Relating the almost-sure Lyapunov exponent of a parabolic SPDE and its coefficients' spatial regularity

by

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Abstract

We derive a lower bound on the large-time exponential behavior of the solution to a stochastic parabolic partial differential equation on $R_+ \times R$ in the case of a spatially homogeneous Gaussian potential that is white-noise in time, and study the relation between the lower bound and the potential's spatial modulus of continuity.

Key words and phrases: parabolic stochastic partial differential equations, Feynman-Kac formula, Lyapunov exponent, Gaussian regularity.

1 Introduction

The goal of this paper is to give a lower bound on the large-time exponential rate of increase of the solution to the stochastic parabolic equation with linear multiplicative potential:

$$\frac{\partial u}{\partial t}(t,x) = \frac{\kappa}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + u(t,x)\frac{\partial}{\partial t}W(t,x) : x \in \mathbf{R};\ t \geq 0, \quad (1)$$
$$u(0,x) = u_0(x) : x \in \mathbf{R},$$

in the case where the diffusivity κ is a small positive constant, u_0 is a function bounded below by a positive constant, and W is a Gaussian random field on $\mathbf{R}_+ \times \mathbf{R}$ that is Brownian in t and homogeneous in x. This problem was first investigated in [4] in the setting of discrete space $x \in \mathbf{Z}^d$. It was found that for any x and for small enough κ, almost-surely, the exponential behavior $\limsup_{t\to\infty} t^{-1}\log u(t,x)$ is bounded above by $C(\log\log\kappa^{-1})/\log\kappa^{-1}$ while $\liminf_{t\to\infty} t^{-1}\log u(t,x)$ is bounded below by $c/\log\kappa^{-1}$, where the constants c and C depend only on the covariance of W. The upper bound was improved in [5] to match the lower bound, up to a constant. We have recently talked with M. Cranston [7], who appears to have established, in the case of space-time white noise in $\mathbf{Z}^d$, that upper and lower bounds can both be written as $C_u(1+\varepsilon(\kappa))/\log\kappa^{-1}$ where C_u is a universal constant, and $\lim_{\kappa\to 0}\varepsilon(\kappa) = 0$. The question of whether similar bounds exist for the continuous-space problem was partially solved in [6] in which it was shown that the upper bound $C/\log\kappa^{-1}$ holds.

In this paper we investigate to what extent lower bounds can be obtained in continuous space, showing that lower bounds depend in fact on the spatial regularity of W; more specifically, we define conditions on W ensuring that it is almost-surely spatially β-Hölder-continuous for all $\beta < \alpha/2$, but not for any $\beta > \alpha/2$, and establish the almost-sure lower bound

$$\frac{c\varepsilon(\kappa)^\alpha}{\log(1/\varepsilon(\kappa))} \leq \liminf_{t\to\infty} t^{-1}\log u(t,x)$$

for small κ and some constant $c > 0$, where $\varepsilon(\kappa)$ is the unique solution of the equation

$$\kappa = \frac{\varepsilon^{\alpha+2}}{(\log\varepsilon^{-1})^2}.$$

One can check that this lower bound is commensurate with[1])

$$\frac{\left(\kappa\log^2\kappa^{-1}\right)^{\alpha/(\alpha+2)}}{\log\kappa^{-1}} = \frac{\kappa^{\alpha/(\alpha+2)}}{\log^{(2-\alpha)/(2+\alpha)}\kappa^{-1}},$$

[1] We have $\kappa = \varepsilon^{\alpha+2}\left(\log\left(\varepsilon^{-1}\right)\right)^{-2}$. Or in other words $\varepsilon = \kappa^{1/(\alpha+2)}\left(\log\left(\varepsilon^{-1}\right)\right)^{2/(\alpha+2)}$. We wish to study $\lambda = \varepsilon^\alpha\left(\log\left(\varepsilon^{-1}\right)\right)^{-1}$. We immediately get $\lambda = \kappa^{\alpha/(\alpha+2)}\left(\log\left(\varepsilon^{-1}\right)\right)^{-(2-\alpha)/(2+\alpha)}$. It is now sufficient to find upper and lower bounds on $\log\left(\varepsilon^{-1}\right)$. Since $\log\left(\varepsilon^{-1}\right) = \frac{1}{\alpha+2}\log\left(\kappa^{-1}\right) - \log\log\left(\varepsilon^{-1}\right) + \log\frac{\alpha+2}{2}$, we get an upper bound $\log\left(\varepsilon^{-1}\right) \leq \frac{1}{\alpha+2}\log\left(\kappa^{-1}\right)$ as soon as ε is smallish ($< \exp -2/(2+\alpha)$). For a lower bound, since $\log\frac{\alpha+2}{2} > 0$ and since for $\varepsilon < 1$ we always have $\log\log\left(\varepsilon^{-1}\right) < \log\left(\varepsilon^{-1}\right)$, we get $\log\left(\varepsilon^{-1}\right) \geq \frac{1}{\alpha+2}\log\left(\kappa^{-1}\right) - \log\left(\varepsilon^{-1}\right)$ so that $\log\left(\varepsilon^{-1}\right) \geq \frac{1}{2(\alpha+2)}\log\left(\kappa^{-1}\right)$. This proves that $\lambda = \kappa^{\alpha/(\alpha+2)}\left(g(\alpha,\kappa)\log\left(\kappa^{-1}\right)\right)^{-(2-\alpha)/(2+\alpha)}$ where $g(\alpha,\kappa)$ is a function bounded by $1/(2(\alpha+2))$ and $1/(\alpha+2)$, and thus is in $[1/6; 1/2]$.

which is the familiar bound $1/\log\kappa^{-1}$ of [5], [6], [7], with an additional factor in the numerator which tends to zero with κ.

Why can one not expect, in general, to establish a lower bound as large as the known upper bound $1/\log\kappa^{-1}$? As an extreme case, take the simplest example of a spatially homogeneous potential: $W(t,x) = W(t)$ a function of t only. We then get a trivial Lyapunov exponent $\lim_{t\to\infty} t^{-1}\log u(t,x) = 0$ for all $\kappa \geq 0$. The lower bound $1/\log\kappa^{-1}$ obtained in discrete space represents the other extreme case, since there W is assumed to be space-time white noise in $\mathbf{R}_+ \times \mathbf{Z}^d$. Such a W has no functional equivalent in continuous space, and we may only expect to show such a large lower bound in continuous space for an extremely spatially irregular W.

The scale of spatially Hölder-continuous potentials which we use here can be expected to have Lyapunov exponents that fill continuously some of the range $[0, 1/\log\kappa^{-1}]$. In particular we note that the lower bound we prove herein reduces to $1/\log\kappa^{-1}$ for $\alpha = 0$, which can be interpreted in some sense as the case of spatially non-Hölder-continuous W. The other extremity of our Hölder scale is the non-constant Lipshitz case $\alpha = 2$. In this case our result shows that the exponential behavior is bounded below by $\sqrt{\kappa}$. [This case includes spatially smooth potentials; the simplest example is $W(t,x) = \cos(x)B_t + \sin(x)B'_t$ where B and B' are independent scalar Brownian motions.] Thus the case of null Lyapunov exponent cannot be understood in our Hölder scale, which indeed does not include $W = W(t)$.

The gap that our lower bound leaves between $\sqrt{\kappa}$ and zero can be explained by the fact we restrict our study to homogeneous potentials; we expect that any Lyapunov exponent between 0 and $\sqrt{\kappa}$ can be obtained with a smooth non-homogeneous W, by choosing W to be spatially degenerate in a subset of $\mathbf{R}^d$ of appropriate size. This will be the subject of another study. Our lower bound is asymptotically sharp in the sense that it reaches the known upper bound in the limit of non-Hölder potential, as measured in the Hölder scale. The question of how far we can improve the upper bound in [6] for a given Hölder exponent will be tackled in a forthcoming article by fully exploiting the assumption of Hölder continuity. A finer analysis, in a non-Hölder, logarithmic continuity scale, will be the subject of further study in another article.

Lastly, we mention other approaches to Lyapunov exponents for different stochastic PDEs than (1), which can be found in [3] and [2], and the recent work of [10] who study the same SPDE as (1) in continuous space and time, and calculate the almost-sure Lyapunov exponent, but with a time-independent potential $W(x)$.

2 Preliminaries

We use $x \in \mathbf{R}$ rather than $x \in \mathbf{R}^d, d > 1$, and an initial data $u_0 \equiv 1$. The extension to multidimension space can be obtained by methods similar to the ones used in [5], [6]. Working with an initial data with compact support should also yield the same lower bound, as long as $u_0(x)$ is bounded away from zero on a set of non-zero Lebesgue measure. Also note that the requirement that W is spatially homogeneous (i.e. that W and $W(\cdot, \cdot + h)$ have the same distribution) is only for convenience. The lower bound should still hold as long as W's almost-sure uniform modulus of continuity is bounded below by some Hölder modulus, which can be signified by requiring that $\mathbf{E}\left[(W(1,x)-W(1,x+h))^2\right] \geq h^{\alpha_0}$ for some $\alpha_0>0$ for small h, although the proof would involve several additional technical difficulties. For the sake of conciseness and readability, we have chosen not to discuss these improvements any further here.

Because $\partial W/\partial t$ is not a bonafide function, equation (1) must be understood in a stochastic integral sense, as follows:

$$u(t,x)=1+\int_0^t \frac{\kappa}{2}\Delta_x u(s,x)\,ds+\int_0^t W(ds,x)\,u(s,x): t\geq 0;\ x\in\mathbf{R}.$$

Here the stochastic integral is interpreted in the Stratonovich sense. Since we assume typically that $\partial W/\partial x$ does not exist, there is no reason to believe that this equation actually has a strong solution. In fact it does not, and we must solve the equation in a weaker sense. As in [6], and as is typical in the modern study of stochastic PDEs ([8], [9], [11], [12]), we choose the evolution sense:

$$u(t,x)=1+\int_{\mathbf{R}} dy\int_0^t p_{\kappa(t-s)}(x-y)\,W(ds,y)\,u(s,y): t\geq 0;\ x\in\mathbf{R}, \tag{2}$$

where $p_\tau(z)=(2\pi\tau)^{-1/2}\exp\left(-z^2/(2\tau)\right)$ is the standard heat kernel. Note finally that, because W is spatially homogeneous, the ratio between the solution to (2) in the Itô and Stratonovich senses is the trivial factor $\exp(-Q(0)\,t/2)$ where $Q(0)=var\,[W(1,x)]$.

