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Itô Formula and Local Time for the Fractional Brownian Sheet

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Abstract. Using the techniques of the stochastic calculus of variations for Gaussian processes, we derive an Itô formula for the fractional Brownian sheet with Hurst parameters bigger than 1/2. As an application, we give a stochastic integral representation for the local time of the fractional Brownian sheet.

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1 Introduction

Let $(B_t^\alpha)_{t\in[0,1]}$ be the fractional Brownian (fBm) motion with Hurst parameter $\alpha \in (0,1)$. When $\alpha = 1/2$, $B^{1/2}$ is the standard Brownian motion. When $\alpha \neq 1/2$, this process is not a semimartingale and in this case the powerful theory and tools of the classical Itô stochastic calculus cannot be applied to it. However, its special properties such as self-similarity, stationarity of increments, and long range dependence make this process a good candidate as a model for different phenomena. Therefore, a development of the stochastic calculus with respect to the fBm was needed. We refer to [1], [6], [9] and [11] for some approaches of this stochastic calculus.

The aim of this paper is to develop a stochastic calculus for the two-parameter fBm introduced in [3], also known as the fractional Brownian sheet. As the one-parameter fBm, the fractional Brownian sheet is not a semimartingale. But the Malliavin calculus for Gaussian processes can be adapted to it, Skorohod stochastic integrals can be defined, and it will allow us to derive an Itô formula when the Hurst parameters are bigger than 1/2 which extends the Itô formula for the two parameters martingales (see [10], [16] and [13] for a complete exposition of this topic). As an application we study the representation of the local time of the fractional Brownian sheet in terms of Skorohod stochastic integrals.

Our paper is organized as follows: Section 2 contains the basic notions of the Malliavin calculus for Gaussian processes and the definition of the fractional Brownian sheet. In Section 3 we give an analogue of the Doob-Meyer decomposition for the square of a martingale, and an Itô formula for the fractional Brownian sheet. The interpretation of a specific new term appearing in the Itô formula is given in Section 4. Finally, as an application of this formula, Section 5 contains the study of the local time of the fractional Brownian sheet using stochastic integrals and an occupation time formula.

2 Preliminaries

2.1 The Malliavin calculus for Gaussian processes

Let $T = [0,1]$. Consider $(B_t)_{t\in T}$ a centered Gaussian process, with covariance $\mathbf{E}(B_t B_s) = R(t,s)$.

We consider the canonical Hilbert space $\mathcal{H}$ associated with the process B, defined as the closure of the linear space generated by the function $\{1_{[0,t]}, t \in T\}$ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]}\rangle = R(t,s)$. Then the mapping $1_{[0,t]} \to B_t$ gives an isometry between $\mathcal{H}$ and the first chaos generated by $\{B_t, t \in T\}$ and $B(\phi)$ denotes the image of a element $\phi \in \mathcal{H}$.

We can develop a stochastic calculus of variations, or a Malliavin calculus, with respect to B. For a smooth $\mathcal{H}$-valued functional $F = f(B(\varphi_1), \cdots, B(\varphi_n))$ with $n \geq 1$, $f \in C_b^\infty(R^n)$ and $\varphi_1, \cdots, \varphi_n \in \mathcal{H}$ we put

$$D^B(F) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(B(\varphi_1), \cdots, B(\varphi_n))\varphi_j$$

and $D^B F$ will be closable from $L^p(\Omega)$ into $L^p(\Omega;\mathcal{H})$. Therefore, we can extend D^B to the closure of smooth functionals on the Sobolev space $D^{k,p}$ defined by the norm

$$\|F\|_{k,p}^p = \mathbf{E}|F|^p + \sum_{j=1}^{k} \|D^{B,j}F\|_{L^p(\Omega;\mathcal{H}^{\otimes j})}^p.$$

Similarly, for a given Hilbert space V we can define Sobolev spaces of V-valued random variables $D^{k,p}(V)$ (see [17]).

Consider the adjoint δ^B of D^B. Then its domain is the class of $u \in L^2(\Omega;\mathcal{H})$ such that

$$\mathbf{E}|\langle D^B F, u\rangle_{\mathcal{H}}| \leq C\|F\|_2$$

and $\delta^B(u)$ is the element of $L^2(\Omega)$ given by

$$\mathbf{E}(\delta^B(u)F) = \mathbf{E}\langle D^B F, u\rangle_{\mathcal{H}}$$

for every F smooth. It is well-known that $D^{1,2}(\mathcal{H})$ is included in the domain of δ^B.

Note that $\mathbf{E}\delta^B(u) = 0$ and the variance of the divergence operator δ^B is given by

$$\mathbf{E}\delta^B(u)^2 = \mathbf{E}\|u\|_{\mathcal{H}}^2 + \mathbf{E}\langle D^B u, (D^B u)^*\rangle_{\mathcal{H}\otimes\mathcal{H}} \tag{1}$$

where $(D^B u)^*$ is the adjoint of $D^B u$ in the Hilbert space $\mathcal{H}\otimes\mathcal{H}$. This last identity implies that

$$\mathbf{E}\delta^B(u)^2 \leq \mathbf{E}\|u\|_{\mathcal{H}}^2 + \mathbf{E}\|D^B u\|_{\mathcal{H}\otimes\mathcal{H}}^2. \tag{2}$$

We will need the property

$$F\delta^B(u) = \delta^B(Fu) + \langle D^B F, u\rangle_{\mathcal{H}} \tag{3}$$

if $F \in D^{1,2}$ and $u \in Dom(\delta^B)$ such that $Fu \in Dom(\delta^B)$.

2.2 Representation of fBm

Let now $B = B^\alpha$ be the fractional Brownian motion with parameter $\alpha \in (0,1)$. This process is centered Gaussian, $B_0 = 0$ and

$$R(t,s) = \frac{1}{2}\left(s^{2\alpha} + t^{2\alpha} - |t-s|^{2\alpha}\right). \tag{4}$$

We know that B admits a representation as Wiener integral of the form

$$B_t = \int_0^t K(t,s)dW_s, \tag{5}$$

where $W = \{W_t, t \in T\}$ is a Wiener process, and $K(t,s)$ is the kernel

$$K(t,s) = d_\alpha (t-s)^{\alpha-\frac{1}{2}} + s^{\alpha-\frac{1}{2}} F_1\left(\frac{t}{s}\right), \tag{6}$$

d_α being a constant and

$$F_1(z) = d_\alpha \left(\frac{1}{2} - \alpha\right) \int_0^{z-1} \theta^{\alpha - \frac{3}{2}} \left(1 - (\theta + 1)^{\alpha - \frac{1}{2}}\right) d\theta. \tag{7}$$

This kernel satisfies the condition (see [15]):

$$\frac{\partial K}{\partial t}(t,s) = d_\alpha(\alpha - \frac{1}{2})(\frac{s}{t})^{\frac{1}{2}-\alpha}(t-s)^{\alpha - \frac{3}{2}}. \tag{8}$$

We will denote by $\mathcal{H}^\alpha$ its canonical Hilbert space and by D^{B^α} and δ^{B^α} the corresponding Malliavin derivative and Skorohod integral with respect to B^α.

