^{2H}$.*

Proof. Since $e^{-v} \asymp 1$, by the change of variables $v' = v/z$, with F defined in (14), we have

$$\begin{aligned} F(z) &= \int_0^z \left(1 - e^{2v-2z}\right) e^{-v} v^{2H-2} dv \\ &\asymp \int_0^z \left(1 - e^{2v-2z}\right) v^{2H-2} dv \\ &= z^{2H-1} \int_0^1 (v')^{2H-2} \left(1 - e^{-2z\left(1-v'\right)}\right) dv'. \end{aligned}$$

Now use the fact that for all $u \in [0,1]$, $1 - e^{-2u} \asymp u$,

$$\begin{aligned} F(z) &\asymp z^{2H-1} \int_0^1 (v')^{2H-2} z\,(1 - v')\, dv' \\ &\asymp z^{2H}, \end{aligned}$$

finishing the proof of the lemma. □

Proof of Theorem 1. To obtain the result, we would like to control $\delta(t_1, t_2)$ from above with order $|t_2 - t_1|^\alpha$ for some $\alpha \le H$. We need to separate the sum in the following

$$\begin{aligned} \delta^2(t_1,t_2) = q_0\, |t_2 - t_1|^{2H} + \sum_{n=1}^{\infty} \frac{q_n c_H}{n^{4H}} \Big\{ &\left(1 - e^{-n^2(t_2-t_1)}\right) F\left(n^2 t_1\right) \\ &+ \left(e^{n^2(t_2-t_1)}\right) F\left(n^2(t_2-t_1)\right) - \left(e^{n^2(t_2-t_1)} - 1\right) F\left(n^2 t_2\right) \Big\} \end{aligned}$$

into two parts, as $n^2(t_2 - t_1) < 1$ (head) and $n^2(t_2 - t_1) \ge 1$ (tail).

First, let us estimate the tail of the series. Since F is increasing, $F\left(n^2(t_2 - t_1)\right) \le F\left(n^2 t_2\right)$; also using the estimate of Lemma 3, we arrive at (C_H being a constant depending only on H and possibly changing from line to line)

$$\begin{aligned} &\sum_{n^2(t_2-t_1)\ge 1} \frac{q_n C_H}{n^{4H}} \Big\{ \left(1 - e^{-n^2(t_2-t_1)}\right) \Gamma(2H-1) \\ &\qquad + \left(e^{n^2(t_2-t_1)}\right) F\left(n^2(t_2-t_1)\right) - \left(e^{n^2(t_2-t_1)} - 1\right) F\left(n^2(t_2-t_1)\right) \Big\} \\ &\le \sum_{n^2(t_2-t_1)\ge 1} \frac{q_n C_H}{n^{4H}} \left\{ \left(1 - e^{-n^2(t_2-t_1)}\right) \Gamma(2H-1) + F\left(n^2(t_2-t_1)\right) \right\} \\ &\le \sum_{n^2(t_2-t_1)\ge 1} \frac{q_n C_H}{n^{4H}} \Gamma(2H-1) \left\{ \left(1 - e^{-n^2(t_2-t_1)}\right) + \left(1 - e^{-2n^2(t_2-t_1)}\right) \right\} \\ &= \sum_{n^2(t_2-t_1)\ge 1} \frac{q_n C_H}{n^{4H}} \left\{ \left(1 - e^{-n^2(t_2-t_1)}\right) \left(1 + 1 + e^{-n^2(t_2-t_1)}\right) \right\} \\ &\le \sum_{n^2(t_2-t_1)\ge 1} \frac{q_n C_H}{n^{4H}} \left(1 - e^{-n^2(t_2-t_1)}\right)^2. \end{aligned} \tag{16}$$