2.1 Feynman-Kac formula

It is well-known that u is representable by a stochastic Feynman-Kac formula. A proof of this fact that is based on the evolution form of the stochastic PDE can be found in [14]. It assumes that W is almost-surely spatially Hölder-continuous.

Theorem 1 *Assume that W is given on some probability space $(\Omega,\mathcal{F},\mathbf{P})$ by its covariance structure $\mathbf{E}[W(t,x)\,W(s,y)]=\min(s,t)Q(x-y)$ where Q is a homogeneous covariance function such that $Q(0)<\infty$, and Q is Hölder-continuous in a neighborhood of 0. Then equation (2) has a unique solution in $L^2(\mathbf{R}_+\times\mathbf{R}\times\Omega)$ and the Feynman-Kac formula holds almost-surely:*

$$u(t,x)=\mathbf{E}_x\left[\exp\int_0^t W(ds,b_t-b_s)\right],\ x\in\mathbf{R};\ t\geq 0, \tag{3}$$

where for each x in $\mathbf{R}$, $(\mathcal{C},\mathcal{G},\mathbf{P}_x)$ is the canonical Wiener space, under which b is a Brownian motion with variance κ started from x.

The meaning of the stochastic integral in the above formula is presumably well understood. However, we recall, as in [6], that it can be defined clearly by introducing the Gaussian spectral representation of W: there exists a Gaussian independently scattered measure M on $\mathbf{R}_+\times\mathbf{R}$ such that

$$W(t,x)=\iint_{\mathbf{R}_+\times\mathbf{R}}\mathbf{1}_{[0,t]}(s)\,e^{i\lambda\cdot x}M(ds,d\lambda)$$

where the law of M is defined by

$$\begin{aligned}
&\mathbf{E}\left[\iint_{\mathbf{R}_+\times\mathbf{R}} f(s,\lambda)\, M(ds,d\lambda)\, \overline{\iint_{\mathbf{R}_+\times\mathbf{R}} g(s,\lambda)\, M(ds,d\lambda)}\right] \\
&= \iint_{\mathbf{R}_+\times\mathbf{R}} f(s,\lambda)\, \overline{g(s,\lambda)}\hat{Q}(d\lambda)\, ds,
\end{aligned}$$

where the finite positive measure $\hat{Q}$ is the Fourier transform of the bounded (covariance) function Q. Thus we may define for any fixed path $b \in \mathcal{C}$:

$$\int_0^t W(ds, b_t - b_s) := \iint_{\mathbf{R}_+\times\mathbf{R}} \mathbf{1}_{[0,t]}(s)\, e^{i\lambda\cdot(b_t-b_s)} M(ds,d\lambda).$$

The measurability of the integral on the right-hand side with respect to the pair $(\omega, b) \in \Omega\times\mathcal{C}$ is obtained as the measurability of an L^2-limit of approximations of this integral.

3 Error estimation

We use a discretization technique that follows that in [6], making some necessary improvements. The idea is to replace the Brownian path in the Feynman-Kac formula (3) by a path that stays in $\varepsilon\mathbf{Z}$ where ε is a positive number that will be chosen as a function of κ. We will study only the exponential behavior of $u(t,0)$, since by homogeneity (and the fact that the coefficients of Δ are constant) it is equal to that of $u(t,x)$ for any fixed x. For fixed $b \in \mathcal{C}$, let $T_0 = 0$ and for $i = 0,1,2,\cdots$ let T_{i+1} be the first time after T_i that $b - b_{T_i}$ exits $[-\varepsilon,\varepsilon]$. We define the discretized path $\tilde{b}$ as the right-continuous path that jumps at each time T_i to the position $x_i := b_{T_i}$, and that is constant between these jump times. For any $t \geq 0$, let N_t be the number of jumps of $\tilde{b}$ before time t, and let $T_{N_t+1} = t$ by convention. The inter-jump times $S_i := T_i - T_{i-1}$ are independent and identically distributed under $\mathbf{P}_0$ and are independent of $(x_i : i = 1,2,\cdots)$. Under $\mathbf{P}_0$, the sequence $(x_i)_i$ is a symmetric nearest-neighbor random walk on $\varepsilon\mathbf{Z}$ started from 0. For $b \in \mathcal{C}$ let $e_{t,b} = \int_0^t W(ds, b_{t-s})$, and let $\tilde{e}_{t,b} = e_{t,\tilde{b}}$ and $\tilde{u}(t) = \mathbf{E}_0 \exp(\tilde{e}_{t,b}/2)$. Let $\gamma = \liminf_{t\to\infty} t^{-1}\log u(t,0)$ and $\tilde{\gamma} = \liminf_{t\to\infty} t^{-1}\log\tilde{u}(t)$. Using Schwartz's inequality, we may write, almost surely:

$$\tilde{u}(t) \leq \left[\mathbf{E}_0 \exp(\tilde{e}_{t,b} - e_{t,b})\right]^{1/2} u(t,0)^{1/2}$$

and thus

$$\gamma \geq 2\tilde{\gamma} - \mathcal{E}. \tag{4}$$

where $\mathcal{E} = \limsup_{t\to\infty} \frac{1}{t}\log \mathbf{E}_0 \exp(\tilde{e}_{t,b} - e_{t,b})$ is the error committed by discretizing b. We thus seek an upper bound on $\mathcal{E}$ and a lower bound on $\tilde{\gamma}$.

The purpose of this section is to estimate $\mathcal{E}$. Using the notation above, we have the following proposition.

Proposition 2 *Let*

$$\mathcal{E}_0 = \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{E}_0 \mathbf{E} \left[\exp \sup_{t\in[n-1,n]} \left(\tilde{e}_{t,b} - e_{t,b} \right) \right].$$

We have almost-surely

$$\mathcal{E} \leq \mathcal{E}_0.$$

Proof. This is a simple application of Chebyshev's inequality, Fubini's theorem, and the Borel-Cantelli lemma. ∎

The estimation of $\mathcal{E}_0$ can be done by using Gaussian regularity estimates. The next proposition follows from the calculations in section 3 of [6] and in section 4 of [14]. We omit the proof for conciseness.

Proposition 3 *There exists a constant K such that*

$$\mathbf{E}_0 \mathbf{E} \left[\exp \sup_{t\in[n-1,n]} \left(\tilde{e}_{t,b} - e_{t,b} \right) \right] \leq \exp K n^{3/4} \mathbf{E}_0 \left[\exp \sigma_{n,b}^2 \right]$$

where

$$\begin{aligned} \sigma_{n,b}^2 &= 2 \sup_{t\in[n-1,n]} \int_0^t \left[Q(0) - Q\left(b_{t-s} - \tilde{b}_{t-s} \right) \right] ds \\ &= 2 \int_0^n \left[Q(0) - Q\left(b_s - \tilde{b}_s \right) \right] ds. \end{aligned}$$

The next proposition is an estimate of $\mathbf{E}_0 \left[\exp \sigma_{n,b}^2 \right]$ which significantly improves the calculations done in [6]. It is the main error-control result; it uses the following assumption.

(H) Assume there exists $\alpha \in (0,2]$ and a constant K_α such that for small x, $Q(0) - Q(x) \leq K_\alpha |x|^\alpha$.

This condition implies that the function $x \mapsto x^{\alpha/2} \left(\log x^{-1} \right)^{1/2}$ is almost surely a uniform modulus of continuity in x for the potential W.

Proposition 4 *Under Hypothesis (H), with the notation introduced above, and with $E_\alpha = (2+\alpha)^{-1}(1+\alpha)^{-1}$, there exists a universal constant $C_0 > 0$ such that for every $B > 0$,*

$$\begin{aligned} \mathbf{E}_0 \left[\exp \sigma_{n,b}^2 \right] \leq& \exp \left[-n \left(-2K_\alpha \varepsilon^\alpha + \left(B \log (B/3) - 1 \right) C_0 \kappa \varepsilon^{-2} \right) \right] \\ &+ \exp \left[16 n C_0 B E_\alpha K_\alpha \kappa^{1+\alpha/2} \varepsilon^\alpha \right]. \end{aligned}$$

The following trivial corollary of Propositions 2, 3, and 4, spells out the final error estimate.

Corollary 5 *For small κ and ε, and every $B > 0$, the discretization error $\mathcal{E}$ is bounded above almost surely as:*

$$\mathcal{E} \leq \max\left[2K_\alpha \varepsilon^\alpha - B\kappa\varepsilon^{-2}; BC_\alpha \kappa^{1+\alpha/2}\varepsilon^\alpha\right]$$

where C_α is a constant that depends only on α.

To estimate $\mathbf{E}_0\left[\exp\sigma^2_{n,b}\right]$, the strategy is to separate the cases of whether N_n is greater that the value $k_0(n) = Bn\kappa\varepsilon^{-2}C_0$. The significance of this value is as follows. The proof of Lemma 8 in [6] shows that there exists a universal constant C_0 such that

$$\mathbf{P}_0\left[N_n = k\right] \leq \left(C_0\kappa\varepsilon^{-2}n\right)^k/k!. \tag{5}$$

Thus, apart for the factor B (and assuming that C_0 is sharp in some sense), $k_0(n)$ appears as an estimate of the expectation of N_n. The value B will be chosen so large that the portion of $\mathbf{E}_0\left[\exp\sigma^2_{n,b}\right]$ corresponding to $N_n \geq k_0(n)$ has a Lyapunov exponent that is smaller than $\tilde{\gamma}$. The remaining possible values of N_n form a set whose cardinality is linear in n, and we will study each possible value of N_n individually.

