# A binomial tree approach to stochastic volatility driven model of the stock price

**Article** *in* Annals of the University of Craiova · January 2005



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Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 32, 2005, Pages 126–142
ISSN: 1223-6934

# A binomial tree approach to stochastic volatility driven model of the stock price

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Abstract. In this article we attempt to deal with the problem of finding option prices when the volatility component of the price is stochastic. The model we use is: $dS_t = \mu S_t dt + \sigma(Y_t)S_t dW_t$, where $Y_t$ is a mean-reverting type process. First, we show how to estimate the distribution of the volatility component, using an algorithm due to Del Moral, Jacod and Protter [6]. Second, using this distributon we are able to construct a binomial tree model which converges to the solution of the given equation. In order to price options on the stock, we use the Monte Carlo method to sample from this tree, and obtain a smaller, recombing tree easier to work with. Finally, we use this method to compute the price of European Call Options on the SP500 index price in April. We use daily data and our method gives good results that are proximate to the reported bid-ask spread.

*2000 Mathematics Subject Classification.* Primary: 90A09; Secondary: 60G35, 65C.
*Key words and phrases.* stochastic volatility, option valuation, binomial model, stochastic filtering.

## 1. Introduction

We should not start an article about option valuation without quoting the most celebrated article in the domain [2]. Despite significant development in the option pricing theory, the Black-Scholes formula for an European call option remains the most widely used application in Finance.

Nevertheless the above quoted formula has significant biases [16]. Its failure to describe the structure of reported option prices is thought to arise from its constant volatility assumption. But if the volatility is allowed to have a random component it becomes stochastic. However, the process of accounting for stochastic volatility within an option valuation formula is not an easy task. Hull and White[12], Chesney and Scott [3], Stein and Stein [18], Heston [10], all have constructed various specific stochastic volatility models. There are no simple formulas for the price of options on stocks driven by such models. When some means of implicit or explicit equations are found, the relations involved are cumbersome at best. Approximations have been constructed to these and other specific volatility models and we will quote here the works of Ritken and Trevor [15], and Hilliard and Schwartz [11].

When Binomial Tree approximation was developed by Sharpe [17], the option pricing model became accesible to a wider audience. Cox, Ross and Rubinstein [4] constructed a binomial model that converged weakly to the lognormal diffusion of Blak-Scholes, and they also showed that the limit of the computed option value was the same as the one given by Black-Scholes valuation. Later Cox and Rubinstein [5] used the same approach to value the American style options on dividend paying

*Received*: October 1, 2004.

stock, and they also relaxed some other assumptions of the Black and Scholes original model.

In trying to implement a real world behaviour of the stock price model, one eventually reaches the ideea of stochastic volatility as the way to do so. There have been attempts to find the option price analytically most notably in this respect see the book by Fouque, Papanicolau, and Sircar [9]. In general though, if one hopes to find any concrete results, one has to retort to numerical methods to solve this problem. This is the path we follow in this article.

We assume that the price process $S_t$ and the volatility driving process $Y_t$ solve the equations:

$$\begin{cases} dS_t &= \mu S_t dt + \sigma(Y_t) S_t dW_t \\ dY_t &= \alpha(\nu - Y_t)dt + \psi(Y_t)dZ_t \end{cases} \tag{1}$$

This model spans all the stochastic volatility models considered previously for different specifications of the functions $\sigma(x)$ and $\psi(x)$. Here $W_t$ and $Z_t$ are two independent Brownian Motions. The case when they are correlated is an extension of our model, but we wil not treat it here. We chose a mean-reverting type process to drive the volatility because this seems to be the most reasonable choice from the practical point of view.

When trying to implement a binomial tree algorithm to price an option on a stock driven by this kind of model, one is faced with 2 problems: Modelling the volatility component and Modelling the price itself. Modeling the volatility is a particularly hard problem because the volatility cannot be observed directly from the market, only the Stock price is going to be known.

This problem has been tackled before, most notably by Leisen [13] who uses the same model as we do. He uses a binomial tree for the volatility and a so-called 8 successors tree for the price. The ideea used is similar with the one applied by Nelson and Ramaswamy [14] for the case when the volatility is deterministic. However, that ideea fails from a theoretical point of view when applied to Leisen's case, since the transfomation used to eliminate the volatility does not work with stochastic volatility. Another article that may be interesting is [1] where the authors use a Markov Chain for the volatility process, but their price tree is not recombining despite what is claimed in the article.

Our method for estimating the volatility distribution uses an algorithm introduced by Del Moral, Jacod and Protter [6] - an ideea taken from genomics. We will describe this ideea in Section 3. For the Price Process we construct a two dimensional tree (recombining in one direction) with all the possible (stock, volatility) pairs, and, by using the Monte Carlo method to sample from it, we get smaller trees. These elementary trees are somewhat recombining, and we use them to find the option price in Section 4. Section 5 contains numerical results obtained when applying our algorithm to SP500 option data.

## 2. The Model and theoretical results

We work under an equivalent martingale measure, and instead of the stock price we work directly with the logarithm of the price (the return). We denote $X_t = \log S_t$. Under this measure the system of equations (1) becomes:

$$\begin{cases} dX_t &= \left(r - \frac{\sigma^2(Y_t)}{2}\right) dt + \sigma(Y_t) dW_t \\ dY_t &= \alpha(\nu - Y_t)dt + \psi(Y_t)dZ_t \end{cases} \tag{2}$$

Here of course we used the same notations $W_t$ and $Z_t$ for the corresponding Brownian Motions under the equivalent martingale measure obtained by applying the Girsanov's theorem. We would like to obtain discrete versions of these two processes so that they converge in distribution to the continuous processes (2). Using the fact that $e^x$ is a continuous function, and that the price of the European Option can be written as a conditional expection of a continuous function of the price, this is enough for the convergence of the option price found using our discrete approximation to the real price of the option.

To achieve this goal, we construct a Markov Chain, and using the theory in chapter 11 of the book by Stroock and Varadhan [19] (more precisely the section 11.2) we show the convergence in distribution of this Markov Chain to the solution of the Diffusion Equation (2). The same theory can also be found in the book by Ethier and Kurtz [8], though in a slightly less general form.

In our case, everything is one dimensional, and the Markov Chain is time homogenous and this fact allows us to apply the theory without any modification.