The last estimate follows from the fact that $(1-e^{-x})(2+e^{-x}) \asymp (1-e^{-x})^2$ when $x \geq 1$. Now since $(t_2-t_1) \in (0,1)$, for any $0 < \alpha \leq H$, (16) can be rewritten as

$$\sum_{n^2(t_2-t_1)\geq 1} \frac{q_n C_H}{n^{4H}} (1-e^{-n^2(t_2-t_1)})^{2\alpha}(1-e^{-n^2(t_2-t_1)})^{2-2\alpha}$$

$$\leq \sum_{n^2(t_2-t_1)\geq 1} \frac{q_n C_H}{n^{4H}} n^{4\alpha}|t_2-t_1|^{2\alpha} \leq \left(C_H \sum_{n^2(t_2-t_1)\geq 1} \frac{q_n}{n^{4H-4\alpha}} \right) |t_2-t_1|^{2\alpha}.$$

Next, we look at the head of the series

$$\sum_{n^2(t_2-t_1)<1} \frac{q_n C_H}{n^{4H}} \Big\{ \left(1-e^{-n^2(t_2-t_1)}\right) F\left(n^2 t_1\right) + \left(e^{n^2(t_2-t_1)}\right) F\left(n^2(t_2-t_1)\right) - \left(e^{n^2(t_2-t_1)}-1\right) F\left(n^2 t_2\right) \Big\}.$$

For $0 < \alpha \leq H$ and $x < 1$, it is clear that $e^x \leq \frac{e}{x^{2H-2\alpha}}$. Now, using Lemma (4), F being increasing and $n^2(t_2-t_1) < 1$, we have

$$\begin{aligned}\left(e^{n^2(t_2-t_1)}\right) F\left(n^2(t_2-t_1)\right) &\leq C_H \frac{1}{n^{4H-4\alpha}(t_2-t_1)^{2H-2\alpha}} n^{4H}(t_2-t_1)^{2H} \\ &\leq C_H n^{4\alpha}(t_2-t_1)^{2\alpha}\end{aligned}$$

and

$$\left(1-e^{-n^2(t_2-t_1)}\right) F\left(n^2 t_1\right) - \left(e^{n^2(t_2-t_1)}-1\right) F\left(n^2 t_2\right) \leq 0.$$

Therefore, the head of the series will have the upper bound

$$C_H \sum_{n^2(t_2-t_1)<1} \frac{q_n}{n^{4H-4\alpha}} |t_2-t_1|^{2\alpha}.$$

Finally, since $|t_2-t_1| < 1$, we have $|t_2-t_1|^{2H} \leq |t_2-t_1|^{2\alpha}$ for any $\alpha \leq H$, and hence,

$$\delta^2(t_1,t_2) \leq \left(q_0 + C_H \sum_{n=1}^{\infty} \frac{q_n}{n^{4H-4\alpha}} \right) |t_2-t_1|^{2\alpha}.$$

The first result of the theorem follows by applying Kolmogorov's Lemma. For the second result, it is sufficient to use the general Gaussian regularity theory (see [11] or [1]). For the corollary, it is sufficient to note that by (15), $\delta^2(t,s) \geq q_0 C_H |t-s|^{2H}$ irrespective of the other values of the q_n's, and to apply the "necessary condition" results in [11]. . □

2.3 General theory of time-regularity

In this section we solve two separate problems. For any given candidate η for the time-modulus of continuity of $X(\cdot,x)$, we give a simple criterion ensuring that

η is an almost-sure modulus of continuity for $X(\cdot, x)$. Then for any sequence $\{q_n\}_{n\in\mathbb{N}}$, we provide a formula for a sharp modulus of continuity for $X(\cdot, x)$.

These results imply that the condition on $\{q_n\}_{n\in\mathbb{N}}$ in Theorem 1 is a sharp criterion to ensure the theorem's conclusion, meaning that any strictly weaker condition on $\{q_n\}_{n\in\mathbb{N}}$ will imply a strictly weaker conclusion.

Theorem 5 *Assume that there exists an increasing positive function ℓ on a neighborhood of* 0 *such that the following series converges:*

$$\sum_n \frac{q_n}{n^{4H}\ell(n^{-2})} < \infty. \tag{Hℓ}$$

Then there exists a constant K depending on $\{q_n\}_{n\in\mathbb{N}}$ and H such that for all s, t sufficiently close, $\delta^2(s,t) \leq K\ell(|t-s|)$. Consequently, the function η defined by

$$\eta(r) = \ell(r)^{1/2}\log^{1/2}(1/r)$$

is an almost-sure uniform modulus of continuity for $X(\cdot, x)$, as long as $\lim_0 \eta = 0$, and as long as $r^{2H} \leq \ell(r)$ in a neighborhood of 0*.*

Proof.