Proof of Proposition 4. Recall that T_i is the i-th jump time of $\tilde{b}$, the interjump times $S_i = T_i - T_{i-1}$ are independent. Also, each process $[0, S_i] \ni s \mapsto b^i_s := b_{T_{i-1}+s} - b_{T_{i-1}}$ is a κ-Brownian motion, and these processes are all independent, and b^i is independent of S_j unless $j = i$, in which case S_i is the first exit time of $[-\varepsilon, \varepsilon]$ for b^i. We obtain by condition (H)

$$\sigma^2_{n,b} \leq 2K_\alpha \int_0^n \left|b_s - \tilde{b}_s\right|^\alpha ds. \tag{6}$$

According to the strategy above, we first investigate the quantity

$$V_n^{(+)} := \mathbf{E}_0\left[\mathbf{1}_{\{N_n \geq k_0(n)\}} \exp 2K_\alpha \sum_{i=1}^{N_n+1} \int_{T_{i-1}}^{T_i} \left|b_s - b_{T_{i-1}}\right|^\alpha ds.\right]$$

We can bound the integral in $V_n^{(+)}$ uniformly in b by using the following obvious bound: for $s \in [T_{i-1}; T_i]$, we have $\left|b_s - b_{T_{i-1}}\right| \leq \varepsilon$. Using (5) and the trivial estimate $\sum_{k\geq m} x^k/k! \leq e^x x^m/m! \leq e^x x^m 3^m m^{-m}$, we obtain, with the notation $x = nC_0\kappa\varepsilon^{-2}$ and thus $k_0(n) = Bx$:

$$\begin{aligned}
V_n^{(+)} &\leq \mathbf{P}_0\left[N_n \geq k_0(n)\right] e^{2nK_\alpha\varepsilon^\alpha} \\
&\leq e^{2nK_\alpha\varepsilon^\alpha} \sum_{k\geq k_0(n)} \left(C_0\kappa\varepsilon^{-2}n\right)^k/k! \\
&\leq \exp\left[2nK_\alpha\varepsilon^\alpha + x\right](3x)^{Bx}(Bx)^{-Bx} \\
&= \exp -n\left[-2K_\alpha\varepsilon^\alpha + \left(B\log(B/3) - 1\right)C_0\kappa\varepsilon^{-2}\right].
\end{aligned} \tag{7}$$

We now consider the remaining portion of $\mathbf{E}_0\left[\exp\sigma^2_{n,b}\right]$, and separate the integral over the last jump interval from the others by Schwartz's inequality:

$$
\begin{aligned}
V_n^{(-)} &:= \mathbf{E}_0\left[\mathbf{1}_{\{N_n\le k_0(n)\}}\exp\left(2K_\alpha\sum_{i=1}^{N_n}\int_{T_{i-1}}^{T_i}\left|b_s-b_{T_{i-1}}\right|^\alpha ds+2K_\alpha\int_{T_{N_n}}^{n}\left|b_s-b_{T_{N_n}}\right|^\alpha ds\right)\right]\\
&\le \mathbf{E}_0\left[\mathbf{1}_{\{N_n\le k_0(n)\}}\exp\left(4K_\alpha\sum_{i=1}^{N_n}\int_{T_{i-1}}^{T_i}\left|b_s-b_{T_{i-1}}\right|^\alpha ds\right)\right]^{1/2}\\
&\qquad\qquad\cdot\mathbf{E}_0\left[\exp 4K_\alpha\int_{T_{N_n}}^{n}\left|b_s-b_{T_{N_n}}\right|^\alpha ds\right]^{1/2}\\
&:= U_n^{1/2}R_n^{1/2}
\end{aligned}
$$

where

$$
\begin{aligned}
U_n &:= \sum_{k=0}^{k_0(n)}\mathbf{E}_0\left[\mathbf{1}_{\{N_n=k\}}\exp\left(4K_\alpha\kappa^{\alpha/2}\sum_{i=1}^{N_n}\int_0^{S_i}\left|B_s^i\right|^\alpha ds\right)\right]\\
R_n &:= \mathbf{E}_0\left[\exp 4K_\alpha\int_{T_{N_n}}^{n}\left|b_s-b_{T_{N_n}}\right|^\alpha ds\right]
\end{aligned}
$$

and where under $\mathbf{P}_0$, $(B^i : i = 1, 2, \cdots)$ is a sequence of IID standard Brownian motions, and the stopping times S_i are their respective exit times from $[-\varepsilon, \varepsilon]$.

We study the first term first.

$$
\begin{aligned}
U_n &:= \sum_{k=0}^{k_0(n)}\mathbf{E}_0\left[\mathbf{1}_{\{N_n=k\}}\exp 2K_\alpha\kappa^{\alpha/2}\sum_{i=1}^{k}\int_0^{S_i}\left|B_s^i\right|^\alpha ds\right]\\
&\le \sum_{k=0}^{k_0(n)}\mathbf{E}_0\left[\exp 2K_\alpha\kappa^{\alpha/2}\sum_{i=1}^{k}\int_0^{S_i}\left|B_s^i\right|^\alpha ds\right]\\
&= \sum_{k=0}^{k_0(n)}\left\{\mathbf{E}_0\left[\exp 2K_\alpha\kappa^{\alpha/2}\int_0^{S_1}\left|B_s^1\right|^\alpha ds\right]\right\}^k. \qquad (8)
\end{aligned}
$$

Therefore we only need to estimate an expectation of the form

$$
v(0)=\mathbf{E}_0\left[\exp c\int_0^S\left|B_s\right|^\alpha ds\right]
$$

where c is a constant, B is a standard Brownian motion, and S is the first exit time of B from $[-\varepsilon, \varepsilon]$. It turns out that this can be calculated explicitly, as the proof of the following lemma shows.

Lemma 6

$$
v(0)=1+\frac{2c\varepsilon^{2+\alpha}}{(2+\alpha)(1+\alpha)}+O\left(c^2\varepsilon^{4+2\alpha}\right).
$$

Proof. Let $x \in [-\varepsilon, \varepsilon]$. Let S^+ be the first hitting time of ε by $B_\cdot + x$, and let S^- be the first hitting time of $-\varepsilon$. Let

$$\begin{aligned} v^+(x) &= \mathbf{E}_0\left[\mathbf{1}_{\{S^+<S^-\}} \exp c \int_0^S |B_s + x|^\alpha \, ds\right] \\ v^-(x) &= \mathbf{E}_0\left[\mathbf{1}_{\{S^+>S^-\}} \exp c \int_0^S |B_s + x|^\alpha \, ds\right]. \end{aligned}$$

Obviously, by symmetry, $v^+(0) = v^-(0)$ and thus $v(0) = 2v^+(0)$. The Feynman-Kac formula for Brownian motion killed at the boundary of $[-\varepsilon, \varepsilon]$ (see Theorem 6.4.3 in [13]) yields that v^+ is twice continuously differentiable on $[-\varepsilon, \varepsilon]$ and satisfies

$$\begin{aligned} \forall x \in [-\varepsilon, \varepsilon] : d^2 v^+/dx^2 + 2c\,|x|^\alpha\, v = 0, \\ v^+(-\varepsilon) = 0; \quad v^+(\varepsilon) = 1. \end{aligned}$$

We can solve this equation explicitly using a series expansion. Let us first renormalize the endpoints. Let $w(y) = v^+(\varepsilon y)$. Thus with $A = 2c\varepsilon^{2+\alpha}$,

$$\begin{aligned} \forall x \in [-1, 1] : w'' + A\,|y|^\alpha\, w = 0, \\ w(-1) = 0; \quad w(1) = 1. \end{aligned}$$

For $y \geq 0$, we look for the solution in the form

$$w(y) = a_0 + b_0 y + a_1 y^{2+\alpha} + b_1 y^{2+\alpha+1} + \cdots + a_p y^{(2+\alpha)p} + b_p y^{(2+\alpha)p+1} + \cdots \tag{9}$$

yielding

$$\begin{aligned} w''(y) &= \sum_{p=1}^{\infty} a_p (2+\alpha)\, p\, ((2+\alpha)\, p - 1)\, y^{(2+\alpha)p-2} \\ &\quad + \sum_{p=1}^{\infty} b_p (2+\alpha)\, p\, ((2+\alpha)\, p + 1)\, y^{(2+\alpha)p-1} \\ A y^\alpha w(y) &= \sum_{p=1}^{\infty} A a_{p-1} y^{(2+\alpha)(p-1)+\alpha} + \sum_{p=1}^{\infty} A b_{p-1} y^{(2+\alpha)(p-1)+\alpha+1}. \end{aligned}$$

Since the powers $(2+\alpha)\,p - 2 = (2+\alpha)(p-1) + \alpha$ and $(2+\alpha)\,p - 1 = (2+\alpha)(p-1) + \alpha + 1$ coincide, we can identify the coefficients, and obtain, for a_0 and b_0 given,

$$\begin{aligned} a_p &= \eta_p (-A)^p a_0; \quad b_p = \delta_p (-A)^p b_0 \\ \eta_p &= \eta_{p-1} / \left[(2+\alpha)\, p\, ((2+\alpha)\, p - 1)\right]; \quad \eta_0 = 1 \\ \delta_p &= \delta_{p-1} / \left[(2+\alpha)\, p\, ((2+\alpha)\, p + 1)\right]; \quad \delta_0 = 1. \end{aligned} \tag{10}$$

For $y < 0$, $|y| = -y$, and so we repeat the above calculation by replacing y by $-y$ in (9). A priori, all coefficients have to be recalculated. Instead of $(a_p, b_p)_p$ we call them $\left(a'_p, b'_p\right)$. Since the second derivative is invariant by this transformation, we obtain the same relations for

(a', b') as in (10), given a'_0 and b'_0. Now we use the fact that w is continuously differentiable at 0. This yields $a_0 = a'_0$, and by identifying the left-hand and right-hand derivatives of w at 0, we obtain $b_0 = -b'_0$. We may now determine a_0 and b_0 using the boundary conditions: let $\eta(z) = \sum_{p=0}^{\infty} \eta_p z^p$ and $\delta(z) = \sum_{p=0}^{\infty} \delta_p z^p$. We have

$$1 = w(1) = a_0 \eta(-A) + b_0 \delta(-A)$$
$$0 = w(-1) = a_0 \eta(-A) - b_0 \delta(-A),$$

and therefore

$$\begin{aligned} w(0) = a_0 &= \frac{1}{2\eta(-A)} \\ &= \frac{1}{2} + \frac{\eta'(0)}{2\eta(0)^2} A + O\left(A^2\right) \\ &= \frac{1}{2} + \frac{A}{2(2+\alpha)(1+\alpha)} + O\left(A^2\right). \end{aligned}$$