Let $T$ be the maturity date of the option we are trying to price and $n$ the number of steps in our binomial tree. Let us denote the time increment by $\Delta t = \frac{T}{n} = h$.

Further, we assume that the martigale problem associated with the diffusion process $X_t$ in (2) has a unique solution starting from $x = \log S_k$, the last data point available. This is equivalent with saying that the equation (2) has a unique solution in the weak sense. In the next section we deal with the convergence issue of the approximating process $Y_t^n$ to $Y_t$.

Let us start with a discrete Markov Chain $(x(ih), \mathcal{F}_{ih})$ with transition probabilities denoted $p_x^z$ of jumping from the point $x$ to the point $z$. These transition probabilities also depend on $h$, but for simplicity of notation we skip that subscript. For each $h$ let $P_x^h$ be the probability measure on $\mathbb{R}$ characterized by:

$$\begin{cases} (i) & P_x^h\left(x(0) = x\right) = 1 \\ (ii) & P_x^h\Big(x(t) = \frac{(i+1)h-t}{h}\, x(ih) + \frac{t-ih}{h}\, x((i+1)h) \\ & \qquad\qquad , \quad ih \le t < (i+1)h\Big) = 1, \quad \forall\, i \ge 0 \\ (iii) & P_x^h\left(x((i+1)h) = z | \mathcal{F}_{ih}\right) = p_x^z, \quad \forall\, z \in \mathbb{R} \text{ and } \forall\, i \ge 0 \end{cases} \tag{3}$$

**Remark 2.1.**

(1) It is easy to see that (i) and (iii) say that $(x(ih), \mathcal{F}_{ih})$, $i \ge 0$ is time-homogenous Markov Chain starting at $x$ with transition probability $p_x^z$ under the probability measure $P_x^h$.
(2) Condition (ii) assures us that the process $x(t)$ is linear between $x(ih)$ and $x((i+1)h)$. In turn, this will later guarantee that the process $x(t)$ we construct is a tree.

Conditional on being at $x$ and on the $Y_t$ distribution, we construct the following quantities:

$$b_h(x) = \frac{1}{h}\,\mathbb{E}^Y\left[\sum_{\text{z successor of x}} p_x^z(z - x)\right]$$

$$a_h(x) = \frac{1}{h}\,\mathbb{E}^Y\left[\sum_{\text{z successor of x}} p_x^z (z-x)^2\right]$$

Here the successor $z$ is determined using both the predecessor $x$ and the $Y^n$ process.

Similarly we define the following quantities corresponding to the infinitesimal generator of the equation (2):

$$b(x) = \mathbb{E}^Y\left[r - \frac{\sigma^2(Y)}{2}\right]$$

$$a(x) = \mathbb{E}^Y\left[\sigma^2(Y)\right]$$

We make the following assumptions: for any $R > 0$,

$$\lim_{h\searrow 0}\ \sup_{\{|x|,|Y| \text{ and } |Y^n| \le R\}} |b_h(x) - b(x)| = 0 \tag{4}$$

$$\lim_{h\searrow 0}\ \sup_{\{|x|,|Y| \text{ and } |Y^n| \le R\}} |a_h(x) - a(x)| = 0 \tag{5}$$

$$\lim_{h\searrow 0}\ \max_{z \text{ successor of } x} |z - x| = 0 \tag{6}$$

**Theorem 2.1.** *Assume that the martigale problem associated with the diffusion process $X_t$ in (2) has a unique solution $P_x$ starting from $x = \log S_k$ and that the functions $a(x,y)$ and $b(x,y)$ are continuous. Then conditions (4), (5) and (6) are sufficient to guarantee that $P_x^h$ as defined in (3) converges to $P_x$ as $h \searrow 0$ uniformly on compacts in $\mathbb{R}$. Or equivalently saying: $x(ih)$ converges in distribution to $X_t$ the unique solution of the equation (2)*

This is the Theorem 11.2.3 in [19] adapted to our particular problem. We must also note that assumption (6), though implying condition (2.6) in the above cited book is a much stronger assumption. However, since it is available to us we shall make use of it.

**Remark 2.2.** *Since in our case the functions $a$ and $b$ do not depend on time, and $a$ is stricly positive definite valued we can relax the assumptions (4) and (5) to a weaker kind of convergence. We direct the reader to Theorem 11.4.2 which details exacly this case.*

## 3. Estimating the Volatility

In this section we describe the method used to find the transition distribution of the volatility component.

We assume that the coefficients $\nu$, $\alpha$ and the functions $\sigma(y)$ and $\psi(y)$ are known or have been already estimated. We are using an algorithm due to [6] adapted to our specific case. The ideea of the algorithm comes from genomics, where it has been used under the name: "The Mutation-Selection Algorithm". We adapt the algorithm for our case, and we rename it Evolution-Selection. For a more general view including proof of the convergence we refer the reader to the above cited article.

The data we work with is a sequence of returns: $\{x_0 = \log S_0, x_1 = \log S_1, \dots, x_k = \log S_k\}$ read from the market. We need an initial distribution for the volatility process $Y_t$. For practical purposes, we use $\delta_{\{\nu\}}$ for this distribution. Here $\delta_{\{x\}}$ is the Dirac Delta function. The only condition we need is that the functions $\sigma(x)$ and $\psi(x)$ have to be twice differentiable with bounded derivatives of all orders up to 2.

Let us define the function:

$$\phi(x) = \begin{cases} 1 - |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

In fact, any probability distribution with finite mean can be used for the function $\phi(x)$. For $n > 0$ we define the contraction coresponding to $\phi(x)$ as:

$$\phi_n(x) = \sqrt[3]{n}\,\phi(x\sqrt[3]{n}) = \begin{cases} \sqrt[3]{n}\,(1 - |x\sqrt[3]{n}|) & \text{if } -\frac{1}{\sqrt[3]{n}} < x < \frac{1}{\sqrt[3]{n}} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

First we choose $m = m_n$ an integer.
Step 1: We start with $X_0 = x_0$ and $Y_0 = y_0 = \nu$