Step 0: setup. Let $h = |t-s|$. We decompose the series giving δ^2 into two parts, omitting the term corresponding to q_0, splitting the series in two according to whether $n^2h \leq c_0$ where c_0 is a constant depending only on H which will be chosen appropriately. Therefore with

$$\begin{aligned}
A_2 &= e^{n^2h}F\left(n^2h\right),\\
A_4 &= \left(e^{n^2h}-1\right)F\left(n^2t\right),\\
A_1 &= \left(1-e^{-n^2h}\right)F\left(n^2s\right),
\end{aligned}$$

we have

$$\delta^2(s,t) - q_0|t-s|^{2H} = G(h) + J(h)$$

where

$$G(h) = \sum_{n^2h\leq c_0}\frac{q_n}{n^{4H}}\left(A_2 - A_4 + A_1\right)$$

and

$$J(h) = \sum_{n^2h>c_0}\frac{q_n}{n^{4H}}\left(A_2 - A_4 + A_1\right).$$

Here we are using an abusive shorthand notation for J and G, which appears to indicate that $\delta^2(s,t)$ only depends on $h = |t-s|$; of course, this is not true, and it would be more correct to denote J by $J_{s,t}(h)$ since A_1 and A_4 depend on s and t respectively. But the estimates we will find on G and J will only depend on h, which is why the abusive notation $J = J(h)$ is meaningful.

Step 1: head of the series. We first observe that $A_1 \leq A_4$ in all cases. When $n^2 h \leq c_0$, if we can prove that $A_2 \geq 2(A_4 - A_1)$, this will ensure that $G(h)$ is commensurate with

$$\tilde{G}(h) := \sum_{n^2 h \leq c_0} A_2 \frac{q_n}{n^{4H}}.$$

Let $z = n^2 h$. Assume $c_0 \leq 1$. Also, in order to be able to formulate sharp lower bounds, we assume that $s > 0$. This means that z is bounded below.

Following Lemmas 4 and 3, for $z \leq c_0$, $F(z) \asymp z^{2H}$, and $e^z \asymp 1 + z \asymp 1$, and $e^z - 1 \asymp z$, where all the commensurability constants are universal or depend only on H. Thus for some constants c_H and C_H

$$c_H z^{2H} \leq A_2 \leq z^{2H} C_H.$$

Now without loss of generality we assume $s < t$ and we let $a = n^2 s$; we calculate

$$\begin{aligned} A_4 - A_1 &= (e^z - 1) F(z + a) - \left(1 - e^{-z}\right) F(a) \\ &= F(a) \left(e^z + e^{-z} - 2\right) + (e^z - 1)(F(z + a) - F(a)). \end{aligned}$$

We use the fact that $(e^z + e^{-z} - 2) \asymp z^2$, also $e^z - 1 \asymp z$, $F(a) \asymp 1$, and $F(z + a) - F(a) \asymp F'(a) z$. These imply

$$A_4 - A_1 \asymp z^2 + z^2 F'(a),$$

where the commensurability constant depend only on H and on a positive lower bound on s. We can now check that

$$F'(a) = 2e^{-2a} \int_0^a e^v v^{2H-2} dv \leq C_H e^{-a}$$

for some constant C_H. This proves that

$$A_4 - A_1 \asymp z^2.$$

Since on the other hand, we already saw that

$$A_2 \asymp z^{2H},$$

there exists a constant c_0 depending only on H such that if $z \leq c_0$, $A_2 \geq 2(A_4 - A_1)$ as announced – indeed, by hypothesis there exist c_1 and c_2 such that $A_4 - A_1 \leq c_1 z^2$ and $A_2 \geq c_2 z^{2H}$; a trivial calculation shows that if $z \geq c_0 := (c_2/(2c_1))^{1/(2-2H)}$, then $A_2 \geq 2(A_4 - A_1)$. We see in particular that this constant c_0 does not depend on a lower bound on s. On the other hand, the lower bound of the commensurability relation on $A_4 - A_1$ does, but this last bound will not be used for the proof of the current theorem.