■

We may now use Lemma 6 in (8) with $A = 2c\varepsilon^{2+\alpha}$ and $c = 4K_\alpha \kappa^{\alpha/2}$, and, denoting $E_\alpha := (2+\alpha)^{-1}(1+\alpha)^{-1}$, we obtain

$$\begin{aligned} U_n &\leq \sum_{k=0}^{k_0(n)} \left\{ \mathbf{E}_0 \left[\exp 4K_\alpha \kappa^{\alpha/2} \int_0^S |B_s|^\alpha \, ds \right] \right\}^k \\ &= \sum_{k=0}^{k_0(n)} \left(1 + 8E_\alpha K_\alpha \kappa^{\alpha/2} \varepsilon^{2+\alpha} + o\left(\kappa^{\alpha/2}\varepsilon^{2+\alpha}\right)\right)^k \\ &\leq k_0(n) \left(1 + 8E_\alpha K_\alpha \kappa^{\alpha/2} \varepsilon^{2+\alpha} + o\left(\kappa^{\alpha/2}\varepsilon^{2+\alpha}\right)\right)^{k_0(n)} \\ &\leq \exp\left[16nC_0 B E_\alpha K_\alpha \frac{\kappa}{\varepsilon^2} \kappa^{\alpha/2} \varepsilon^{2+\alpha}\right]. \end{aligned}$$

for n large.

The last remaining term to control the error is $R_n := \mathbf{E}_0 \left[\exp 4K_\alpha \int_{T_{N_n}}^n \left|b_s - b_{T_{N_n}}\right|^\alpha ds\right]^{1/2}$. By conditioning by $\mathcal{G}_{T_n}$, the events up to the last jump time before n, we can bound R_n as

$$R_n^2 \leq \mathbf{E}_0 \left[\exp 4K_\alpha \kappa^{\alpha/2} \int_0^R |B_s|^\alpha \, ds\right]$$

where under $\mathbf{P}_0$, B is a standard Browian motion and R is a random time which is bounded above by S, the first exit time of B from $[-\varepsilon, \varepsilon]$. Therefore R_n is bounded above by $(U_n)^{1/k_0(n)}$, and thus is altogether negligible, and Proposition 4 is proved. ■

4 Lyapunov exponent estimation

The strategy for estimating $\tilde{\gamma} = \liminf_{t\to\infty} t^{-1} \log \tilde{u}(t)$ where $\tilde{u}(t) = \mathbf{E}_0 \exp(\tilde{e}_{t,b}/2)$ is to bound $\tilde{u}(t)$ below by throwing away all trajectories b whose jump times do not fall in prescribed, regularly spaced, small intervals, and to further throw away all trajectories except the one that maximizes $\tilde{e}_{t,b} = \int_0^t W\left(ds, \tilde{b}_{t-s}\right)$. We face several difficulties in addition to those which were dealt with in the proof of the lower bound in Proposition 4.1.1 in [4] in discrete space:

- estimating the distribution of suprema of stochastic processes with continuous *space* parameters
- estimating the distribution of suprema of stochastic processes with continuous *time* parameters (although the authors of [4] work with a continuous-time model, the proof of Proposition 4.1.1 therein is only actually spelled out for a fixed sequence of times tending to infinity).
- in continuous space, the substitute for the exponentially distributed jump times of a random walk, are times whose density near zero is not bounded below, making it impossible to implement the choices made in [4].

The first two difficulties are dealt with below using Gaussian suprema estimates. The third is fundamental, and requires a careful analysis. Hereafter, we ignore the factor 1/2 in the definition of $\tilde{u}$.

Let n be a positive integer and let $t \in [n-1, n]$. Let $k = (n-1) f$ be an integer, where the frequency f will be chosen later as a function of κ. Let h be a fixed value which will also be chosen later as a function of κ, with the restriction that $h \in (0, f^{-1})$. For $j = 0, \cdots, k$, let $t_j = (n-1)j/k$. Let $\mathcal{S}(t,k)$ be the simplex

$$\mathcal{S}(t,k) = \left\{ s \in [0,t]^k : 0 = s_0 \leq s_1 \leq \cdots \leq s_k \leq t = s_{k+1} \right\}$$

and denote by $\mathcal{L}(ds) = \mathcal{L}_{t,k}(ds)$ the probability measure on $\mathcal{S}(t,k)$ equal to the law of the jump times T_i of $\tilde{b}$ under $\mathbf{P}_0$ given that the number of jumps N_t before time t is equal to k. Let $p(t,k) = \mathbf{P}_0[N_t = k]$. For fixed $k \in \mathbf{N}$ and $s \in \mathcal{S}(t,k)$, let $\mathbf{P}_{0,k,s}$ be the measure $\mathbf{P}_0$ conditional on $N_t = k$ and $(T_j)_{j=1,\cdots,k} = s$. Let us introduce the notation, for $0 \leq a \leq b$ and $z \in \mathbf{R}$,

$$W([a,b];z) := W(b,z) - W(a,z)$$

By definition we have

$$\tilde{u}(t) = \sum_{k=0}^{\infty} p(t,k) \int_{\mathcal{S}(t,k)} \mathcal{L}(ds)\, \mathbf{E}_{0,k,s}\left[\exp \sum_{l=0}^{k} W([t-s_{l+1}, t-s_l]; x_l)\right]$$

where we recall that $x_l = b_{T_l}$. For convenience of notation, we operate the isometric change of variable in $(\mathcal{S}(t,k), \mathcal{L})$ defined by $s'_l = t - s_{k-l+1}$, which has the effect of eliminating the time reversal and replacing W by $W' = -W$. We abusively ignore the variable name changes, to obtain

$$\tilde{u}(t) = \sum_{k=0}^{\infty} p(t,k) \int_{\mathcal{S}(t,k)} \mathcal{L}(ds)\, \mathbf{E}_{0,k,s}\left[\exp \sum_{l=0}^{k} W([s_l, s_{l+1}]; x_{k-l})\right].$$

Now we obtain a lower bound by restricting the time integral to the subset of $\mathcal{S}(t,k)$ defined as

$$A(t,k) = A([t],k) = \{s \in \mathcal{S}(t,k) : \forall j = 1, \cdots, k : s_j \in [t_j - h, t_j]\},$$

and by throwing away all possibilities for $(x_{k-l})_{l=0,\cdots,k}$ except for one fixed nearest-neighbor trajectory $m = (m_l)_{l=0,\cdots,k}$ in $\varepsilon\mathbf{Z}$ with $m_k = 0$:

$$\tilde{u}(t) \geq \sum_{k=0}^{\infty} p(t,k) \int_{A([t],k)} \mathcal{L}(ds)\, \mathbf{P}_{0,t,k}[x_{k-l} = m_l : l = 1, \cdots, k] \exp \sum_{l=0}^{k} W([s_l, s_{l+1}]; m_l). \tag{11}$$

Under $\mathbf{P}_{0,t,k}$, the probability of the trajectory m is of course 2^{-k}. Also note we are still using the convention $s_0 = 0$, $s_{k+1} = t$. We now make a special (W-dependent) choice for m, defined recursively from $m_k = 0$. For $l = k, \cdots, 1$, if m_l^* is chosen and measurable with respect to the increments of W in $\cup_{j=l}^{k-1}[t_j; t_{j+1} - h]$, let m_{l-1}^* be the value μ in $\{m_l^* - \varepsilon; m_l^* + \varepsilon\}$ which maximizes the random variable $W([t_{l-1}, t_l - h]; \mu)$. By the independence of increments of W, given m_l^*, the resulting maximum

$$W_{l-1}^* := \max\{W([t_{l-1}, t_l - h]; m_l^* - \varepsilon); W([t_{l-1}, t_l - h]; m_l^* + \varepsilon)\} \tag{12}$$

depends only on the increments of W in the interval $[t_{l-1}, t_l - h]$, and the same is true of $m_{l-1}^* - m_l^*$. With the convention $t_{k+1} = n$, for each $s \in A(t,k)$ and $l = 1, \cdots, k-1$ now introduce the remainder

$$\begin{aligned} R_l(s) &:= W([s_l, t_l]; m_l^*) + W([t_{l+1} - h, s_{l+1}]; m_l^*]), \\ R_0(s) &:= W([s_0, t_0]; m_0^*) + W([t_1 - h, s_1]; m_0^*) = W([t_1 - h, s_1]; m_0^*) \\ R_k(s,t) &:= W([s_k, t_k]; 0) + W([t_{k+1} - 1, s_{k+1}]; 0) = W([s_k, t]; 0) \end{aligned} \tag{13}$$

Thus for $m = m^*$, we can rewrite (11) as

$$\begin{aligned} \tilde{u}(t) &\geq \sum_{k=0}^{\infty} p(t,k)\, 2^{-k} \exp\left(\sum_{l=0}^{k-1} W_l^*\right) \int_{A(t,k)} \mathcal{L}(ds) \exp\left(\sum_{l=0}^{k} R_l(s)\right) \\ &\geq \sum_{k=0}^{\infty} p(t,k)\, 2^{-k} \exp\left(\sum_{l=0}^{k-1} W_l^*\right) \mathcal{L}(A(t,k)) \exp\left(\inf\left\{\sum_{l=0}^{k} R_l(s) : s \in A(t,k)\right\}\right) \\ &= \sum_{k=0}^{\infty} p(t,k)\, 2^{-k} \exp(W^*)\, \mathcal{L}(A(t,k)) \exp(\inf\{R(s) : s \in A(t,k)\}). \end{aligned}$$

where we introduced the notation $\sum_{l=0}^{k-1} W_l^* = W^*$ and $\sum_{l=0}^{k} R_l(s) = R(s) = R(s,t)$. The advantage of this decomposition of W's increments into the maximized part W^* and the remainder R lies in the fact that R is a centered Gaussian process indexed by $A(t,k) \times [n-1, n]$, whose supremum can be estimated using Gaussian regularity tools, and the distribution of W^* can be calculated. The strategy is thus to

1. estimate $p(t,k)$ and $\mathcal{L}(A(t,k))$ via an analysis of the jump times T_j.