**Evolution step:** This part approximates a random variable with the same distribution as $(X_1, Y_1)$ using the well known Euler scheme for the equation (2). More precisely we set:

$$\begin{aligned} Y(m, y_0)_{i+1} &\stackrel{\text{def}}{=} Y_{i+1} = Y_i + \frac{1}{m}\,\alpha(\nu - Y_i) + \frac{1}{\sqrt{m}}\psi(Y_i)U_i \\ X(m, x_0)_{i+1} &\stackrel{\text{def}}{=} X_{i+1} = X_i + \frac{1}{m}(r - \frac{\sigma^2(Y_i)}{2}) + \frac{1}{\sqrt{m}}\sigma(Y_i)U'_i \end{aligned} \tag{8}$$

Here $U_i$ and $U'_i$ are iid Normal random variates with mean 0 and variance 1.
At the end of the first evolution step we obtain:

$$\begin{aligned} X_1 &= X(m, x_0)_m \\ Y_1 &= Y(m, y_0)_m \end{aligned} \tag{9}$$

**Selection step:** We repeat the evolution step n times to obtain n pairs: $\{(X_1^j, Y_1^j)\}_{j=\overline{1,n}}$
Now we introduce the discrete probability measure:

$$\Phi_1^n = \begin{cases} \frac{1}{C}\sum_{j=1}^n \phi_n(X_1^j - x_1)\delta_{\{Y_1^j\}} & \text{if } C > 0 \\ \delta_{\{0\}} & \text{otherwise} \end{cases} \tag{10}$$

Here the constant $C$ is choosen to make $\Phi_1^n$ a probability measure, i.e. $C = \sum_{j=1}^n \phi_n(X_1^j - x_1)$. Basically, the ideea is to "select" only the values of $Y_1$ which corespond to values of $X_1$ not far away from the realization $x_1$.

We conclude the first Selection step by simulating n iid variables $\{{Y'}_1^j\}_{j=\overline{1,n}}$.

Steps 2 to k: For each step $i = \overline{2,k}$, first we apply the evolution step to each of the variables selected at the end of the previous step, that is, starting with $X_0 = x_{i-1}$ and $Y_0 = {Y'}_{i-1}^j$ for each $j = \overline{1,n}$ in (8). Thus, we obtain n pairs $\{(X_i^j, Y_i^j)\}_{j=\overline{1,n}}$. Then we apply the selection step to these pairs. That is: we use them in the distribution (10) instead of the $\{(X_1^j, Y_1^j)\}_{j=\overline{1,n}}$ pairs and $x_i$ instead of $x_1$.

At the end of each step $i$ we obtain a discrete distribution $\Phi_i^n$, and this is our estimate for the transition probability of the process $Y_t$. In our construction of the binomial tree we use only the latest estimated transition probability, i.e., $\Phi_k^n$.

**Remark 3.1** (Results of the Approximation)**.** *With the choice $m_n = \sqrt[3]{n}$ the mean error of approximating $Y_i$ is of the order $O\left(\frac{1}{\sqrt[3]{n}}\right)$ for each $i = \overline{1,k}$.*

For exact estimates, including an estimate of the probability of deviation from the continuous distribution, the reader is advised to consult Theorem 5.1 in [6]

## 4. Constructing the tree

We assume that we have an option with maturity $T$. Our purpose in this section is to construct a discrete binomial tree which serves to put a price on the given option. The data we have is the stock price today $S$, together with a history of earlier stock prices which are used to compute the discrete transition probability of the approximating volatility process $Y^h$, as we did in the previous section. We assume that we have done that and we have obtained a discrete distribution $\{Y_1, Y_2, \ldots, Y_K\}$ with respective probabilities $\{\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_K\}$.

Let us divide the interval $[0,T]$ into $n$ subintervals each of length $\Delta t = \frac{T}{n} = h$. At each of the points $i\Delta t = ih$ the tree is branching. The knots on the tree represent possible return values $X_t = \log S_t$.

Now, assume that we are at a point $x$. What are the possible successors of $x$?

In the following we refer to the Figure 1 on page 131.

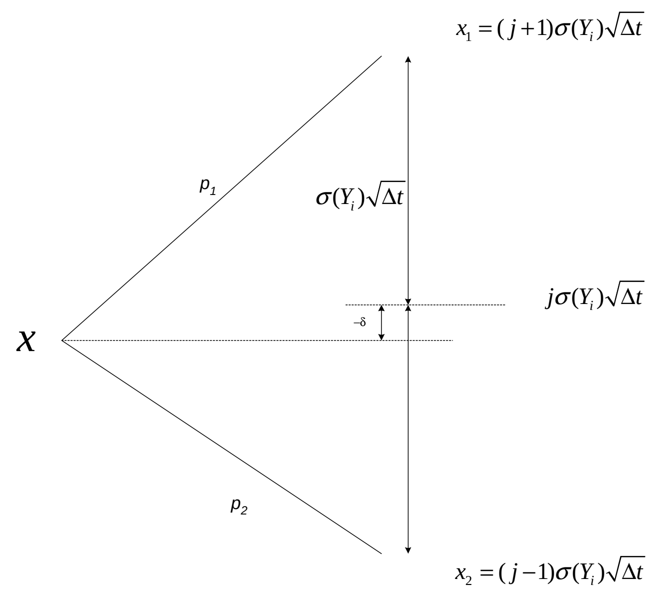



FIGURE 1. The basic successors for a given volatility value

For any choice of the volatility, we have two respective successors. In total we have $2K$ for any starting $x$. Although not specifically stated, we can think of each of the points of the tree as coresponding to a pair $(X_t, Y_t)$.

Let us see what are the successors for a specific value of the volatility process $Y_i$ with corresponding probability $\bar{p}_i$. We consider a grid of points of the form $l\,\sigma(Y_i)\sqrt{\Delta t}$

with $l$ taking integer values. In this grid let $j$ be the **odd** integer that corresponds to the point nearest to $x$. Mathematically $j$ is the point that attains:

$$\inf_{\{l\in\mathbb{N};l\equiv 1(mod2)\}}\left|x - l\,\sigma(Y_i)\sqrt{\Delta t}\right|$$

Let us also denote $\delta = x - j\,\sigma(Y_i)\sqrt{\Delta t}$. Note that $\delta$ is negative in the Figure 1. That is why we represented the distance by the number $-\delta$. The situation when $\delta$ is positive is going to give us the same results.

One of the assumptions we need to verify is the assumption (4), which asks that the mean of the increment needs to converge to the drift of the $X_t$ process in (2). In order to simplify this requirement we simply add the drift quantity to each of the successors. This trick modifies the assumption to ask now the convergence of the mean increment to zero. This clever ideea has been used by Leisen in his article as well as by Nelson & Ramaswamy.