With the upper bound on A_2 we now have proved that there exists constants c_H, C_H, and c_0 depending only on H (and c_H depends on a positive lower bound on s), such that

$$c_H h^{2H} \sum_{n^2 h \leq c_0} q_n \leq G(h) \leq C_H h^{2H} \sum_{n^2 h \leq c_0} q_n. \tag{17}$$

Step 2: control the estimate of Step 1 using Condition (Hℓ) of Theorem 5. By the result of Corollary 2, we know that we can assume that $\ell(r) \geq r^{2H}$. Then without loss of generality, we can assume that $n^{4H}\ell\left(n^{-2}\right)$ is increasing. Thus using Condition (Hℓ)

$$+\infty > \sum_{n^2h \leq c_0} \frac{q_n}{n^{4H}\ell\left(n^{-2}\right)} \geq \left(h/c_0\right)^{2H} \ell^{-1}\left(h/c_0\right) \sum_{n^2h \leq c_0} q_n.$$

Also since $\ell(r) \geq r^{2H}$, we can assume without loss of generality that $\ell(h/c_0) \leq C\ell(h)$ for some constant C depending only on C. We have thus proved, for some constant C depending on H and $\{q_n\}_{n\in\mathbb{N}}$:

$$G(h) \leq C\ell(h).$$

Step 3: tail of the series. When $n^2h \geq c_0$, we have $1 - e^{-n^2h} \asymp 1$. The estimate (16) still holds if one replaces t_2 by t, t_1 by s and $n^2(t_2 - t_1) > 1$ by $n^2(t - s) > c_0$. Now using Condition (Hℓ), we can write

$$\begin{aligned} J(h) &\leq C_H \sum_{n^2h \geq c_0} \frac{q_n}{n^{4H}} \left(1 - e^{-n^2h}\right)^2 \\ &\asymp \sum_{n^2h \geq c_0}^{\infty} \frac{q_n}{n^{4H}} \qquad (18) \\ &\leq \ell\left(h/c_0\right) \sum_{n^2h \geq c_0}^{\infty} \frac{q_n}{n^{4H}\ell\left(n^{-2}\right)} \asymp \ell(h). \end{aligned}$$

We have thus proved that for some constant C depending on H and $\{q_n\}_{n\in\mathbf{N}}$:

$$J(h) \leq C\ell(h).$$

We leave it to the reader to show the following fact, which will not be needed here, but will become convenient in the next theorem, that for some other constant c,

$$J(h) \geq c \sum_{n^2h \geq c_0}^{\infty} \frac{q_n}{n^{4H}} \qquad (19)$$

Conclusion. The conclusions of Step 2 and Step 3 finishes the proof of the estimate on δ^2, while the other claim is again a direct consequence of regularity theory of Gaussian processes (see [11]), with the caveat due to the fact, which we have noticed before, that $\delta^2(s,t) \geq q_0 h^{2H}$. □

The proof of the above Theorem 5 has enabled us to identify two fundamental quantities in the estimation of δ^2. We summarize the results of lines (17), (18), and (19):

$$\delta^2(s,t) = q_0 |t-s|^{2H} + G(h) + J(h),$$
$$G(h) \asymp G_0(h) := h^{2H} \sum_{n^2 h \leq c_0} q_n,$$
$$J(h) \asymp J_0(h) := \sum_{n^2 h \geq c_0}^{\infty} \frac{q_n}{n^{4H}}.$$

The lower bounds in the above statements hold only for s,t bounded away from 0. The next result follows immediately, its last statement being a consequence of the necessary condition results of [11].

Corollary 6 *Let G_0 and J_0 be the two functions defined above. Then the function η defined by*

$$\eta(r) := (G_0(r) + J_0(r))^{1/2} \log^{1/2}(1/r)$$

is an almost-sure uniform modulus of continuity for $X(\cdot,x)$ on any closed interval in $(0,1]$, as long as $\lim_0 \eta = 0$ and $G_0(r) + J_0(r) \geq r^{2H}$ near 0. Moreover η is sharp in the sense that if $\zeta \ll \eta$, then ζ is not an almost-sure uniform modulus of continuity for $X(\cdot,x)$.