Fix now $\alpha \in (\frac{1}{2}, 1)$ and define the operator $K^\star$ from the set of step functions on T (denoted by $\mathcal{E}$) to $L^2(T)$ by

$$(K^\star f)(s) = \int_s^1 f(r) \frac{\partial K^\alpha}{\partial r}(r,s) dr.$$

It was proved in [1] that the space $\mathcal{H}^\alpha$ of the fBm can be represented as the closure of $\mathcal{E}$ with respect to the norm $\|f\|_{\mathcal{H}^\alpha} = \|K^\star f\|_{L^2(T)}$ and $K^\star$ is an isometry between $\mathcal{H}^\alpha$ and a closed subspace of $L^2(T)$. This fact implies the following relation between the Skorohod integration with respect to B^α and W, the Wiener process:

$$\delta^{B^\alpha}(u) = \delta^W(K^\star u) \tag{9}$$

for any $\mathcal{H}^\alpha$-valued random variable u in $Dom(\delta^{B^\alpha})$. We will call $\delta^{B^\alpha}(u)$ the Skorohod integral of the process u with respect to B. Since $\alpha > 1/2$, we have the following representation of the scalar product in $\mathcal{H}^\alpha$:

$$\langle f, g \rangle_{\mathcal{H}^\alpha} = \alpha(2\alpha - 1) \int_0^1 \int_0^1 f(a) g(b) |a-b|^{2\alpha - 2} da db. \tag{10}$$

This formula holds as long as the integral above is finite when f and g are replaced by $|f|$ and $|g|$.

2.3 Fractional Brownian sheet

Consider $(W^{\alpha,\beta}_{s,t})_{s,t\in[0,1]}$ a fractional Brownian sheet with parameters $\alpha, \beta \in (0,1)$. This process is Gaussian, it starts from 0 and its covariance is

$$\mathbf{E}\left(W^{\alpha,\beta}_{t,s} W^{\alpha,\beta}_{u,v}\right) = R^\alpha(t,u) R^\beta(s,v)$$

$$= \frac{1}{2}\left(t^{2\alpha} + u^{2\alpha} - |t-u|^{2\alpha}\right) \frac{1}{2}\left(s^{2\beta} + v^{2\beta} - |s-v|^{2\beta}\right).$$

This process self similar and with stationary increments and it admits a continuous version. We refer to [3] for the basic properties of $W^{\alpha,\beta}$. The fractional Brownian sheet with Hurst parameters $\alpha, \beta \in (0,1)$ can be defined as (see [4])

$$W^{\alpha,\beta}_{s,t} = \int_0^t \int_0^s K^\alpha(t,u) K^\beta(s,v) dW_{u,v}$$

where $(W_{u,v})_{u,v\in[0,1]}$ is the Brownian sheet.

The canonical Hilbert Space $\mathcal{H}^{\alpha,\beta}$ of the fractional Brownian sheet is the closure of linear space generated by the indicator functions on $[0,1]^2$ with respect to the scalar product

$$\langle 1_{[0,t]\times[0,s]}, 1_{[0,u]\times[0,v]}\rangle_{\mathcal{H}^{\alpha,\beta}} = R^{\alpha}(t,u)R^{\beta}(s,v).$$

Fix $\alpha,\beta > \frac{1}{2}$. Notice that in this case, by tensorization of (10), we have for every $f,g\in\mathcal{H}^{\alpha,\beta}$

$$\langle f,g\rangle_{\mathcal{H}^{\alpha,\beta}} = c(\alpha)c(\beta)\int_0^1\int_0^1\int_0^1\int_0^1 f(a,b)g(m,n)\left|a-m\right|^{2\alpha-2}\left|b-n\right|^{2\beta-n} dadbdmdn$$

and $c(\alpha) = \alpha(2\alpha-1)$. Define the operator $K^{*,2}_{\alpha,\beta}$ on the space of step function on $[0,1]^2$ to $L^2([0,1]^2)$ given by

$$\left(K^{*,2}_{\alpha,\beta}f\right)(s,t) = \int_t^1\int_s^1 f(r,r')\frac{\partial K^{\alpha}}{\partial r}(r,t)\frac{\partial K^{\beta}}{\partial r'}(r',s)drdr'. \tag{11}$$

We have

$$\langle K^{*,2}_{\alpha,\beta}f, K^{*,2}_{\alpha,\beta}g\rangle_{L^2([0,1]^2)} = \langle f,g\rangle_{\mathcal{H}^{\alpha,\beta}}. \tag{12}$$

Indeed,

$$\begin{aligned}
&\langle K^{*,2}_{\alpha,\beta}f, K^{*,2}_{\alpha,\beta}g\rangle_{L^2([0,1]^2)}\\
&= \int_0^1\int_0^1\left(\int_u^1\int_v^1 f(a,b)\frac{\partial K^{\alpha}}{\partial a}(a,u)\frac{\partial K^{\beta}}{\partial b}(b,v)dadb\right)\\
&\times\left(\int_u^1\int_v^1 g(m,n)\frac{\partial K^{\alpha}}{\partial m}(m,u)\frac{\partial K^{\beta}}{\partial n}(n,v)dmdn\right)dudv\\
&= \int_0^1\int_0^1\int_0^1\int_0^1 f(a,b)g(m,n)\left(\int_0^{a\wedge m}\int_0^{b\wedge n}\frac{\partial K^{\alpha}}{\partial m}(m,u)\frac{\partial K^{\beta}}{\partial n}(n,v)dmdndudv\right)dadbdmdn\\
&= \int_0^1\int_0^1\int_0^1\int_0^1 f(a,b)g(m,n)\frac{\partial^2 R^{\alpha}}{\partial a\partial m}(a,m)\frac{\partial^2 R^{\beta}}{\partial b\partial n}(b,n)\\
&= c(\alpha)c(\beta)\int_0^1\int_0^1\int_0^1\int_0^1 f(a,b)g(m,n)\left|a-m\right|^{2\alpha-2}\left|b-n\right|^{2\beta-n} dadbdmdn\\
&= \langle f,g\rangle_{\mathcal{H}^{\alpha,\beta}}.
\end{aligned}$$