Explicitly we take the 2 successors to be:

$$\begin{cases} x_1 &= (j+1)\sigma(Y_i)\sqrt{\Delta t} + \left(r - \frac{\sigma^2(Y_i)}{2}\right)\Delta t \\ x_2 &= (j-1)\sigma(Y_i)\sqrt{\Delta t} + \left(r - \frac{\sigma^2(Y_i)}{2}\right)\Delta t \end{cases} \tag{11}$$

First notice that condition (6) is trivially satisfied by this choice of successors. What we hope for is to match the mean condition (4) exacltly (i.e. to set $b_h(x,Y) = b(x,Y)$), and use this condition to find the joint probabilities $p_1$ and $p_2$. Then we must verify the assumption about the variance (5). If everything is accurate we will obtain our convergence result.

Algebrically we write: $j\,\sigma(Y_i)\sqrt{\Delta t} = x - \delta$, and using this we obtain that the increments over the perion $\Delta t$ are:

$$\begin{cases} x_1 - x &= \sigma(Y_i)\sqrt{\Delta t} - \delta + \left(r - \frac{\sigma^2(Y_i)}{2}\right)\Delta t \\ x_2 - x &= -\sigma(Y_i)\sqrt{\Delta t} - \delta + \left(r - \frac{\sigma^2(Y_i)}{2}\right)\Delta t \end{cases} \tag{12}$$

The condition (4) translates here as:

$$\mathbb{E}[\Delta x|Y_i] = \left(r - \frac{\sigma^2(Y_i)}{2}\right)\Delta t$$

, where by $\Delta x$ we denoted the increment over the period $\Delta t$.

We will solve the following system of equations with respect to $p_1$ and $p_2$:

$$\begin{cases} \left(\sigma(Y_i)\sqrt{\Delta t} - \delta\right)\frac{p_1}{p_1+p_2} + & \left(-\sigma(Y_i)\sqrt{\Delta t} - \delta\right)\frac{p_2}{p_1+p_2} = 0 \\ & p_1 + p_2 = \bar{p}_i \end{cases} \tag{13}$$

The first equation in the system becomes:

$$\sigma(Y_i)\sqrt{\Delta t}\,\frac{p_1 - p_2}{\bar{p}} - \delta = 0$$

or:

$$p_1 - p_2 = \frac{\bar{p}\,\delta}{\sigma(Y_i)\sqrt{\Delta t}} \tag{14}$$

And now it is very easy to see that (14) together with the second equation in (13) give the following solution:

$$\begin{cases} p_1 &= \frac{\bar{p}_i}{2}\left(1 + \frac{\delta}{\sigma(Y_i)\sqrt{\Delta t}}\right) \\ p_2 &= \frac{\bar{p}_i}{2}\left(1 - \frac{\delta}{\sigma(Y_i)\sqrt{\Delta t}}\right) \end{cases} \tag{15}$$

**Remark 4.1.** *By construction* $-\sigma(Y_i)\sqrt{\Delta t} < \delta < \sigma(Y_i)\sqrt{\Delta t}$

Using the Remark 4.1 it is easy to see that $p_1$ and $p_2$ are guaranteed to be numbers between 0 and 1.

**Remark 4.2.** *The assumptions* (4) *and* (5) *will be satisfied by our choice of* $x_1, x_2, p_1$ *and* $p_2$

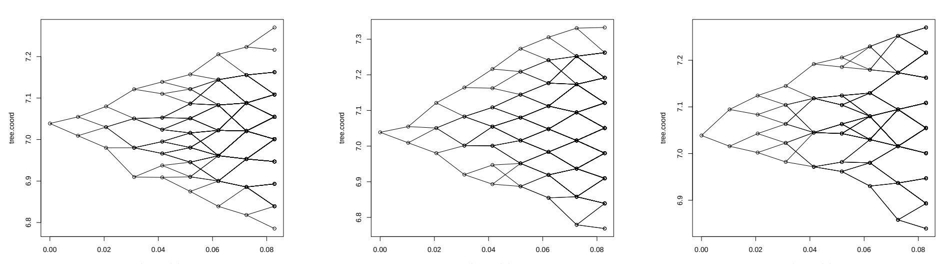
## 5. Using the Bootstrap Method to construct a manageable tree

One potential drawback of the method described in the previous section is that the constructed binomial tree is quite difficult to work with. For example assuming that the volatility has $k$ levels, every point on the tree has 2k successors. Even with our construction, which insures that inside the same volatility level the points on the tree are recombining this tree becomes quickly unmanageable.

In fact, it is very difficult to draw the tree even for small values of n.

Therefore to make this problem manageable we select an n sized bootstrap sample from the discrete volatility distribution, and construct a much smaller tree only for those volatility levels. More precisely, say $\{Y_1, Y_2, \ldots, Y_n\}$ is a sample drawn with replacement from the discrete distribution $Y$ - a bootstrap sample. We start with the initial value $x_0$. Then we compute the successors of $x_0$ according to the method in Section 4 but only for the volatility $Y_1$. Note that we do not have conditional distribution anymore, $Y_1$ behaves as if it were the volatility value. In other words, $\bar{p}_i$ is replaced by 1 in all the formulas in Section 4. After this, for each one of the 2 successors we compute their respective successors for the volatility equal with $Y_2$.

And we continue like this for every step from 1 to n until we construct a price tree. We can see some examples of the " bootstrap trees" in Figure 2 on page 133. They are $n = 8$ steps trees.




(a) Tree 1 (b) Tree 2 (c) Tree 3

FIGURE 2. Example of trees

Once we have the price tree constructed we compute the option value at the terminal nodes then work backwards on the tree to find the value at the first node on the tree as the expected value of the terminal nodes as in the usual binomial tree method.

Consequently our estimated option price is the average of the computed values for each bootstrap sample.

The convergence of this estimated price to the option price value given by the complete tree is assured by the general Bootstrap theory. For example, see [7].

**Remark 5.1.** *¿From the above cited book a good choice for the number of bootstrap samples is between 20 and 200.*

## 6. Real world example: Working with SP500 option data

In this section we present results obtained using Standard & Poor 500 companies stock index data from April 2004. We are using daily data from Jan 1st 1999 to April 21 2004 to compute the discrete volatility distribution according to the method described in Section 3. Figure 3 represents the evolution of the S&P500 stock price over the time period mentioned above.