We turn to some examples.

- *Hölder scale.*
  - The situation of Theorem 1 can be precisely achieved as follows. Assume $0 < \alpha \leq H$ and

    $$q_n = n^{4H-1-4\alpha}.$$

    Then one can easily check that $G_0(r) \asymp r^{2\alpha} \asymp J_0(r)$, so that by the preceding corollary, the function

    $$\eta(r) = r^{\alpha} \log^{1/2}(1/r)$$

    is almost surely a sharp uniform modulus of continuity for $X(\cdot,x)$.
  - In this example, we may recall the results of [11], which state that the regularity is twice as good in space: $\zeta(r) = r^{2\alpha} \log^{1/2}(1/r)$ is an almost-sure modulus of continuity of $X(t,\cdot)$ for any fixed t.

- *Logarithmic scale.*
  - Assume there exists $\beta > 1$ such that

    $$q_n = \frac{n^{4H-1}}{\log^{2\beta} n}.$$

Then one can check that $J_0(r) \asymp \log^{-(2\beta-1)}(1/r)$ while $G_0(r) \leq \log^{-2\beta}(1/r)$ (the "tail" term of the series defining δ^2 is dominant). Hence

$$\eta(r) = \log^{-(\beta-1)}(1/r)$$

is almost surely a sharp uniform modulus of continuity for $X(\cdot, x)$.

– On the other hand, the space-regularity of $X(t, \cdot)$ given in [11] for this class of examples is significantly higher, especially for β close to 1: we find that the following function is an almost-sure modulus of continuity for $X(t, \cdot)$

$$\zeta(r) = \log^{-(\beta-1/2)}(1/r);$$

it is interesting to see here that the increase in regularity between time and space moduli of continuity is not proportional to the space regularity; it is always equal to the constant factor $\log^{1/2}(1/r)$ for all β.

3 Regularity of the solution as a function-valued process

Let $\mathcal{B}$ be the Banach space of Hölder continuous functions on S^1 with parameter $K < H$, endowed with the norm

$$\|f\| = \sup_{x \in S^1} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^K}.$$

Now considering (10) as a $\mathcal{B}$-valued process on $[0, 1]$:

$$[0,1] \ni t \to X(t, \cdot) \in \mathcal{B},$$

we will deduce the continuity of X in the norm of $\mathcal{B}$,

$$\begin{aligned}\|X(t_2, \cdot) - X(t_1, \cdot)\| &= \sup_{x \in S^1} |X(t_2, x) - X(t_1, x)| \\ &+ \sup_{x \neq y} \frac{|X(t_2, x) - X(t_1, x) - X(t_2, y) + X(t_1, y)|}{|x-y|^K}.\end{aligned}$$

To estimate the second supremum, let us write

$$\begin{aligned}&X(t_2, x) - X(t_1, x) - X(t_2, y) + X(t_1, y) \\ &= (X(t_2, x) - X(t_2, y)) - (X(t_1, x) - X(t_1, y)) \\ &= \sum_{n=1}^{\infty} \sqrt{q_n} (\cos nx - \cos ny) \left\{ \int_0^{t_2} e^{-n^2(t_2-s)} \beta_n^H(ds) - \int_0^{t_1} e^{-n^2(t_1-s)} \beta_n^H(ds) \right\} \\ &+ \sum_{n=1}^{\infty} \sqrt{q_n} (\sin nx - \sin ny) \left\{ \int_0^{t_2} e^{-n^2(t_2-s)} \widetilde{\beta}_n^H(ds) - \int_0^{t_1} e^{-n^2(t_1-s)} \widetilde{\beta}_n^H(ds) \right\}.\end{aligned}$$