One can develop a Malliavin stochastic calculus with respect to the fractional Brownian sheet following the method of Section 2.1, using rectangles in $[0,1]^2$ instead of intervals in $[0,1]$. Denote by $\delta^{W^{\alpha,\beta}}$ and $D^{W^{\alpha,\beta}}$ its associated Skorohod integral and Malliavin derivative respectively. A consequence of the identity (12) is that the following expression of the Skorohod integral $\delta^{W^{\alpha,\beta}}$ holds: For $f\in Dom(\delta^{W^{\alpha,\beta}})\subset L^2(\Omega,\mathcal{H}^{\alpha,\beta})$,

$$\begin{aligned}
\int_0^1\int_0^1 f(u,v)dW^{\alpha,\beta}_{u,v} &= \int_0^1\int_0^1 (K^{*,2}_{\alpha,\beta}f)(u,v)dW_{u,v}\\
&= \int_0^1\int_0^1\left(\int_u^1\int_v^1 f(a,b)\frac{\partial K^{\alpha}}{\partial a}(a,u)\frac{\partial K^{\beta}}{\partial b}(b,v)dadb\right)dW_{u,v}.
\end{aligned}$$

All properties of the anticipating Skorohod integration with respect to Gaussian processes will hold in this case.

Consider now the processes $s \to W^{\alpha,\beta}_{s,t}$ and $t \to W^{\alpha,\beta}_{s,t}$. These processes are real fractional Brownian motions with the same law as $t^{\beta}W^{\alpha}$ and $s^{\alpha}W^{\beta}$ respectively and with covariances

$$R_1(s_1,s_2) = t^{2\beta}\frac{1}{2}\left(s_1^{2\alpha}+s_2^{2\alpha}-|s_1-s_2|^{2\alpha}\right) \text{ and } R_2(t_1,t_2) = s^{2\alpha}\frac{1}{2}\left(t_1^{2\beta}+t_2^{2\beta}-|t_1-t_2|^{2\beta}\right) \tag{13}$$

respectively. Since these processes are fBm's themselves, we have the Malliavin calculus with respect to them. We denote by δ_1^t the Skorohod integral with respect to the fractional Brownian motion $s \to W^{\alpha,\beta}_{s,t}$ and by δ_2^s the Skorohod integral with respect to the fractional Brownian motion $t \to W^{\alpha,\beta}_{s,t}$. We will only use the integration by parts formulas

$$f(W^{\alpha,\beta}_{s,t})\delta_1^t(u) = \delta_1^t\left(f(W^{\alpha,\beta}_{s,t})u\right) + t^{2\beta}f'(W^{\alpha,\beta}_{s,t})\langle u, 1_{[0,s]}\rangle_{\mathcal{H}^\alpha} \tag{14}$$

and

$$f(W^{\alpha,\beta}_{s,t})\delta_2^s(u) = \delta_2^s\left(f(W^{\alpha,\beta}_{s,t})u\right) + s^{2\alpha}f'(W^{\alpha,\beta}_{s,t})\langle u, 1_{[0,t]}\rangle_{\mathcal{H}^\beta}. \tag{15}$$

3 The Itô formula for the fractional Brownian sheet

We will start this section with three technical Lemmas that will be used below, and that rely on the following exponential quadratic growth assumption on f and/or its derivatives:

(H) We say that a function g on $\mathbf{R}$ satisfies Condition (H) if, for $|x|$ large enough, we have $|g(x)| \le M \exp ax^2$ where a is a constant that depends only on α and/or β, and M is an arbitrary constant.

Lemma 1 *Let $s,t \in [0,1]$ and $\pi^1 : 0 = s_0 < s_1 < \ldots < s_n = s$ and $\pi^2 : 0 = t_0 < t_1 < \ldots < t_m = t$ be two partitions of the intervals $[0,s]$ and $[0,t]$ respectively. Assume f, f', and f'' satisfy Condition (H). Then we have the following convergences in $L^2(\Omega)$ when the partition meshes both tend to zero:*

1 *If $(W^\alpha)_{t\in[0,1]}$ is a fBm with parameter $\alpha \in (\frac{1}{2},1)$, then for any $f \in C_b^2(\mathbf{R})$,*

$$\sum_{j=0}^{m-1}\delta\left(f(W^\alpha_{t_j})t_j^{2\alpha}1_{[t_j,t_{j+1}]}(\cdot)\right) = \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} f(W^\alpha_{t_j})t_j^{2\alpha}dW^\alpha_u \to \int_0^t f(W^\alpha_u)u^{2\alpha}dW^\alpha_u. \tag{16}$$

2 *If $(W^{\alpha,\beta}_{s,t})_{s,t\in[0,1]}$ is a fractional Brownian sheet with parameters $\alpha,\beta \in (\frac{1}{2},1)$, then for any $f \in C_b^2(\mathbf{R})$,*

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\delta\left(1_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}(\cdot)f(W^{\alpha,\beta}_{s_i,t_j})\right) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}} f(W^{\alpha,\beta}_{s_i,t_j})dW^{\alpha,\beta}_{u,v} \tag{17}$$

$$\to \int_0^t\int_0^s f(W^{\alpha,\beta}_{u,v})dW^{\alpha,\beta}_{u,v}. \tag{18}$$

Proof: We use the notation $\Delta_j = [t_j, t_{j+1}]$. Concerning point 1, we prove first the convergence in $L^2(\Omega; \mathcal{H}^\alpha)$ of the sum

$$\sum_{j=0}^{m-1} \left(f(W_{t_j}^\alpha) t_j^{2\alpha} 1_{\Delta_j}(u) \right) \tag{19}$$

to the process

$$f(W_u^\alpha) u^{2\alpha} 1_{[0,t]}(u) = \sum_{j=0}^{m-1} f(W_u^\alpha) u^{2\alpha} 1_{\Delta_j}(u). \tag{20}$$

The L^2-norm of the difference D between the two terms can be estimated as follows

$$\begin{aligned} D &= \mathbf{E} \left| \sum_{j=0}^{m-1} \left(f(W_{t_j}^\alpha) t_j^{2\alpha} - f(W_\cdot^\alpha) (\cdot)^{2\alpha} \right) 1_{\Delta_j} \right|_{\mathcal{H}^\alpha}^2 \\ &\leq 2\mathbf{E} \left| \sum_{j=0}^{m-1} \left| f(W_{t_j}^\alpha) - f(W_\cdot^\alpha) \right| t_j^{2\alpha} 1_{\Delta_j} \right|_{\mathcal{H}^\alpha}^2 \\ &+ 2\mathbf{E} \left| \sum_{j=0}^{m-1} |f(W_\cdot^\alpha)| \, |(\cdot)^{2\alpha} - t_j^{2\alpha}| 1_{\Delta_j} \right|_{\mathcal{H}^\alpha}^2 \end{aligned}$$