2. estimate $\sup\{R(s) : s \in A(t,k)\}$ by calculating the canonical metric and entropy of R.

3. calculate the law of W^*

4. put all theses elements together to apply a standard Borel-Cantelli argument, and choose the parameters optimally.

As a last preparatory step, we throw away all terms in the above series except for one value of $k = k(n,\kappa)$ which will be chosen later, and we bound $\inf_{t\in[n-1,n]} \tilde{u}(t)$ as follows:

$$\inf_{t\in[n-1,n]} \tilde{u}(t) \tag{14}$$

$$\geq \inf_{t\in[n-1,n]} \left[p(t,k)\, 2^{-k}\mathcal{L}(A(t,k))\right] \exp(W^*) \exp\left(\inf\{R(s,t) : s \in A([t],k); t \in [n-1,n]\}\right) \tag{15}$$

4.1 Analysis of jump times

We begin with an elementary result, which expresses the fact that the first exit time T_1 of Brownian motion from $[-1,1]$ has a density that decays like the exponential density with parameter the first eigenvalue of Δ in $[-1,1]$.

Lemma 7 *T_1 has a density ψ. There exist universal constants c_0 and c_0' such that, with $b = \sqrt{\pi/2}$,*

- *for all $t \geq 0$, $\psi(t) \leq c_0' \exp(-bt)$,*
- *for all $t \geq 1/2$, $\psi(t) \geq c_0 \exp(-bt)$.*

Using the notation introduced above, and the scaling property of Brownian motion, the sequence $\tau_j = T_j - T_{j-1}$ of interjump times for $\tilde{b}$ are independent and identically distributed as the first exit time of standard Brownian motion B from $[-\sqrt{\kappa/\varepsilon^2}, \sqrt{\kappa/\varepsilon^2}]$. From Lemma 7 we have

$$\begin{aligned}
\mathbf{P}_0[\tau_j \in [t, t+dt]] &= \mathbf{P}_0\left[|B_t| < \sqrt{\kappa/\varepsilon^2}; |B_{t+dt}| > \sqrt{\kappa/\varepsilon^2}\right] \\
&= \mathbf{P}_0\left[|B_{t\kappa\varepsilon^{-2}}| < 1; \left|B_{(t+dt)\kappa\varepsilon^{-2}}\right| > 1\right] \\
&= \mathbf{P}_0\left[T_1 \in \left[\kappa\varepsilon^{-2}t, \kappa\varepsilon^{-2}t + \kappa\varepsilon^{-2}dt\right]\right] \\
&\geq c_0\kappa\varepsilon^{-2}e^{-b\kappa\varepsilon^{-2}t}dt
\end{aligned} \tag{16}$$

for all t such that $\kappa\varepsilon^{-2}t \geq 1/2$, and similarly, for all $t \geq 0$,

$$\mathbf{P}_0[\tau_j \in [t, t+dt]] \leq c_0'\kappa\varepsilon^{-2}e^{-b\kappa\varepsilon^{-2}t}dt. \tag{17}$$

By a simple conditioning argument, (see Proposition 9 in [6]), with the notation $F(0) = 0$ and $F(ds) = \mathbf{P}_0[\tau_j \in ds]$, we get

$$p(t,k) := \mathbf{P}_0[N_t = k] = \int_{\{s_1+\cdots+s_k \leq t\}} F(ds_1)\cdots F(ds_k)\left(1 - F(t - s_1 - \cdots - s_k)\right). \tag{18}$$

We can now prove the next lemma.

Lemma 8 *With the constants introduced in Lemma 7, for all t, k,*

$$p(t,k) \leq c_0' b^{-1} e^{-b\kappa\varepsilon^{-2}t} \left(c_0' \kappa\varepsilon^{-2} t\right)^k / k!.$$

There exists a universal constant $c_1 > 0$ such that for all $t \geq 0$ and the special choice $k = [\kappa\varepsilon^{-2}[t]]$ (where $[x]$ denotes the greatest integer less than x), we have the similar lower bound

$$p(t,k) \geq c_0 b^{-1} e^{-b\kappa\varepsilon^{-2}t} \left(c_0 c_1 \kappa\varepsilon^{-2} t\right)^k / k!.$$

Proof. Let $\lambda = b\kappa\varepsilon^{-2}$. First note that by (17), $1 - F(u) = \int_u^\infty F(ds) \leq c_0' b^{-1} e^{-\lambda u}$. Then using (18), and (17) we can write

$$\mathbf{P}_0[N_t = k] \leq c_0' b^{-1} (c_0'/b)^k \int_{\{s_1+\cdots+s_k \leq t\}} \lambda^k e^{-\lambda(s_1+\cdots+s_k)} e^{-\lambda(t-s_1-\cdots-s_k)} ds_1 \cdots ds_k.$$

The integral in the last expression is exactly equal to the probability that a Poisson process with intensity λ be equal to k at time t. Therefore we have

$$\mathbf{P}_0[N_t = k] \leq \left(c_0' b^{-1}\right)^{k+1} e^{-\lambda t} (\lambda t)^k / k!$$

which proves the Lemma's upper bound.

For the lower bound, we replace the domain of integration in (18) by

$$\left\{s : \forall j = 1, \cdots, k : s_j \in \left[\varepsilon^2/(2\kappa), \varepsilon^2/\kappa\right]\right\}.$$

This set is included in $\{s_1 + \cdots + s_k \leq t\}$ since $k\varepsilon^2/\kappa \leq t$ by our choice of k. Moreover this restriction implies $\kappa\varepsilon^{-2}s_j \geq 1/2$ for all j, allowing us to use the lower bound (16). We thus obtain

$$\begin{aligned}
\mathbf{P}_0[N_t = k] &= \int_{s_1+\cdots+s_k \leq t} \prod_{j=1}^k F(ds_j) \int_{t-s_1-\cdots-s_k}^\infty F(ds_{k+1}) \\
&\geq \prod_{j=1}^k \left[\frac{c_0}{b} \int_{\frac{\varepsilon^2}{2\kappa}}^{\frac{\varepsilon^2}{\kappa}} \lambda e^{-\lambda s_j} ds_j\right] \int_t^\infty \frac{c_0}{b} \lambda e^{-\lambda s} ds \\
&= \left(\frac{c_0}{b}\right)^{k+1} \left(e^{-b/2} - e^{-b}\right)^k e^{-\lambda t} \\
&\geq \left(\frac{c_0}{b}\right)^{k+1} \left(e^{-b} b/2\right)^k e^{-\lambda t}.
\end{aligned}$$

By Stirling's formula, we have $(\lambda t)^k / k! \leq (eb)^k$. Therefore,

$$\mathbf{P}_0[N_t = k] \geq \left(\frac{c_0}{b}\right)^{k+1} 2^{-k} e^{k(-1-b)} e^{-\lambda t} (\lambda t)^k / k!$$

which is the lower bound of the Lemma. ■

The next proposition estimates $\mathcal{L}(A(t,k))$. Recall that $t \in [n-1, n]$ where $n \in \mathbf{N}$, and that $f = k/(n-1)$. Note that the condition $0 < h < f^{-1}$ is built into this proposition's hypothesis.

Proposition 9 *Assuming that $c_0'\kappa\varepsilon^{-2}(1+h) \le 2/3$, and that $\kappa\varepsilon^{-2}f^{-1} \ge \kappa\varepsilon^{-2}h+1/2$, there exists a universal constant c_2 such that*

$$\mathcal{L}(A(t,k)) \ge c_2 \left(\frac{c_0}{c_0'}\frac{h}{t}\right)^k k!$$

Proof. Using equation (18) we have

$$\mathcal{L}(A(t,k)) = \frac{1}{p(t,k)} \int_{\substack{s_1+\cdots+s_k \le t \\ s_1+\cdots+s_j \in [t_j-h,t_j]:j=1,\cdots,k}} (1-F(t-(s_1+\cdots+s_k)))\, F(ds_1)\cdots F(ds_k)$$
$$:= I(t,h,k)/p(t,k).$$

Denote $T_k = s_1+\cdots+s_k$. To estimate $1-F(t-T_k)$, we notice first that in the domain of integration, $t-T_k < h+t-t_k \le h+1$.

By the upper bound (17) we find that $1-F(t-T_k) \ge 1-c_0'(t-T_k)\kappa\varepsilon^{-2}$. By the hypothesis imposed on h, $c_0'(t-T_k)\kappa\varepsilon^{-2} < 2/3$, and thus $1-F(t-T_k) \ge e^{-2c_0'(t-T_k)\kappa\varepsilon^{-2}}$ and thus, for some universal constant c_0'',

$$1-F(t-T_k) \ge c_0'' e^{-b(t-T_k)\kappa\varepsilon^{-2}}. \tag{19}$$

Now the condition $s_1+\cdots+s_j \in [t_j-h,t_j] : j=1,\cdots,k$ implies that $s_j \in [f^{-1}-h, f^{-1}+h]$. By hypothesis $(f^{-1}-h)\kappa\varepsilon^{-2}$ exceeds $1/2$, and therefore we can use the lower bound (16), which, together with (19), and the notation $\lambda = b\kappa\varepsilon^{-1}$, yields

$$I(t,h,k) \ge c_0''\left(c_0 b^{-1}\right)^k \int_{\substack{s_1+\cdots+s_k \le t \\ s_1+\cdots+s_j \in [t_j-h,t_j]:j=1,\cdots,k}} e^{-\lambda(t-(s_1+\cdots+s_k))} \prod_{j=1}^k \lambda e^{-\lambda s_j} ds_j.$$