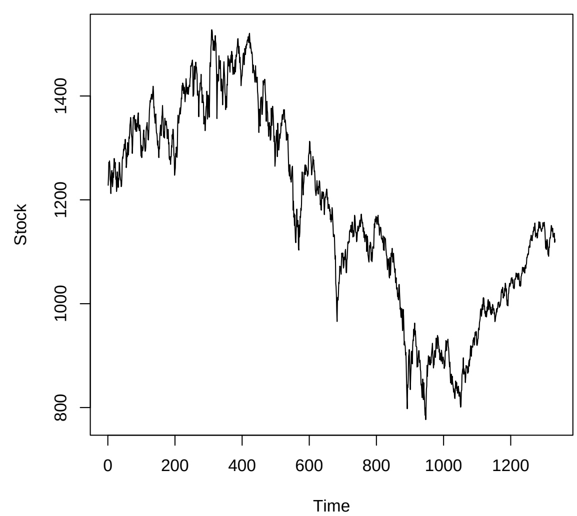


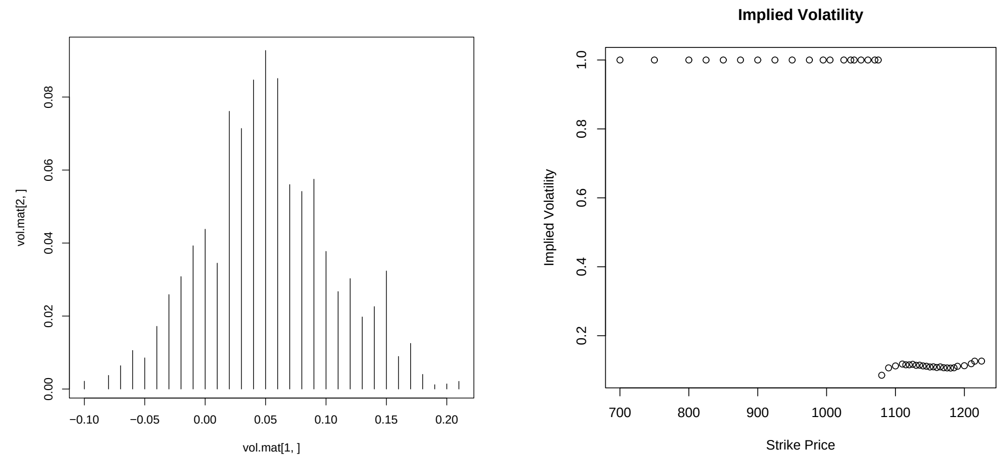

Figure 3. The S&P500 stock price over time

We are working with the Hull-White model in (2) with $\phi(y) = \alpha y$ and $\sigma(y) = \sqrt{|y|}$, using as parameters for the volatility equation $\alpha = .1$, $m = .12$ and $r = .05$ for the price. The parameters have been estimated from the data. However, a discussion about the method used is beyond the scope of the present article.

We estimate the discrete volatility distribution using the Del Moral& all. method presented in Section 3. A plot of this distribution can bee seen in Figure 4(a).

To compare our method we also estimate the implied volatility on April 21 for a range of strike prices from the option data available that day. We use a simple bisection method to do so. Figure 4(b) shows the implied volatility's behaviour for various strike prices.

Using the stock price for the next day, April 22, we compute our estimates of the Call option prices for that day. Tables 1 and 2 present our results. Each row in the




(a) Estimated discrete Volatility Distribution (b) Implied Volatility

FIGURE 4. Estimates from historical data

table corresponds to a price for a specific strike value. Columns 3 and 4 show the bid-ask spread read from the market for each corresponding strike price.

Columns 5 and 6 present the estimated price computed with the Cox-Ross-Rubinstein binomial tree respectively with the Black-scholes formula for a fixed volatility value of 0.12 which is the median for the implied volatility values.

Columns 7 and 8 present the estimated price computed with the same methods but using the previous day volatility from the second column.

Finally the last column presents the estimated price obtained with our method.

Remarkably, our method calculates option prices that are much closer to the bid-ask interval than the other traditional methods. Sometimes our estimated price falls inside the interval.

To illustrate better how the various methods perform we separate the options in groups depending on the range of the strike prices (out of the money, at the money, and in the money) and plotted the estimated prices given by the various methods.

We can see the estimated prices for the 29 day Maturity date in the Figures 5, 6 and 7 and