We can write that $\| \sum_j a_j \|_{\mathcal{H}^\alpha}^2 \leq \sum_{j,k} |a_j|_{\mathcal{H}^\alpha} |a_k|_{\mathcal{H}^\alpha}$. Also we have the inequality

$$\left| f(W_{t_j}^\alpha) - f(W_u^\alpha) \right| \leq \sup_{x \in [0,1]} \left| f'(W_x^\alpha) \right| \left| W_{t_j}^\alpha - W_u^\alpha \right|.$$

This follows from the mean value theorem and the fact that W is almost-surely continuous. Since here we will have $u \in \Delta_j$, so that $|u - t_j| \leq \left| \pi^2 \right|$, it also follows that

$$\left| f(W_{t_j}^\alpha) - f(W_u^\alpha) \right| \leq \sup_{x \in [0,1]} \left| f'(W_x^\alpha) \right| \sup_{a,b: |a-b| \leq |\pi^2|} |W_a^\alpha - W_b^\alpha|.$$

Therefore, using Hölder's inequality,

$$\begin{aligned} D &\leq 2\mathbf{E} \left[\sup_{x \in [0,1]} \left| f'(W_x^\alpha) \right|^4 \right] \mathbf{E} \left[\sup_{a,b: |a-b| \leq |\pi^2|} |W_a^\alpha - W_b^\alpha|^4 \right] \sum_{j,k} \langle 1_{\Delta_j}, 1_{\Delta_k} \rangle_{\mathcal{H}^\alpha} \\ &+ 2\mathbf{E} \left[\sup_{x \in [0,1]} |f(W_x^\alpha)|^2 \right] |\pi^2|^{4\alpha} \sum_{j,k} \langle 1_{\Delta_j}, 1_{\Delta_k} \rangle_{\mathcal{H}^\alpha}. \end{aligned}$$

Since W is Gaussian and almost-surely continuous, the general Dudley-Fernique theory guarantees that $\mathbf{E} \left[\sup_{a,b: |a-b| \leq r} \mathbf{E} \, |W_a^\alpha - W_b^\alpha|^4 \right]$ converges to 0 as r tends to 0. To guarantee that D goes to zero as $|\pi^2| \to 0$, we only need to assume that condition (H) holds for f and

f' with $a < 8^{-1}Var(Z)^{-1}$ where $Z = \sup_{x\in[0,1]} |W^\alpha_x|$. Indeed, since Z is the supremum of a centered continuous Gaussian process, a classical result of Fernique implies that $\mathbf{E}\left[\exp aZ^2\right]$ is finite for $a < Var(Z)^{-1}2^{-1}$. Using the inequality (2), the proof of point 1 will be finished if we show that the derivative of (19) converges to the derivatives of (20). It holds that

$$D_s f(W^\alpha_{t_j}) = f'(W^\alpha_{t_j})1_{[0,t_j]}(s) \text{ and } D_s f(W^\alpha_u) = f'(W^\alpha_u)1_{[0,u]}(s)$$

and

$$\begin{aligned}
&\left|\sum_{j=0}^{m-1}\left(D_s f(W^\alpha_{t_j})t_j^{2\alpha} - D_s f(W^\alpha_u)u^{2\alpha}\right)1_{\Delta_j}(u)\right| \\
&\leq \mathbf{E}\left|\sum_{j=0}^{m-1}\left(f'(W^\alpha_{t_j}) - f'(W^\alpha_u)\right)1_{[0,t_j]}(s)t_j^{2\alpha}1_{\Delta_j}(u)\right| \\
&+\mathbf{E}\left|\sum_{j=0}^{m-1} f'(W^\alpha_u)(t_j^{2\alpha} - u^{2\alpha})1_{[0,t_j]}(s)1_{\Delta_j}(u)\right| \\
&+\mathbf{E}\left|\sum_{j=0}^{m-1} f'(W^\alpha_u)u^{2\alpha}1_{[t_j,u]}(s)1_{\Delta_j}(u)\right|.
\end{aligned}$$

We seek convergence of these three terms as functions of (s,u) in the space $(\mathcal{H}^\alpha)^{\otimes 2}$. The convergence of the first two terms to zero can be done using the bounds presented above in this proof, using Condition (H) on f' and f''. Finally, using the definition of the scalar product in $(\mathcal{H}^\alpha)^{\otimes 2}$, including the fact that we can use the inequality $1_{[t_j,u]}(s) \leq 1_{\Delta_j}(s)$, we obtain

$$\begin{aligned}
&\mathbf{E}\left|\sum_{j=0}^{m-1} f'(W^\alpha_\cdot)(\cdot)^{2\alpha}1_{[t_j,\cdot]}(\star)1_{\Delta_j}(\cdot)\right|^2_{(\mathcal{H}^\alpha)^{\otimes 2}} \\
&\leq \mathbf{E}\left[\sup_{x\in[0,1]}\left|f'(W^\alpha_x)\right|^2\right]\sum_{j,k}\langle 1_{\Delta_j}(\cdot), 1_{\Delta_k}(\cdot)\rangle_{\mathcal{H}^\alpha}\left\langle 1_{\Delta_j}(\star), 1_{\Delta_k}(\star)\right\rangle_{\mathcal{H}^\alpha} \\
&\leq \mathbf{E}\left[\sup_{x\in[0,1]}\left|f'(W^\alpha_x)\right|^2\right]\left(\sum_j \left\|1_{\Delta_j}\right\|^2_{\mathcal{H}^\alpha}\right)^2.
\end{aligned}$$

Condition (H) on f' guarantees that $\mathbf{E}\left[\sup_{x\in[0,1]}\left|f'(W^\alpha_x)\right|^2\right]$ is finite, and the other factor converges to 0 with $\left|\pi^2\right|$ since $\sum_j \|1_{\Delta_j}\|^2_{\mathcal{H}^\alpha} \leq \left|\pi^2\right|\sum_j \|1_{\Delta_j}\|_{\mathcal{H}^\alpha}$ converges to zero.