Then, we can obtain the following estimation, uniformly in space:

$$\frac{|X(t_2,x)-X(t_1,x)-X(t_2,y)+X(t_1,y)|}{|x-y|^K}$$
$$\le \sum_{n=1}^{\infty}\sqrt{q_n}\frac{|\cos nx-\cos ny|}{|x-y|^K}\left|\int_0^{t_2}e^{-n^2(t_2-s)}\beta_n^H(ds)-\int_0^{t_1}e^{-n^2(t_1-s)}\beta_n^H(ds)\right|$$
$$+\sum_{n=1}^{\infty}\sqrt{q_n}\frac{|\sin nx-\sin ny|}{|x-y|^K}\left|\int_0^{t_2}e^{-n^2(t_2-s)}\widetilde{\beta}_n^H(ds)-\int_0^{t_1}e^{-n^2(t_1-s)}\widetilde{\beta}_n^H(ds)\right|$$
$$\le \sum_{n=1}^{\infty}\sqrt{q_n}Cn^K\left\{\left|\int_0^{t_2}e^{-n^2(t_2-s)}\beta_n^H(ds)-\int_0^{t_1}e^{-n^2(t_1-s)}\beta_n^H(ds)\right|\right.$$
$$\left.+\left|\int_0^{t_2}e^{-n^2(t_2-s)}\widetilde{\beta}_n^H(ds)-\int_0^{t_1}e^{-n^2(t_1-s)}\widetilde{\beta}_n^H(ds)\right|\right\}$$

where $C>0$ depends only on K. We made use of the fact that for arbitrary $\gamma\in[0,1]$ there exists $c_\gamma>0$ such that

$$|\sin x-\sin y|\le c_\gamma|x-y|^\gamma,$$

for all $x,y\ge 0$ (which holds for cosine as well).

For the first supremum, note that

$$X(t_2,x)-X(t_1,x)=\sqrt{q_0}\left(\beta_0^H(t_2)-\beta_0^H(t_1)\right)$$
$$+\sum_{n=1}^{\infty}\sqrt{q_n}\cos nx\left\{\int_0^{t_2}e^{-n^2(t_2-s)}\beta_n^H(ds)-\int_0^{t_1}e^{-n^2(t_1-s)}\beta_n^H(ds)\right\}$$
$$+\sum_{n=1}^{\infty}\sqrt{q_n}\sin nx\left\{\int_0^{t_2}e^{-n^2(t_2-s)}\widetilde{\beta}_n^H(ds)-\int_0^{t_1}e^{-n^2(t_1-s)}\widetilde{\beta}_n^H(ds)\right\}.$$

Therefore,

$$\|X(t_2,\cdot)-X(t_1,\cdot)\|\le\sqrt{q_0}\left|\beta_0^H(t_2)-\beta_0^H(t_1)\right|$$
$$+\sum_{n=1}^{\infty}\sqrt{q_n}(1+Cn^K)\left\{\left|\int_0^{t_2}e^{-n^2(t_2-s)}\beta_n^H(ds)-\int_0^{t_1}e^{-n^2(t_1-s)}\beta_n^H(ds)\right|\right.$$
$$\left.+\left|\int_0^{t_2}e^{-n^2(t_2-s)}\widetilde{\beta}_n^H(ds)-\int_0^{t_1}e^{-n^2(t_1-s)}\widetilde{\beta}_n^H(ds)\right|\right\}$$

and

$$\mathbb{E}\,\|X(t_2,\cdot)-X(t_1,\cdot)\|^2\le q_0|t_2-t_1|^{2H}$$
$$+2\sum_{n=1}^{\infty}q_n\left(1+Cn^K\right)^2\mathbb{E}\left\{\int_0^{t_2}e^{-n^2(t_2-s)}\beta_n^H(ds)-\int_0^{t_1}e^{-n^2(t_1-s)}\beta_n^H(ds)\right\}^2.$$

Now using the estimates in the proof of Theorem 1, the next result follows immediately.

Theorem 7 *Let $H > \frac{1}{2}$ and $L, K \leq H$ be such that*

$$\sum_{n=1}^{\infty} \frac{q_n}{n^{4H-4L-2K}} < \infty.$$

Then X is a.s. β-Hölder continuous as a $\mathcal{B}$-valued process $\forall \beta \in (0, L)$.

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