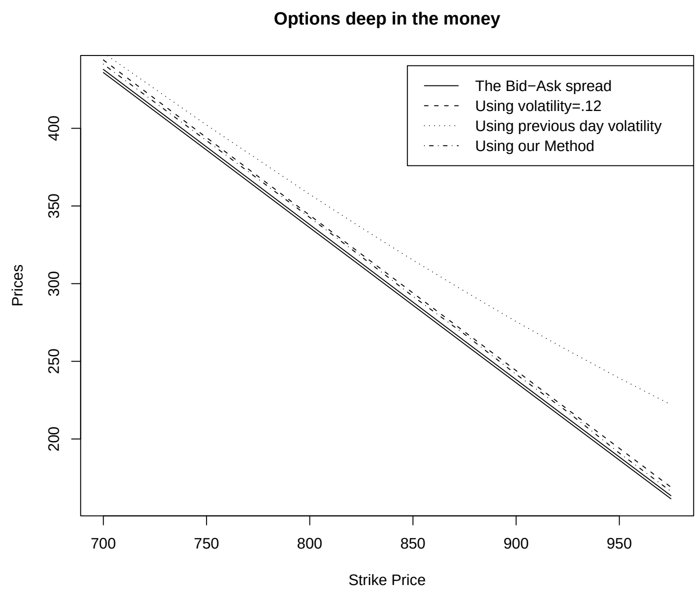



Figure 5. Estimated 29 day option prices: Deep in the money

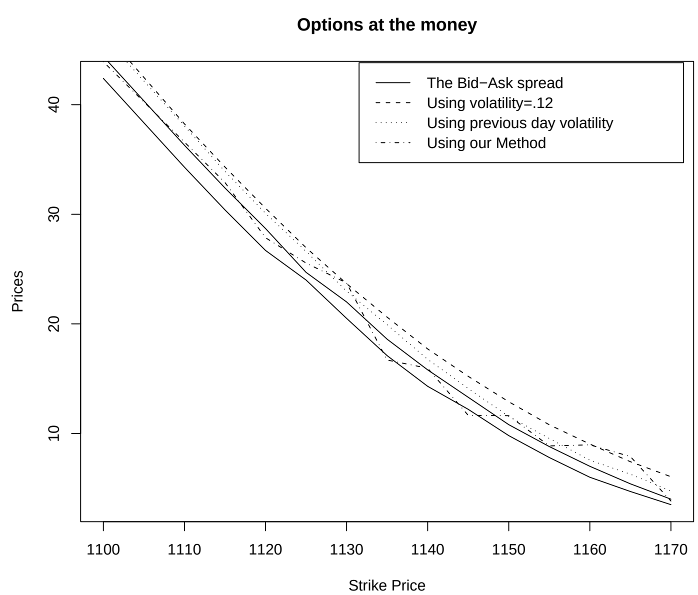



Figure 6. Estimated 29 day option prices: At the money

Table 1. Results for 29 day SP500 Call Option on April 22

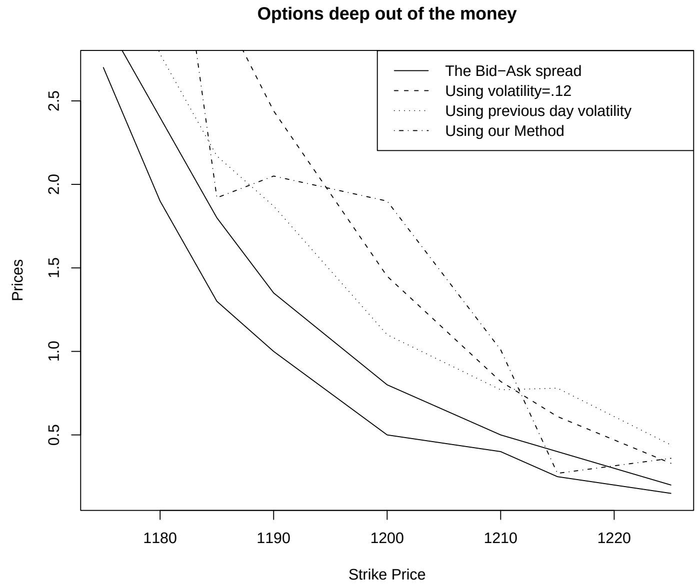



FIGURE 7. Estimated 29 day option prices: Deep out of the money

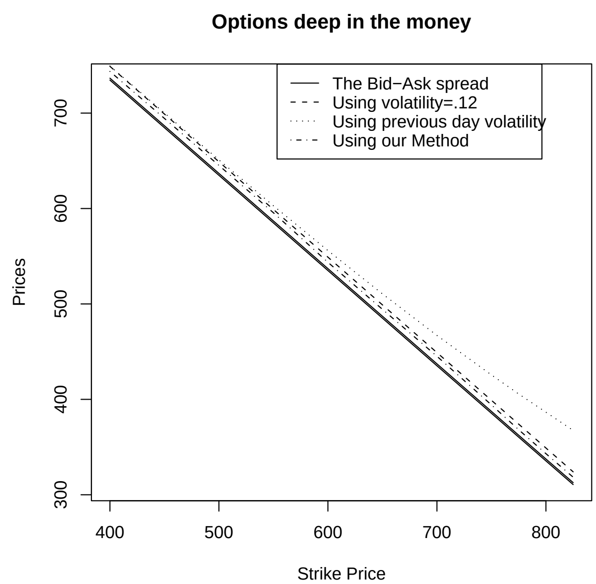



FIGURE 8. Estimated 58 day option prices: Deep in the money

| Strike Price | Implied Volatility | Bid-Ask Spread | | Binomial tree and Black Scholes formula vol = 0.12 | | Binomial tree and Black Scholes formula vol = prev. day | | Our Method |
|---|---|---|---|---|---|---|---|---|
| 700 | 0.99999994 | 435.9 | 437.9 | 444.00 | 443.00 | 448.57 | 446.80 | 441.27 |
| | | | | | | | | *continued on next page* |

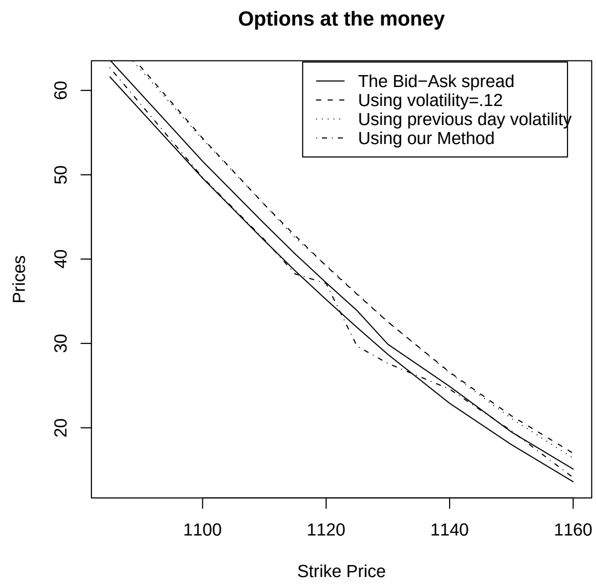



Figure 9. Estimated 58 day option prices: At the money

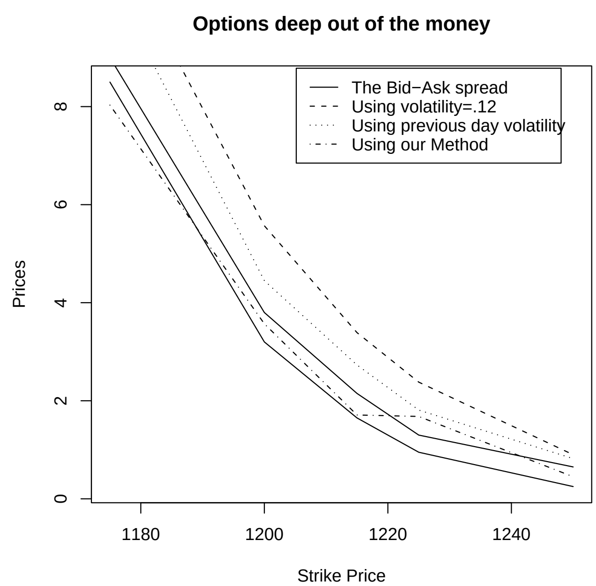



Figure 10. Estimated 58 day option prices: Deep out of the money

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| Strike Price | Implied Volatility | Bid-Ask Spread | | Binomial tree and Black Scholes formula vol = 0.12 | | Binomial tree and Black Scholes formula vol = prev. day | | Our Method |
|---|---|---|---|---|---|---|---|---|
| 750 | 0.99999994 | 386 | 388 | 394.00 | 393.00 | 402.09 | 400.48 | 392.07 |