Concerning point 2, we need first to prove that

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} 1_{\Delta_i\times\Delta_j}(u,v)\left(f(W^{\alpha,\beta}_{s_i,t_j}) - f(W^{\alpha,\beta}_{u,v})\right) \tag{21}$$

goes to zero in $L^2\left(\Omega;\mathcal{H}^{\alpha,\beta}\right)$. We have, as before,

$$
\begin{aligned}
&\mathbf{E}\left|\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} 1_{\Delta_i\times\Delta_j}(u,v)\left(f(W^{\alpha,\beta}_{s_i,t_j})-f(W^{\alpha,\beta}_{u,v})\right)\right|^2_{\mathcal{H}^{\alpha,\beta}}\\
&\leq \mathbf{E}^{1/2}\left[\sup_{x,y\in[0,1]}\left|f'\left(W^{\alpha,\beta}_{x,y}\right)\right|^4\right]\mathbf{E}^{1/2}\left[\sup_{|a_1-b_1|\leq|\pi^1|,|a_2-b_2|\leq|\pi^2|}\left|W^{\alpha,\beta}_{a_1,a_2}-W^{\alpha,\beta}_{b_1,b_2}\right|^2\right]\\
&\quad\cdot\sum_{i,i',j,j'}\langle 1_{\Delta_i},1_{\Delta_{i'}}\rangle_{\mathcal{H}^\alpha}\langle 1_{\Delta_j},1_{\Delta_{j'}}\rangle_{\mathcal{H}^\beta}
\end{aligned}
$$

which, assuming condition (H) on f', converges to zero as $\left|\pi^1\right|$ and $\left|\pi^2\right|$ tend to zero, due to the almost-sure continuity of $W^{\alpha,\beta}$ as a centered Gaussian process on $[0,1]^2$. The convergence of the Malliavin derivatives of (21) is no more difficult than the other proofs above; we leave it to the reader. The proof of the lemma is completed. $\square$

The next lemma is trivial.

Lemma 2 *Let $s,t\in[0,1]$, π^1,π^2 as in Lemma 1 and A,B two finite variation processes. Then, for every $f\in C(\mathbf{R})$ and $g\in C(\mathbf{R}^2)$ the following convergences hold in $L^1(\Omega)$.*

$$\sum_{j=0}^{m-1} f(t_j)\left(B_{t_{j+1}}-B_{t_j}\right)\rightarrow\int_0^t f(u)dB_u \tag{22}$$

and

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} g(s_i,t_j)\left(A_{s_{i+1}}-A_{s_i}\right)\left(B_{t_{j+1}}-B_{t_j}\right)\rightarrow\int_0^s\int_0^t g(u,v)dA_u dB_v \tag{23}$$

The next lemma is useful for dealing with certain specific double Skorohod integrals that we will encounter.

Lemma 3 *For $s,t\in[0,1]$ and for fixed integers n and m, let the partitions π^1,π^2 be again as in Lemma 1. Denote $\pi=\pi_{n,m}=\left(\pi^1_n,\pi^2_m\right)$. Also let*

$$a^{\pi}_{(u_1,v_1);(u_2,v_2)}=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} 1_{[s_i,s_{i+1}]\times[0,t_j]}((u_1,v_1))1_{[0,s_i]\times[t_j,t_{j+1}]}((u_2,v_2)) \tag{24}$$

for every $u=(u_1,u_2), v=(v_1,v_2)\in[0,1]^2$. Then the sequence $(a^\pi)_\pi$ converges in $\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes 2}$ as the mesh of π tends to 0 (when n and m both tend to infinity) to the function a defined for every $(u_1,v_1),(u_2,v_2)\in[0,1]^2$ by:

$$a\left((u_1,v_1);(u_2,v_2)\right)=1_{[0,s]}\left(u_1\right)1_{[0,t]}\left(v_2\right)1_{[0,u_1]}\left(u_2\right)1_{[0,v_2]}\left(v_1\right).$$

Proof: We denote $s_0 = t_0 = 0$ and $\Delta_i = \Delta_i^n = [s_i; s_{i+1}]$, $\Delta_j = \Delta_j^m = [t_j; t_{j+1}]$, as well as $D_i = D_i^n = [0, s_i]$ and $D_j = D_j^m = [0, t_j]$. Although the names of these intervals are ambiguous, the names of their indices allow us to identify which variables they refer to, which lightens the notation. To further lighten notation, we allow $D_j(t)$ to denote the indicator function $1_{D_j}(t)$, and similarly for the other intervals. We only need to show that the doubly indexed sequence of functions

$$\begin{aligned}
&[a_{n,m}((u_1,v_1),(u_2,v_2)) - a((u_1,v_1),(u_2,v_2))] \\
&\cdot \left[a_{n,m}\left(\left(u_1',v_1'\right),\left(u_2',v_2'\right)\right) - a\left(\left(u_1',v_1'\right),\left(u_2',v_2'\right)\right)\right] \\
&= \left[\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} D_j(v_1)\Delta_i(u_1)D_i(u_2)\Delta_j(v_2) - a((u_1,v_1),(u_2,v_2))\right] \\
&\cdot \left[\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} D_j\left(v_1'\right)\Delta_i\left(u_1'\right)D_i\left(u_2'\right)\Delta_j\left(v_2'\right) - a\left(\left(u_1',v_1'\right),\left(u_2',v_2'\right)\right)\right] \qquad (25)
\end{aligned}$$

defined on $\left([0,1]^2 \times [0,1]^2\right)^2$ converges to 0 as m and n tend to infinity in the space $L^1(\mu)$ where μ is the measure defined by

$$\begin{aligned}
&\mu\left(du_1du_2dv_1dv_2du_1'du_2'dv_1'dv_2'\right) \\
&= \left|u_1-u_1'\right|^{2\alpha-2}\left|u_2-u_2'\right|^{2\alpha-2}\left|v_1-v_1'\right|^{2\beta-2}\left|v_2-v_2'\right|^{2\beta-2} du_1du_2dv_1dv_2du_1'du_2'dv_1'dv_2'.
\end{aligned}$$

We first note that, by virtue of the partitions, for every pair of points $(u_1,v_1),(u_2,v_2) \in [0,1]^2$, except for the countably many points in the partitions π^1 and π^2, we have convergence of $a_{n,m}((u_1,v_1);(u_2,v_2))$ to $a((u_1,v_1);(u_2,v_2))$. In other words, $a_{n,m}$ converges to a Lebesgue-almost everywhere in $[0,1]^2 \times [0,1]^2$. Thus the quantity in (25) converges in $\left([0,1]^2 \times [0,1]^2\right)^2$ almost everywhere to 0 with respect to the Lebesgue measure, and also with respect to the measure μ since μ is Lebesgue-absolutely continuous. Since the sequence in (25) is bounded by 1, and since 1 is integrable with respect to μ, by dominated convergence, the lemma is proved. □

We will also need the basic lemma for the convergence of the Skorohod integral.