With the upper bound of Lemma 8, we obtain

$$\mathcal{L}(A(t,k)) \ge \frac{c_0''\left(c_0 b^{-1}\right)^k}{c_0' b^{-1} e^{-\kappa\varepsilon^{-2}bt}\left(c_0'\kappa\varepsilon^{-2}t\right)^k/k!} \int_{\substack{s_1+\cdots+s_k \le t \\ s_1+\cdots+s_j \in [t_j-h,t_j]:j=1,\cdots,k}} e^{-\lambda(t-(s_1+\cdots+s_k))} \prod_{j=1}^k \lambda e^{-\lambda s_j} ds_j$$
$$= \frac{c_0'' b}{c_0'}\left(\frac{c_0}{c_0'}\right)^k \frac{1}{\exp(-\lambda t)\frac{(\lambda t)^k}{k!}} \int_{\substack{s_1+\cdots+s_k \le t \\ s_1+\cdots+s_j \in [t_j-h,t_j]:j=1,\cdots,k}} e^{-\lambda(t-(s_1+\cdots+s_k))} \prod_{j=1}^k \lambda e^{-\lambda s_j} ds_j.$$

We recognize the conditional distribution of the jump times $\{T_j : j=1,\cdots,k\}$ of a Poisson process N^λ with intensity λ conditioned on the event that $N_t = k$, and we see that this distribution is integrated over the set $\mathcal{A}(t,k) = \{T_j \in [t_j-h,t_j] : j=1,\cdots,k\}$. However, it is well-known that these times are uniformly distributed in the simplex $\mathcal{S}(t,k)$. Therefore, since $\mathcal{A}(t,k) \subset \mathcal{S}(t,k)$ and the Lebesgue measure of $\mathcal{A}(t,k)$ is h^k, we obtain

$$\mathcal{L}(A(t,k)) \ge \frac{c_0'' b}{c_0'}\left(\frac{c_0}{c_0'}\right)^k \frac{h^k}{t^k/k!}.$$

■

4.2 Controlling the supremum of the remainder

Our purpose here is to prove the following proposition.

Proposition 10 *There exists a universal constant C_u such that if $k \geq 1$, $h \geq 1$ and $z > (hkQ(0))^{-1/2}$,*

$$\begin{aligned} &\mathbf{E} \exp \left[z \sup \left\{ R(s,t) : s \in A([t], k); t \in [n-1, n] \right\} \right] \\ &\leq \exp \left[2Q(0) z^2 hk + C_u z (k+1) (Q(0) h)^{1/2} \right]. \end{aligned}$$

Proof. When calculating the above expectation, one can first calculate the expectation conditional on the values of the path m^*, and then calculate an expectation with respect to m^*. But by the independence of the increments of W, the distribution of R is independent of the values m^*. Therefore, one can fix a non-random m^* arbitrarily and calculate one expectation only. For fixed $n \in \mathbf{N}$, according to (13), we can rename the domain and variables of $R = R(s)$ over which the supremum is to be taken by letting $s = (s_0, s_1, \cdots, s_k, s_{k+1}) \in \{0\} \times A(n,k) \times [n-1, n] = D_0$. Let

$$D = [-h, 0] \times [t_1 - h, t_1] \times \cdots \times [t_l - h, t_l] \times \cdots \times [t_k - h, t_k] \times [t_k, t_k + h].$$

Here W is extended to $[-h, 0]$ in the standard way. Clearly, $D \supset D_0$ and it is sufficient to estimate the supremum of R over D. Moreover, by the stationarity of W, the fact that the intervals which form D are not contiguous is irrelevant in terms of the distribution of R's supremum over D. Therefore, we can redefine (D, R) as follows: under $\mathbf{P}$, let $\left(\left(\bar{B}_l, \hat{B}_l \right) \right)_{l=0,\cdots,k+1}$ be a family of independent pairs of independent Brownian motions with variance $Q(0)$; let

$$\begin{aligned} &D = [0, h]^{k+2} \\ &\forall s \in D : R(s) = \sum_{l=0}^{k} \bar{B}_l (h - s_l) + \hat{B}_l (s_{l+1}). \end{aligned}$$

By a standard argument of Gaussian boundedness using the Borell-type inequality (see [1] and equation (18) in [6]) we obtain

$$\mathbf{E} \left[\exp \left(z \sup_D R \right) \right] \leq \left(\exp \mathbf{E} \left[\sup_D zR \right] \right) \left(1 + \left(8\pi\sigma^2 \right)^{1/2} \exp \left(\sigma^2 / 2 \right) \right)$$

where

$$\sigma^2 = \sup_D \mathbf{E} \left[z^2 R^2 \right] = z^2 Q(0) hk.$$

To calculate $\mathbf{E}[\sup_D zR]$, in principle, the so-called Dudley theorem of Gaussian regularity must be used (see [1]), but here the calculation can be made nearly explicitly. Indeed, using the fact that $\sup_D (A_s + B_s) \le \sup_D A_s + \sup_D B_s$, and the fact that for Brownian motion B, $\mathbf{E} \sup_{[0,h]} B = \sqrt{2h/\pi}$, we obtain

$$\begin{aligned}
\mathbf{E}\left[\sup_D zR\right] &= z\mathbf{E}\left[\sup_{s\in D}\sum_{l=0}^{k}\bar{B}_l\left(h-s_l\right)+\hat{B}_l\left(s_{l+1}\right)\right] \\
&\le z\mathbf{E}\left[\sup_{s\in D}\sum_{l=0}^{k}\bar{B}_l\left(h-s_l\right)\right]+z\mathbf{E}\left[\sup_{s\in D}\sum_{l=0}^{k}\hat{B}_l\left(s_{l+1}\right)\right] \\
&\le z\mathbf{E}\left[\sum_{l=0}^{k}\sup_{s_l\in[0,h]}\bar{B}_l\left(h-s_l\right)\right]+z\mathbf{E}\left[\sum_{l=0}^{k}\sup_{s_{l+1}\in[0,h]}\hat{B}_l\left(s_{l+1}\right)\right] \\
&= 2z\left(k+1\right)\sqrt{2Q\left(0\right)h/\pi}.
\end{aligned}$$

Therefore for $k \ge 1$ and z such that $\sigma^2 > 1$

$$\mathbf{E}\left[\exp\left(z\sup_D R\right)\right] \le \exp\left[2z\left(k+1\right)\left(2Q\left(0\right)h/\pi\right)^{1/2}+2z^2Q\left(0\right)hk\right]$$

and the proposition is proved. ■

4.3 The law of W^*

By definition (12), and by W's homogeneity, $W^* = \sum_{l=0}^{k-1} W_l^*$ is a sum of identically distributed random variables, and given m_l^*, W_{l-1}^* is independent of W_j^* for $j \ge l$, and its distribution does not depend on the value of m_l^*. This implies that the W_l's are in fact independent, as the following calculation shows: let $l < l' \le k$,

$$\begin{aligned}
\mathbf{E}g\left(W_l^*\right)h\left(W_{l'}^*\right) &= \mathbf{E}\left[\mathbf{E}\left[g\left(W_l^*\right)h\left(W_{l'}^*\right)\middle|\,\mathcal{F}_{[t_{l+1},n]}\right]\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[g\left(W_l^*\right)\middle|\,\mathcal{F}_{[t_{l+1},n]}\right]h\left(W_{l'}^*\right)\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[g\left(W_l^*\right)\middle|\,m_{l+1}^*\right]h\left(W_{l'}^*\right)\right] \\
&= \mathbf{E}\left[g\left(W_l^*\right)\right]\mathbf{E}\left[h\left(W_{l'}^*\right)\right].
\end{aligned}$$

Therefore

$$\mathbf{E}\exp zW^* = \left(\mathbf{E}\left[\exp zM^*\right]\right)^k \tag{20}$$

where we may define the random variable M as

$$\begin{aligned}
M^* &= \left(\frac{1}{f}-h\right)^{1/2}\max\left(W\left(1,0\right),W\left(1,2\varepsilon\right)\right) \\
&= \left(\frac{n-1}{k}-h\right)^{1/2}\max\left(W\left(1,0\right),W\left(1,2\varepsilon\right)\right).
\end{aligned}$$

Let

$$r = \mathbf{E}\left[W\left(1,0\right),W\left(1,2\varepsilon\right)\right]/\mathrm{var}\left[W\left(1,0\right)\right] = Q\left(2\varepsilon\right)/Q\left(0\right).$$

We only need to study the distribution of $X = \max\left(Z; rZ + \theta Z'\right)$ where Z, Z' are independent standard normals and $\theta = \sqrt{1-r^2}$.

Lemma 11 *For all $\delta \in [0,1/2)$, for $G = (1-2\delta)^{-1/2}$, we have*

$$\mathbf{E}\left[\exp\left(\delta X^2\right)\right] \in [G; 2G]. \tag{21}$$

For the mean of X we get

$$\mathbf{E}[X] = \sqrt{(1-r)/\pi} := \mu.$$

Proof. To find μ, let $Y = rZ + \theta Z'$. Since $X = \max(Y,Z) = (Y + Z + |Y - Z|)/2$, we have $\mu = \mathbf{E}|Y - Z|/2$. Since $Y - Z = \theta Z' - (1-r)Z$ is a Gaussian r.v. with variance $2(1-r)$ we obtain the announced result. For the quadratic exponential moment we write

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(\delta X^2\right)\right] \\
&= \iint_{(1-r)x>\theta y} \frac{dxdy}{2\pi} e^{\delta x^2 - \left(x^2+y^2\right)/2} + \iint_{(1-r)x<\theta y} \frac{dxdy}{2\pi} e^{\delta(rx+\theta y)^2 - \left(x^2+y^2\right)/2} \\
&= \int_{-\infty}^{\infty} \frac{dy}{(2\pi)^{1/2}} e^{-y^2/2} N\left(\frac{\theta y}{(1-r)G}\right) G \\
&+ \int_{-\infty}^{\infty} \frac{dy}{(2\pi)^{1/2}} e^{-y^2/2} \int_{-\infty}^{\theta y/(1-r)} \frac{dx}{(2\pi)^{1/2}} e^{-x^2/2} e^{\delta(rx+\theta y)^2 - y^2/2}.
\end{aligned}$$

where $N(z) = \int_z^\infty e^{-u^2/2}(2\pi)^{-1/2}\,du$. The lower bound in (21) is obtained by replacing the second term by 0 and in the first term, by integrating only from $y = 0$ to $y = +\infty$. The upper bound in (21) is obtained by replacing N by 1 in the first term, and by noticing that the second term is bounded by (replacing the upper endpoint for x by $+\infty$) $\mathbf{E}\exp(\delta(rZ + \theta Z')) = G$. ■