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| Strike Price | Implied Volatility | Bid-Ask Spread | | Binomial tree and Black Scholes formula vol = 0.12 | | Binomial tree and Black Scholes formula vol = prev. day | | Our Method |
|---|---|---|---|---|---|---|---|---|
| 800 | 0.99999994 | 336 | 338 | 344.00 | 343.00 | 357.46 | 356.03 | 342.63 |
| 825 | 0.99999994 | 311.1 | 313.1 | 319.00 | 318.00 | 336.01 | 334.66 | 317.36 |
| 850 | 0.99999994 | 286.1 | 288.1 | 294.00 | 293.00 | 315.14 | 313.91 | 291.85 |
| 875 | 0.99999994 | 261.1 | 263.1 | 269.00 | 268.00 | 295.10 | 293.85 | 267.45 |
| 900 | 0.99999994 | 236.2 | 238.2 | 244.00 | 243.00 | 275.59 | 274.51 | 241.61 |
| 925 | 0.99999994 | 211.3 | 213.3 | 219.00 | 219.00 | 257.05 | 255.93 | 217.40 |
| 950 | 0.99999994 | 186.4 | 188.4 | 194.00 | 194.00 | 239.14 | 238.13 | 190.94 |
| 975 | 0.99999994 | 161.5 | 163.5 | 169.00 | 169.00 | 222.11 | 221.15 | 166.28 |
| 995 | 0.99999994 | 141.7 | 143.7 | 149.47 | 148.88 | 209.21 | 208.16 | 146.93 |
| 1005 | 0.99999994 | 131.9 | 133.9 | 139.47 | 138.92 | 202.76 | 201.86 | 137.48 |
| 1025 | 0.99999994 | 112.2 | 114.2 | 119.47 | 119.00 | 190.36 | 189.67 | 117.21 |
| 1035 | 0.99999994 | 102.5 | 104.5 | 109.48 | 109.05 | 184.67 | 183.77 | 105.55 |
| 1040 | 0.99999994 | 97.6 | 99.6 | 104.49 | 104.08 | 181.83 | 180.87 | 102.73 |
| 1050 | 0.99999994 | 88 | 90 | 94.53 | 94.16 | 176.15 | 175.17 | 91.56 |
| 1060 | 0.99999994 | 78.5 | 80.5 | 84.62 | 84.28 | 170.46 | 169.61 | 81.55 |
| 1070 | 0.99999994 | 69.1 | 71.1 | 74.79 | 74.50 | 164.78 | 164.17 | 72.06 |
| 1075 | 0.99999994 | 64.5 | 66.5 | 69.94 | 69.66 | 161.93 | 161.50 | 66.45 |
| 1080 | 0.08534008 | 59.9 | 61.9 | 65.12 | 64.87 | 64.54 | 64.28 | 62.03 |
| 1090 | 0.10675174 | 51 | 53 | 55.74 | 55.51 | 55.21 | 54.99 | 53.65 |
| 1100 | 0.11239082 | 42.4 | 44.4 | 46.73 | 46.54 | 46.27 | 46.08 | 43.92 |
| 1110 | 0.11785072 | 34.3 | 36.3 | 38.25 | 38.12 | 38.06 | 37.94 | 36.59 |
| 1115 | 0.11562651 | 30.4 | 32.4 | 34.31 | 34.16 | 33.91 | 33.76 | 32.90 |
| 1120 | 0.11560673 | 26.7 | 28.7 | 30.55 | 30.41 | 30.09 | 29.96 | 27.86 |
| 1125 | 0.11706859 | 24 | 24.7 | 26.95 | 26.87 | 26.61 | 26.54 | 25.55 |
| 1130 | 0.11411554 | 20.5 | 22 | 23.67 | 23.56 | 22.98 | 22.86 | 23.73 |
| 1135 | 0.11454815 | 17.1 | 18.6 | 20.61 | 20.49 | 19.92 | 19.82 | 16.70 |
| 1140 | 0.11247212 | 14.3 | 15.8 | 17.71 | 17.68 | 16.76 | 16.72 | 15.95 |
| 1145 | 0.11145037 | 12.2 | 13.3 | 15.21 | 15.12 | 14.10 | 14.03 | 11.65 |
| 1150 | 0.10943896 | 9.8 | 10.8 | 12.89 | 12.82 | 11.55 | 11.48 | 11.62 |
| 1155 | 0.10980803 | 7.8 | 8.8 | 10.79 | 10.77 | 9.54 | 9.51 | 8.86 |
| 1160 | 0.10793394 | 6 | 7 | 9.03 | 8.96 | 7.56 | 7.54 | 8.96 |
| 1165 | 0.10959452 | 4.7 | 5.4 | 7.42 | 7.39 | 6.28 | 6.23 | 7.89 |
| 1170 | 0.10715657 | 3.5 | 4 | 6.05 | 6.03 | 4.75 | 4.72 | 3.83 |
| 1175 | 0.1066758 | 2.7 | 3 | 4.92 | 4.88 | 3.63 | 3.65 | 3.34 |
| 1180 | 0.10621303 | 1.9 | 2.4 | 3.90 | 3.90 | 2.78 | 2.77 | 4.34 |
| 1185 | 0.10705012 | 1.3 | 1.8 | 3.11 | 3.09 | 2.17 | 2.16 | 1.92 |
| 1190 | 0.11137563 | 1 | 1.35 | 2.44 | 2.42 | 1.87 | 1.87 | 2.05 |
| 1200 | 0.11297244 | 0.5 | 0.8 | 1.45 | 1.44 | 1.10 | 1.12 | 1.90 |
| 1210 | 0.11875373 | 0.4 | 0.5 | 0.82 | 0.83 | 0.77 | 0.78 | 1.01 |
| 1215 | 0.12618941 | 0.25 | 0.4 | 0.61 | 0.61 | 0.78 | 0.80 | 0.27 |