Lemma 4 *Let u_n be a sequence of elements in Dom(δ) which converges to u in $L^2\left(\Omega;\mathcal{H}^{\alpha,\beta}\right)$. Suppose that $\delta(u_n)$ converges in $L^2(\Omega)$ to some square integrable random variable G. Then u belongs to the domain of δ and $\delta(u) = G$. This result also holds for sequences in $\mathcal{H}^\alpha$, or in $\mathcal{H}^\beta$, and their respective Skorohod integrals.*

3.1 A decomposition for $(W_{s,t}^{\alpha,\beta})^2$

Let, from now on, $s,t \in [0,1]$ and $\pi^1 : 0 = s_0 < s_1 < \ldots < s_n = s$ and $\pi^2 : 0 = t_0 < t_1 < \ldots < t_m = t$ be two partitions of the intervals $[0,s]$ and $[0,t]$ respectively. Without loss of generality, we can choose the dyadic partitions.

Proposition 1 *Let* $\left(W^{\alpha,\beta}_{s,t}\right)_{s,t\in[0,1]}$ *a fractional Brownian sheet with parameters* $\alpha,\beta \in (\frac{1}{2},1)$. *Then it holds*

$$\left(W^{\alpha,\beta}_{s,t}\right)^2 = 2\int_0^t\int_0^s W^{\alpha,\beta}_{u,v}dW^{\alpha,\beta}_{u,v} + 2\tilde{M}_{s,t} + s^{2\alpha}t^{2\beta} \tag{26}$$

where, with a *and* a^π *as in Lemma 3, and with* $\delta^{(2)}$ *the double Skorohod integral with respect to* $W^{\alpha,\beta}$ *on* $[0,1]^2\times[0,1]^2$,

$$\begin{aligned}\tilde{M}_{s,t} &:= \lim_{|\pi|\to 0}\delta^{(2)}\left[\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}1_{[s_i,s_{i+1}]\times[0,t_j]}(\cdot)1_{[0,s_i]\times[t_j,t_{j+1}]}(\star)\right] \\ &= \lim_{|\pi|\to 0}\delta^{(2)}(a^\pi) = \delta^{(2)}(a).\end{aligned} \tag{27}$$

Although this result is contained in the Itô formula that we prove independently in the next section, we give a detailed self-contained proof of this decomposition for $\left(W^{\alpha,\beta}\right)^2$ as a dydactic tool that enables us to introduce some of the essential ingredients and techniques used in proving Itô's formula, including the random field $\tilde{M}$. This section helps us make the next section more concise and easier to read.

Proof of the proposition: First write Taylor's formula

$$(W^{\alpha,\beta}_{s,t})^2 = 2\sum_{i=0}^{n-1}\bar{W}^{\alpha,\beta}_{s_i,t}\left(W^{\alpha,\beta}_{s_{i+1},t} - W^{\alpha,\beta}_{s_i,t}\right)$$

where $\bar{W}^{\alpha,\beta}_{s_i,t}$ is an arbitrary point located between $W^{\alpha,\beta}_{s_i,t}$ and $W^{\alpha,\beta}_{s_{i+1},t}$. Observe first that the $L^2(\Omega)$-limit as $|\pi^1|\to 0$ of the difference

$$\sum_{i=0}^{n-1}\left(\bar{W}^{\alpha,\beta}_{s_i,t} - W^{\alpha,\beta}_{s_i,t}\right)\left(W^{\alpha,\beta}_{s_{i+1},t} - W^{\alpha,\beta}_{s_i,t}\right)$$

is equal to 0. Indeed, by Hölder inequalities,

$$\begin{aligned}&\mathbf{E}\left|\sum_{i=0}^{n-1}\left(\bar{W}^{\alpha,\beta}_{s_i,t} - W^{\alpha,\beta}_{s_i,t}\right)\left(W^{\alpha,\beta}_{s_{i+1},t} - W^{\alpha,\beta}_{s_i,t}\right)\right|^2 \\ &\le \mathbf{E}\sum_{i,l=0}^{n-1}\left|W^{\alpha,\beta}_{s_{i+1},t} - W^{\alpha,\beta}_{s_i,t}\right|\left|W^{\alpha,\beta}_{s_{l+1},t} - W^{\alpha,\beta}_{s_l,t}\right|\left(W^{\alpha,\beta}_{s_{i+1},t} - W^{\alpha,\beta}_{s_i,t}\right)\left(W^{\alpha,\beta}_{s_{l+1},t} - W^{\alpha,\beta}_{s_l,t}\right) \\ &\le \sup_{|a-b|\le|\pi^1|}\left(\mathbf{E}\left|W^{\alpha,\beta}_{a,t} - W^{\alpha,\beta}_{b,t}\right|^2\right)^{\frac{1}{2}}\sum_{i,l=0}^{n-1}\left(\mathbf{E}\left|(W^{\alpha,\beta}_{s_{i+1},t} - W^{\alpha,\beta}_{s_i,t})(W^{\alpha,\beta}_{s_{l+1},t} - W^{\alpha,\beta}_{s_l,t})\right|^2\right)^{\frac{1}{2}} \\ &\le \sup_{|a-b|\le|\pi^1|}\left(\mathbf{E}\left|W^{\alpha,\beta}_{a,t} - W^{\alpha,\beta}_{b,t}\right|^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^{n-1}\left(\mathbf{E}\left|W^{\alpha,\beta}_{s_{i+1},t} - W^{\alpha,\beta}_{s_i,t}\right|^4\right)^{\frac{1}{4}}\right)^2 \le |\pi^1|^{4\alpha-2}\end{aligned}$$

and this goes to 0 when $\alpha > \frac{1}{2}$. Therefore it suffices to study the limit of the sum

$$T = 2\sum_{i=0}^{n-1} W^{\alpha,\beta}_{s_i,t}\left(W^{\alpha,\beta}_{s_{i+1},t} - W^{\alpha,\beta}_{s_i,t}\right).$$