We now write $M^* = \nu + \bar{M}$ where $\bar{M}$ is a centered random variable, and ν is the mean of M^*, namely

$$\nu = \mu\left(Q(0)\left((n-1)/k - h\right)\right)^{1/2}. \tag{22}$$

By the quadratic exponential moment (21) of Lemma 11, $\bar{M}$ is a subgaussian random variable, since

$$\begin{aligned}
\mathbf{E}\left[\exp\left(\eta \bar{M}^2\right)\right] &\le \exp(2\eta\nu)\,\mathbf{E}\left[\exp\left(2\eta M^{*2}\right)\right] \\
&= \exp(2\eta\nu)\,\mathbf{E}\left[\exp\left(2\eta\left(Q(0)\left((n-1)/k - h\right)\right)X^2\right)\right]
\end{aligned} \tag{23}$$

is finite as soon as $\eta < \left(4Q(0)\left((n-1)/k - h\right)\right)^{-1}$. In that case we also have

$$\exp(2\eta\nu) \le \exp\left(\mu^2/2\right) \le e^{1/(2\pi)}. \tag{24}$$

We can now prove the main estimation of W^*.

Proposition 12 *For all $z \ge 0$, if $hk < n - 1$,*

$$\begin{aligned}
&\mathbf{E}\left[\exp(-zW^*)\right] \\
&\le \sqrt{8}e^{1/(2\pi)} \exp\left(-z\left(\left((n-1)k - hk^2\right)\left(Q(0) - Q(2\varepsilon)\right)\pi^{-1}\right)^{1/2}\right) \\
&\qquad \exp\left(z^2 2Q(0)(n-1-hk)\right).
\end{aligned}$$

Proof. For $\eta = \left(8Q\left(0\right)\left(\left(n-1\right)/k-h\right)\right)^{-1}$, by Lemma 11, (23), and (24) we have

$$\mathbf{E}\left[\exp\left(\eta \bar{M}^2\right)\right] \leq e^{1/(2\pi)}\sqrt{8}.$$

Since the function $g\left(x\right) = \exp\left(-zx - \eta x^2\right)$ is bounded by $\exp\left(z^2/\left(4\eta\right)\right)$ we obtain

$$\begin{aligned}\mathbf{E}\left[\exp\left(-z\bar{M}\right)\right] &= \mathbf{E}\left[\exp\left(-z\bar{M} - \eta \bar{M}^2\right)\exp\left(\eta \bar{M}^2\right)\right] \\ &\leq e^{1/(2\pi)}\sqrt{8}\exp\left(2z^2 Q\left(0\right)\left(\left(n-1\right)/k-h\right)\right)\end{aligned}$$

and the proposition follows by (20) and the value of μ and ν in Lemma 11 and (22). ∎

Evidently, for this proposition to be useful, a lower bound on $Q\left(0\right) - Q\left(2\varepsilon\right)$ is required. We thus make the following crucial assumption as a companion to Hypothesis (H):

(H') Assume that for the same α as in Hypothesis (H), there exists a constant K'_α such that for small x, $Q\left(0\right) - Q\left(x\right) \geq K'_\alpha \left|x\right|^\alpha$.

Remark 13 *Using Gaussian regularity theorems (see [1]) one can prove that assumption (H'), together with (H), imply that $x \mapsto \left(x^{\alpha/2}\log x^{-1}\right)^{1/2}$ is a modulus of continuity for $x \mapsto W\left(t,x\right)$, but that W is not β-Hölder-continuous for any $\beta > \alpha/2$.*

We obtain the following useful result.

Corollary 14 *Under Hypothesis (H'), for small ε, for all $z \geq 0$, if $hk < n-1$,*

$$\begin{aligned}&\mathbf{E}\left[\exp\left(-zW^*\right)\right] \\ &\leq e^{1/(2\pi)}\sqrt{8}\exp\left(-z\left(\left(\left(n-1\right)k - hk^2\right)\pi^{-1}K'_\alpha\left(2\varepsilon\right)^\alpha\right)^{1/2}\right)\exp\left(z^2 2Q\left(0\right)\left(n-1-hk\right)\right).\end{aligned}$$

4.4 The lower bound for $\tilde{u}$

Let $n, \kappa, \varepsilon, k = \left(n-1\right)f$ and h be fixed. We first estimate $\inf_{t\in[n-1,n]}\left[p\left(t,k\right)2^{-k}\mathcal{L}\left(A\left(t,k\right)\right)\right]$. To ensure that the assumptions in Lemma 8 and Proposition 9 are satisfied for all $t \in [n-1,n]$, we need:

C1 $(n-1)f = \left[\kappa\varepsilon^{-2}\left(n-1\right)\right]$,

C2 $\kappa\varepsilon^{-2}\left(1+h\right) \leq 2/3$

C3 $\kappa\varepsilon^{-2}f^{-1} \geq \kappa\varepsilon^{-2}h + 1/2$

Note that C1 implies $(n-1)f \leq \kappa\varepsilon^{-2}(n-1)$. Therefore $\kappa\varepsilon^{-2}f^{-1} \geq 1$. Assume for a moment that $\kappa\varepsilon^{-2}$ and $h\kappa\varepsilon^{-2}$ can be made small when κ is small. Our choices made below will show that this is indeed the case. Then C2 and C3 become trivial when C1 holds true. We make this assumption formally:

C0 $\kappa\varepsilon^{-2} \ll 1$, $h\kappa\varepsilon^{-2} \ll 1$.

We may now use Lemma 8 and Proposition 9 which imply

$$\begin{aligned}
&\inf_{t\in[n-1,n]}\left[p(t,k)\,2^{-k}\mathcal{L}(A(t,k))\right]\\
&\geq \inf_{t\in[n-1,n]}\left[c_0 b^{-1}e^{-b\kappa\varepsilon^{-2}t}\left(c_0c_1\kappa\varepsilon^{-2}t\right)^k/k!\cdot 2^{-k}\cdot c_2\left(\frac{c_0}{c_0'}\frac{h}{t}\right)^k k!\right]\\
&= e^{-b\kappa\varepsilon^{-2}n}c_0c_2b^{-1}\left(C_2\kappa\varepsilon^{-2}h\right)^{[\kappa\varepsilon^{-2}(n-1)]}\\
&\geq e^{-b\kappa\varepsilon^{-2}(n-1)}C_1\left(C_2\kappa\varepsilon^{-2}h\right)^{\kappa\varepsilon^{-2}(n-1)},
\end{aligned}\tag{25}$$

where $C_1 = c_0c_2/(be^b)$ and $C_2 = c_0^2c_1/(2c_0')$, and the last inequality follows from the fact that, by condition C0, $\kappa\varepsilon^{-2} < 1$ and $C_2\kappa\varepsilon^{-2}h < 1$.

Note that C_2 is a universal constant. Let λ be a fixed real number that we will choose later. Let

$$R^* = -\sup\left\{-R(s,t) : s\in A(n-1,k)\,; t\in[n-1,n]\right\}.$$

By inequalities (14) and (25), and using Chebyshev's and Schwarz's inequalities we obtain

$$\begin{aligned}
&\mathbf{P}\left[\inf_{t\in[n-1,n]}\tilde{u}(t) < e^{\lambda(n-1)}\right]\\
&\leq \mathbf{P}\left[e^{-b\kappa\varepsilon^{-2}(n-1)}C_1\left(C_2\kappa\varepsilon^{-2}h\right)^{\kappa\varepsilon^{-2}(n-1)}\exp(W^*)\exp(R^*) < e^{\lambda(n-1)}\right]\\
&= \mathbf{P}\Big[\exp(-W^*)\exp(-R^*)\\
&\qquad > C_1\exp\left(-\lambda(n-1) - b\kappa\varepsilon^{-2}(n-1) + \kappa\varepsilon^{-2}\log\left(C_2\kappa\varepsilon^{-2}h\right)(n-1)\right)\Big]\\
&\leq C_1^{-z}\mathbf{E}\left[\exp(-zW^*)\exp(-zR^*)\right]\\
&\qquad \exp\left(z\left(\lambda(n-1) + b\kappa\varepsilon^{-2}(n-1) - \kappa\varepsilon^{-2}\log\left(C_2\kappa\varepsilon^{-2}h\right)(n-1)\right)\right)\\
&\leq C_1^{-z}\mathbf{E}\left[\exp(-2zW^*)\right]^{1/2}\mathbf{E}\left[\exp(-2zR^*)\right]^{1/2}\\
&\qquad \exp\left(z(n-1)\left(\lambda + \left(b + \log\left(\left(C_2\kappa\varepsilon^{-2}h\right)^{-1}\right)\right)\kappa\varepsilon^{-2}\right)\right).
\end{aligned}\tag{26}$$

To apply Proposition 10 and Corollary 14, we must comply with the following additional conditions:

C4 $k\geq 1$, $h\geq 1$, and $z > (hkQ(0))^{-1/2}$

C5 $hk < n-1$.

In C4, $k = [\kappa\varepsilon^{-2}(n-1)] \geq 1$ for n large enough. For large n, we can also replace the condition on z by $z > 2\left(h\kappa\varepsilon^{-2}Q(0)\,n\right)^{-1/2}$. Since we are assuming that $h\kappa\varepsilon^{-2} \ll 1$, C5 is automatically satisfied. We can summarize the conditions C0–C5 that we need as:

D n large, $\kappa\varepsilon^{-2} \ll 1$, $h\kappa\varepsilon^{-2} \ll 1$, and $z > 2\left(h\kappa\varepsilon^{-2}Q(0)\,n\right)^{-1/2}$.

Assuming D, we may now use Proposition 10 and Corollary 14 (since R and $-R$ have the same distribution, as centered Gaussian processes). We use the bounds $\kappa\varepsilon^{-2}(n-1)-1<k\leq\kappa\varepsilon^{-2}(n-1)$.