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|---|---|---|---|---|---|---|---|---|
| Strike Price | Implied Volatility | Bid-Ask Spread | | Binomial tree and Black Scholes formula vol = 0.12 | | Binomial tree and Black Scholes formula vol = prev. day | | Our Method |
| 1225 | 0.12638921 | 0.15 | 0.2 | 0.33 | 0.33 | 0.44 | 0.45 | 0.36 |

Table 2. Results for 58 day SP500 Call Option on April 22

| Strike Price | Implied Volatility | Bid-Ask Spread | | Binomial tree and Black Scholes formula vol = 0.12 | | Binomial tree and Black Scholes formula vol = prev. day | | Our Method |
|---|---|---|---|---|---|---|---|---|
| 400 | 0.9999999 | 734.6 | 736.6 | 749.02 | 743.10 | 749.34 | 743.43 | 743.66 |
| 500 | 0.9999999 | 634.8 | 636.8 | 649.02 | 643.89 | 650.99 | 645.86 | 645.20 |
| 550 | 0.9999999 | 584.9 | 586.9 | 599.02 | 594.28 | 602.90 | 598.19 | 595.56 |
| 600 | 0.9999999 | 535 | 537 | 549.02 | 544.68 | 555.95 | 551.68 | 543.62 |
| 650 | 0.9999999 | 485.1 | 487.1 | 499.02 | 495.07 | 510.54 | 506.66 | 494.37 |
| 700 | 0.9999999 | 435.2 | 437.2 | 449.02 | 445.47 | 467.00 | 463.42 | 445.19 |
| 750 | 0.9999999 | 385.4 | 387.4 | 399.02 | 395.87 | 425.62 | 422.24 | 394.21 |
| 775 | 0.9999999 | 360.4 | 362.4 | 374.02 | 371.06 | 405.65 | 402.48 | 368.18 |
| 800 | 0.9999999 | 335.5 | 337.5 | 349.02 | 346.26 | 386.54 | 383.31 | 343.11 |
| 825 | 0.9999999 | 310.7 | 312.7 | 324.02 | 321.46 | 367.44 | 364.75 | 318.41 |
| 850 | 0.9999999 | 285.8 | 287.8 | 299.02 | 296.66 | 349.73 | 346.80 | 293.97 |
| 875 | 0.9999999 | 261 | 263 | 274.02 | 271.85 | 332.27 | 329.47 | 267.41 |
| 900 | 0.9999999 | 236.2 | 238.2 | 249.02 | 247.05 | 315.00 | 312.78 | 244.15 |
| 925 | 0.9999999 | 211.6 | 213.6 | 224.02 | 222.25 | 299.36 | 296.71 | 218.77 |
| 950 | 0.9999999 | 187 | 189 | 199.02 | 197.45 | 283.72 | 281.28 | 195.00 |
| 975 | 0.9999999 | 162.7 | 164.7 | 174.03 | 172.65 | 268.33 | 266.48 | 169.60 |
| 995 | 0.9999999 | 143.4 | 145.4 | 154.04 | 152.82 | 257.37 | 255.09 | 149.16 |
| 1005 | 0.9999999 | 133.8 | 135.8 | 144.06 | 142.92 | 251.89 | 249.54 | 139.23 |
| 1025 | 0.9999999 | 114.9 | 116.9 | 124.16 | 123.19 | 240.94 | 238.74 | 117.54 |
| 1050 | 0.9999999 | 91.9 | 93.9 | 99.63 | 98.84 | 227.24 | 225.77 | 93.21 |
| 1075 | 0.1064617 | 70 | 72 | 76.00 | 75.40 | 75.19 | 74.61 | 70.69 |
| 1085 | 0.1138119 | 61.6 | 63.6 | 67.02 | 66.50 | 66.53 | 65.99 | 62.68 |
| 1100 | 0.1189136 | 49.6 | 51.6 | 54.33 | 53.88 | 54.20 | 53.75 | 49.69 |
| 1110 | 0.1200046 | 42.2 | 44.2 | 46.47 | 46.09 | 46.47 | 46.09 | 42.31 |
| 1115 | 0.1180641 | 38.6 | 40.6 | 42.77 | 42.40 | 42.49 | 42.12 | 38.24 |
| 1120 | 0.1204317 | 35.2 | 37.2 | 39.19 | 38.87 | 39.26 | 38.94 | 37.11 |
| 1125 | 0.1205274 | 31.9 | 33.9 | 35.83 | 35.51 | 35.92 | 35.59 | 29.68 |
| 1130 | 0.1201625 | 28.7 | 29.9 | 32.55 | 32.31 | 32.57 | 32.34 | 27.63 |
| 1140 | 0.1194584 | 22.9 | 24.9 | 26.59 | 26.44 | 26.50 | 26.34 | 24.58 |
| 1150 | 0.1183253 | 18 | 19.5 | 21.42 | 21.28 | 21.11 | 20.98 | 19.63 |
| 1160 | 0.1168272 | 13.6 | 15.1 | 16.98 | 16.84 | 16.39 | 16.27 | 14.09 |

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|---|---|---|---|---|---|---|---|---|
| Strike Price | Implied Volatility | Bid-Ask Spread | | Binomial tree and Black Scholes formula vol = 0.12 | | Binomial tree and Black Scholes formula vol = prev. day | | Our Method |
| 1170 | 0.1158288 | 10 | 11.5 | 13.22 | 13.10 | 12.49 | 12.39 | 13.65 |
| 1175 | 0.1139731 | 8.5 | 9 | 11.56 | 11.47 | 10.59 | 10.49 | 8.03 |
| 1200 | 0.1107671 | 3.2 | 3.8 | 5.57 | 5.52 | 4.45 | 4.42 | 3.57 |
| 1215 | 0.1131049 | 1.65 | 2.15 | 3.39 | 3.36 | 2.73 | 2.74 | 1.71 |
| 1225 | 0.1126395 | 0.95 | 1.3 | 2.38 | 2.36 | 1.81 | 1.83 | 1.68 |
| 1250 | 0.1181863 | 0.25 | 0.65 | 0.90 | 0.90 | 0.82 | 0.82 | 0.45 |

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