Using that $g(t) = g(0) + \sum_{j=0}^{m-1}\left(g(t_{j+1}) - g(t_j)\right)$ we will obtain

$$\begin{aligned}
T &= 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[W^{\alpha,\beta}_{s_i,t_{j+1}}\left(W^{\alpha,\beta}_{s_{i+1},t_{j+1}} - W^{\alpha,\beta}_{s_i,t_{j+1}}\right) - W^{\alpha,\beta}_{s_i,t_j}\left(W^{\alpha,\beta}_{s_{i+1},t_j} - W^{\alpha,\beta}_{s_i,t_j}\right)\right] \\
&= 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} W^{\alpha,\beta}_{s_i,t_{j+1}}\left(W^{\alpha,\beta}_{s_{i+1},t_{j+1}} - W^{\alpha,\beta}_{s_i,t_{j+1}} - W^{\alpha,\beta}_{s_{i+1},t_j} + W^{\alpha,\beta}_{s_i,t_j}\right) \\
&+ 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(W^{\alpha,\beta}_{s_i,t_{j+1}} - W^{\alpha,\beta}_{s_i,t_j}\right)\left(W^{\alpha,\beta}_{s_{i+1},t_j} - W^{\alpha,\beta}_{s_i,t_j}\right) \\
&= 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} W^{\alpha,\beta}_{s_i,t_{j+1}}\delta\left(1_{\Delta_i\times\Delta_j}\right) + 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(W^{\alpha,\beta}_{s_i,t_{j+1}} - W^{\alpha,\beta}_{s_i,t_j}\right)\left(W^{\alpha,\beta}_{s_{i+1},t_j} - W^{\alpha,\beta}_{s_i,t_j}\right) \\
&= I + J
\end{aligned}$$

where we denoted by $\Delta_i = [s_i, s_{i+1}]$ and $\Delta_j = [t_j, t_{j+1}]$. Now, by property (3), the summand I becomes

$$\begin{aligned}
I &= 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\delta\left(W^{\alpha,\beta}_{s_i,t_{j+1}}1_{\Delta_i\times\Delta_j}\right) + 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\langle 1_{[0,s_i]\times[0,t_{j+1}]}, 1_{\Delta_i\times\Delta_j}\rangle_{\mathcal{H}^{\alpha,\beta}} \\
&= 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\delta\left(W^{\alpha,\beta}_{s_i,t_{j+1}}1_{\Delta_i\times\Delta_j}\right) + 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\langle 1_{[0,s_i]}, 1_{\Delta_i}\rangle_{\mathcal{H}^{\alpha}}\langle 1_{[0,t_{j+1}]}, 1_{\Delta_j}\rangle_{\mathcal{H}^{\beta}}.
\end{aligned}$$

By Lemma 1 point 2, and Lemma 2, we have the following convergences in $L^2(\Omega)$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\delta\left(W^{\alpha,\beta}_{s_i,t_{j+1}}1_{\Delta_i\times\Delta_j}\right) \to \int_0^t\int_0^s W^{\alpha,\beta}_{u,v}dW^{\alpha,\beta}_{u,v} \tag{28}$$

and

$$\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\langle 1_{[0,s_i]}, 1_{\Delta_i}\rangle_{\mathcal{H}^{\alpha}}\langle 1_{[0,t_{j+1}]}, 1_{\Delta_j}\rangle_{\mathcal{H}^{\beta}} \\
&= \frac{1}{4}\sum_i\left(s_{i+1}^{2\alpha} - s_i^{2\alpha} - (s_{i+1}-s_i)^{2\alpha}\right)\sum_j\left(t_{j+1}^{2\beta} - t_j^{2\beta} - (t_{j+1}-t_j)^{2\beta}\right) \to \frac{1}{4}s^{2\alpha}t^{2\beta}. \qquad (29)
\end{aligned}$$

The term J admits the following decomposition

$$J = 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\delta\left[\left(W^{\alpha,\beta}_{s_i,t_{j+1}} - W^{\alpha,\beta}_{s_i,t_j}\right)1_{\Delta_i\times[0,t_j]}\right] + 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\langle 1_{[0,s_i]}, 1_{\Delta_i}\rangle_{\mathcal{H}^\alpha}\langle 1_{[0,t_j]}, 1_{\Delta_j}\rangle_{\mathcal{H}^\beta}$$

$$= 2M(\pi) + 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\langle 1_{[0,s_i]}, 1_{\Delta_i}\rangle_{\mathcal{H}^\alpha}\langle 1_{[0,t_j]}, 1_{\Delta_j}\rangle_{\mathcal{H}^\beta}$$

where

$$M(\pi) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\delta\left[\left(W^{\alpha,\beta}_{s_i,t_{j+1}} - W^{\alpha,\beta}_{s_i,t_j}\right)1_{\Delta_i\times[0,t_j]}\right]$$

$$= \delta^{(2)}\left[\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}1_{\Delta_i\times[0,t_j]}(\cdot)1_{[0,s_i]\times\Delta_j}(\star)\right] = \delta^{(2)}(a^\pi) \tag{30}$$

where $\delta^{(2)}$ denotes the double Skorohod integral and the process a^π is defined in Lemma 3. As before, we can show that

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\langle 1_{[0,s_i]}, 1_{\Delta_i}\rangle_{\mathcal{H}^\alpha}\langle 1_{[0,t_j]}, 1_{\Delta_j}\rangle_{\mathcal{H}^\beta} \to_{|\pi|\to 0} \frac{1}{4}s^{2\alpha}t^{2\beta} \text{ in } L^2 . \tag{31}$$

Lemma 4 can be used in combination with Lemma 3 to assert immediately that $\delta^{(2)}(a^\pi)$ converges in $L^2(\Omega)$ to $\delta^{(2)}(a)$; indeed the convergence of the deterministic integrands in $\left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes 2}$ is in Lemma 3, and since the corresponding Skorohod integrals are equal to $(W^{\alpha,\beta}_{s,t})^2$ minus other terms that were already proved to converge in $L^2(\Omega)$, they converge in $L^2(\Omega)$. Note that Lemma 4 is not stated for the convergence of the double Skorohod integral, but the simple one; however it can be still applied because a double integral can be written as a simple one in which the integrand is convergent in $L^2(\Omega;\mathcal{H}^{\alpha,\beta})$. Combining this convergence and (28), (29), (31), we find the result of the proposition. □

3.2 Itô formula

In the sequel of this article we will make use of the notation $d_u W^{\alpha,\beta}_{u,v}$ to denote the differential of the fBm given by $u \mapsto W^{\alpha,\beta}_{u,v}$ for fixed v, and similarly for $d_v W^{\alpha,\beta}_{u,v}$. As before, $dW^{\alpha,\beta}_{u,v}$ denote the differential of the fractional Brownian sheet with respect to both parameters.