Let $a<1$. Since we assume in **D** that we can make $h\kappa\varepsilon^{-2}$ arbitrarily small, for n large enough, we obtain:

$$
\begin{aligned}
&\mathbf{E}\left[\exp\left(-2zW^*\right)\right]\\
&\leq e^{\pi/2}\sqrt{8}\exp\left(-z\left(\left(\left(n-1\right)k-hk^2\right)K'_\alpha\pi^{-1}\left(2\varepsilon\right)^\alpha\right)^{1/2}+2z^2Q\left(0\right)\left(n-1-hk\right)\right)\\
&\leq e^{\pi/2}\sqrt{8}\exp\Bigg[-z\Bigg\{\left(\left(\left(n-1\right)\left(\kappa\varepsilon^{-2}\left(n-1\right)-1\right)-h\left(\kappa\varepsilon^{-1}\right)^2\left(n-1\right)^2\right)K'_\alpha\pi^{-1}\left(2\varepsilon\right)^\alpha\right)^{1/2}\\
&\qquad\qquad -2z\left(n-1\right)^2Q\left(0\right)\Bigg\}\Bigg]\\
&\leq e^{\pi/2}\sqrt{8}\exp\Bigg[-z\left(n-1\right)\Bigg\{\left(\left(\left(\kappa\varepsilon^{-2}-\left(n-1\right)^{-1}\right)-h\kappa\varepsilon^{-1}\kappa\varepsilon^{-1}\right)K'_\alpha\pi^{-1}\left(2\varepsilon\right)^\alpha\right)^{1/2}\\
&\qquad\qquad -2zQ\left(0\right)\Bigg\}\Bigg]\\
&\leq e^{\pi/2}\sqrt{8}\exp\left[-z\left(n-1\right)\left\{\left(a\kappa\varepsilon^{-2}K'_\alpha\pi^{-1}\left(2\varepsilon\right)^\alpha\right)^{1/2}-2zQ\left(0\right)\right\}\right].
\end{aligned}
$$

Similarly we have

$$
\mathbf{E}\left[\exp\left(-2zR^*\right)\right]\leq\exp\left[z\left(n-1\right)\left\{C_ua^{-1}\kappa\varepsilon^{-2}\left(hQ\left(0\right)\right)^{1/2}+2zQ\left(0\right)\kappa\varepsilon^{-2}h\right\}\right].
$$

Since we can choose $2^\alpha a>1$ we obtain from (26), still assuming D,

$$
\begin{aligned}
&\mathbf{P}\left[\inf_{t\in[n-1,n]}\tilde{u}\left(t\right)<e^{\lambda n}\right]\\
&\leq C_1^{-z}8^{1/4}\exp\left[-z\left(n-1\right)\left\{\begin{array}{c}\left(\kappa\varepsilon^{-2}\varepsilon^\alpha\pi^{-1}K'_\alpha\right)^{1/2}\\-\left(\lambda+\left(b+\log\left(\left(C_2\kappa\varepsilon^{-2}h\right)^{-1}\right)\right)\kappa\varepsilon^{-2}\right)\\-2zQ\left(0\right)\left(1+\kappa\varepsilon^{-2}h\right)-C_ua^{-1}\kappa\varepsilon^{-2}\left(hQ\left(0\right)\right)^{1/2}\end{array}\right\}\right].
\end{aligned}
\tag{27}
$$

In order to apply the Borel-Cantelli lemma, it is sufficient to ensure that the right-hand side is summable in n, to conclude that **P**-almost surely, there exists an $n_1(\omega)<\infty$ such that for all $n>n_1(\omega)$ we have $\inf_{t\in[n-1,n]}\tilde{u}(t)\geq e^{\lambda n}$. This would indeed imply that

$$
\tilde{\gamma}=\liminf_{t\to\infty}t^{-1}\log\tilde{u}\left(t\right)\geq\lambda.
$$

We may choose

$$
\lambda=\frac{1}{4}\left(\kappa\varepsilon^{-2}\varepsilon^\alpha\pi^{-1}K'_\alpha\right)^{1/2}.
$$

Since $z>2\left(h\kappa\varepsilon^{-2}Q\left(0\right)n\right)^{-1/2}$ is the only restriction on z, it can be made arbitrarily small for n large enough. Therefore the term involving $zQ(0)$ can be ignored, without requiring that z depend on n. Thus the term C_1^{-z} is independent of n, and can also be ignored. We now need to choose the value h. A good choice is such that $\kappa\varepsilon^{-2}h^{1/2}$ is a (small) fraction of $\left(\kappa\varepsilon^{-2}\varepsilon^{\alpha}\right)^{1/2}$. Therefore we can take

$$h = c\varepsilon^{\alpha+2}\kappa^{-1}$$

where c is a constant small enough that the term $C_u a^{-1}\kappa\varepsilon^{-2}\left(hQ\left(0\right)\right)^{1/2} \leq \lambda$. It is now apparent that the optimal choice for $\varepsilon = \varepsilon\left(\kappa\right)$ is such that $\kappa\varepsilon^{-2}\log\left(\left(C_2\kappa\varepsilon^{-2}h\right)^{-1}\right) = c'\left(\kappa\varepsilon^{-2}\varepsilon^{\alpha}\right)^{1/2}$ where c' is another small constant, chosen so that $\left(b + \log\left(\left(C_2\kappa\varepsilon^{-2}h\right)^{-1}\right)\right)\kappa\varepsilon^{-2} \leq \lambda$. Indeed the constant b is negligible compared to the log term since $\kappa\varepsilon^{-2}h \ll 1$. We can write our choice as

$$\kappa = \frac{c'^2\varepsilon^{\alpha+2}}{\left(\log\left(\left(cC_2\right)^{-1}\right) + \alpha\log\varepsilon^{-1}\right)^2}.$$

For small κ, the solution ε to this equation is unique, albeit non-explicit, and clearly, for any $\beta < \left(\alpha+2\right)^{-1}$, $\varepsilon = o\left(\kappa^{\beta}\right)$ so that we can rewrite our choice for ε as follows:

$$\kappa = c'\frac{\varepsilon^{\alpha+2}}{\left(\log\varepsilon^{-1}\right)^2},$$

where we abusively use the letter c' to denote a new constant that depends only on Q. Therefore our choice for h becomes:

$$h = \frac{c}{c'}\left(\log\varepsilon^{-1}\right)^2.$$

We now check that all the conditions we need are satisfied: for small κ, we do have $\kappa\varepsilon^{-2} = c'\varepsilon^{\alpha}\left(\log\varepsilon^{-1}\right)^{-2} \ll 1$, and $h\kappa\varepsilon^{-2} = c\varepsilon^{\alpha} \ll 1$. In other words there exist constants $\kappa_0 > 0$ and $n_0 < \infty$ that depend only on Q, such that for $\kappa < \kappa_0$ and $n > n_0$, condition D is satisfied. Then, up to a constant, the right-hand side of (27) becomes

$$\begin{aligned} &\exp\left[-z\left(n-1\right)\left\{\left(\kappa\varepsilon^{-2}\varepsilon^{\alpha}\pi^{-1}K'_{\alpha}\right)^{1/2} - 3\lambda\right\}\right] \\ &= \exp\left[-z\left(n-1\right)\frac{1}{4}\left(\kappa\varepsilon^{-2}\varepsilon^{\alpha}\pi^{-1}K'_{\alpha}\right)^{1/2}\right] \end{aligned}$$

which is summable in n. We have proved that there are deterministic constant C_{α} and κ_0 depending only on Q (and α) such that for $\kappa < \kappa_0$, and with ε uniquely defined by

$$\kappa = \varepsilon^{\alpha+2}\left(\log\varepsilon^{-1}\right)^{-2}, \tag{28}$$

the exponential behavior $\liminf_{t\to\infty}\frac{1}{t}\log\tilde{u}\left(t\right)$ is almost-surely bounded below by

$$\lambda = C_{\alpha}\left(\kappa\varepsilon^{-2+\alpha}\right)^{1/2} = C_{\alpha}\frac{\varepsilon^{\alpha}}{\log\varepsilon^{-1}}$$

4.5 Comparison with the error

¿From the error inequality (4), Corollary 5, and the calculation of the previous section, we have the existence of constants $C_\alpha, \kappa_0 > 0$ such that if $\kappa < \kappa_0$, with ε defined by (28) and for every B positive, $\mathbf{P}$-almost surely

$$\liminf_{t\to\infty} \frac{1}{t} \log u(t,x) \geq C_\alpha \varepsilon^\alpha / \log \varepsilon^{-1} - \max\left[2K_\alpha \varepsilon^\alpha - B\kappa\varepsilon^{-2}; BC_\alpha \kappa^{1+\alpha/2}\varepsilon^\alpha\right].$$

We need to choose B such that $2K_\alpha\varepsilon^\alpha \leq B\kappa\varepsilon^{-2}$. Thus say

$$B = 2K_\alpha \varepsilon^{\alpha+2}\kappa^{-1} = 2K_\alpha \left(\log \varepsilon^{-1}\right)^2.$$

Then the above max reduces to $BC_\alpha\kappa^{1+\alpha/2}\varepsilon^\alpha = C'_\alpha \left(\log \varepsilon^{-1}\right)^2 \kappa^{1+\alpha/2}\varepsilon^\alpha$ for some constant C'_α. This is negligible compared to λ. Indeed we saw that $\varepsilon = o\left(\kappa^\beta\right)$ for some $\beta > 0$, and therefore, for small κ, $\left(\log \varepsilon^{-1}\right)^3 \kappa^{1+\alpha/2} \ll 1$. Thus we can assert our final result.

Theorem 15 *There exist constants $C_\alpha > 0$ and $\kappa_0 > 0$ that depend only on Q, such that for $\kappa < \kappa_0$, and with ε defined by (28), almost surely,*

$$\liminf_{t\to\infty} \frac{1}{t} \log u(t,x) \geq C_\alpha \frac{\varepsilon^\alpha}{\log \varepsilon^{-1}}.$$

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