Theorem 1 *Let $f \in C^4(R)$ and $(W^{\alpha,\beta}_{s,t})_{s,t\in[0,1]}$ a fractional Brownian sheet with $\alpha, \beta \in$*

$(\frac{1}{2}, 1)$. Also assume that f, f', f'', f''' and $f^{(iv)}$ satisfy Condition (H). Then it holds

$$\begin{aligned} f(W^{\alpha,\beta}_{s,t}) = f(0) &+ \int_0^t \int_0^s f'(W^{\alpha,\beta}_{u,v}) dW^{\alpha,\beta}_{u,v} \\ &+ 2\alpha\beta \int_0^t \int_0^s f''(W^{\alpha,\beta}_{u,v}) u^{2\alpha-1} v^{2\beta-1} du dv + \int_0^t \int_0^s f''(W^{\alpha,\beta}_{u,v}) d\tilde{M}_{u,v} \\ &+ \alpha \int_0^t \int_0^s f'''(W^{\alpha,\beta}_{u,v}) u^{2\alpha-1} v^{2\beta} du d_v W^{\alpha,\beta}_{u,v} + \beta \int_0^t \int_0^s f'''(W^{\alpha,\beta}_{u,v}) u^{2\alpha} v^{2\beta-1} d_u W^{\alpha,\beta}_{u,v} dv \\ &+ \alpha\beta \int_0^t \int_0^s f^{iv}(W^{\alpha,\beta}_{u,v}) u^{4\alpha-1} v^{4\beta-1} du dv \end{aligned} \tag{32}$$

where, by definition,

$$\int_0^t \int_0^s f''(W^{\alpha,\beta}_{u,v}) d\tilde{M}_{u,v} = \lim_{|\pi| \to 0} \delta^{(2)} \left[\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f'' \left(W^{\alpha,\beta}_{s_i,t_j} \right) 1_{[0,s_i] \times [t_j, t_{j+1}]}(\cdot) 1_{[s_i, s_{i+1}] \times [0, t_j]}(\star) \right]$$

where the last limit exists in $L^2(\Omega)$. More specifically, we have the following formula, which can be taken as a definition of the integral with respect to $\tilde{M}$ which would otherwise only be a formal notation:

$$\int_0^t \int_0^s f''(W^{\alpha,\beta}_{u,v}) d\tilde{M}_{u,v} := \delta^{(2)}(N) \tag{33}$$

where for every $(u_1, v_1), (u_2, v_2) \in [0,1]^2$ the function N is defined by:

$$N((u_1, v_1); (u_2, v_2)) = f'' \left(W^{\alpha,\beta}_{u_1, v_2} \right) 1_{[0,s]}(u_1) 1_{[0,t]}(v_2) 1_{[0,u_1]}(u_2) 1_{[0,v_2]}(v_1). \tag{34}$$

Proof: Let $s, t \in [0,1]$ and write Taylor's formula with fixed t. It holds that

$$\begin{aligned} f\left(W^{\alpha,\beta}_{s,t}\right) = f(0) &+ \sum_{i=0}^{n-1} f'\left(W^{\alpha,\beta}_{s_i,t}\right) \left(W^{\alpha,\beta}_{s_{i+1},t} - W^{\alpha,\beta}_{s_i,t}\right) \\ &+ \sum_{i=0}^{n-1} f''\left(\bar{W}^{\alpha,\beta}_{s_i,t}\right) \left(W^{\alpha,\beta}_{s_{i+1},t} - W^{\alpha,\beta}_{s_i,t}\right)^2 \end{aligned}$$

where $W^{\bar{\alpha},\beta}_{s_i,t}$ is a point on the segment from $W^{\alpha,\beta}_{s_{i+1},t}$ to $W^{\alpha,\beta}_{s_i,t}$. First, we can prove that the last term goes to zero in $L^2(\Omega)$. Indeed, by Hölder's inequality,

$$
\begin{aligned}
&\mathbf{E}\left|\sum_i f''\left(W^{\bar{\alpha},\beta}_{s_i,t}\right)\left(W^{\alpha,\beta}_{s_{i+1},t}-W^{\alpha,\beta}_{s_i,t}\right)^2\right|^2 \\
&\leq \mathbf{E}\sum_{i,l=0}^{n-1}\left|f''\left(W^{\bar{\alpha},\beta}_{s_i,t}\right)f''\left(W^{\bar{\alpha},\beta}_{s_l,t}\right)\right|\left(W^{\alpha,\beta}_{s_{i+1},t}-W^{\alpha,\beta}_{s_i,t}\right)^2\left(W^{\alpha,\beta}_{s_{l+1},t}-W^{\alpha,\beta}_{s_l,t}\right)^2 \\
&\leq \mathbf{E}^{1/2}\left[\sup_{x\in[0,1]^2}\left|f''\left(W^{\alpha,\beta}_x\right)\right|^4\right]\sum_{i,l}\mathbf{E}^{1/4}\left[\left|W^{\alpha,\beta}_{s_{i+1},t}-W^{\alpha,\beta}_{s_i,t}\right|^8\right]\mathbf{E}^{1/4}\left[\left|W^{\alpha,\beta}_{s_{l+1},t}-W^{\alpha,\beta}_{s_l,t}\right|^8\right] \\
&\leq CK\sum_{i,l}\left|s_{i+1}-s_i\right|^{2\alpha}\left|s_{l+1}-s_l\right|^{2\alpha}
\end{aligned}
$$

where K is a constant obtained by using Condition (H) on f'', C is a universal constant; this all tends clearly to zero since $2\alpha > 1$. It remains to study the limit of the term

$$T=\sum_{i=0}^{n-1}f'\left(W^{\alpha,\beta}_{s_i,t}\right)\left(W^{\alpha,\beta}_{s_{i+1},t}-W^{\alpha,\beta}_{s_i,t}\right).$$

Writing the fact that $g(t)=g(0)+\sum_{j=0}^{m-1}\left(g(t_{j+1})-g(t_j)\right)$ we get

$$
\begin{aligned}
T &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f'\left(W^{\alpha,\beta}_{s_i,t_{j+1}}\right)\left(W^{\alpha,\beta}_{s_{i+1},t_{j+1}}-W^{\alpha,\beta}_{s_i,t_{j+1}}\right)-f'\left(W^{\alpha,\beta}_{s_i,t_j}\right)\left(W^{\alpha,\beta}_{s_{i+1},t_j}-W^{\alpha,\beta}_{s_i,t_j}\right)\right] \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f'\left(W^{\alpha,\beta}_{s_i,t_{j+1}}\right)\left(W^{\alpha,\beta}_{s_{i+1},t_{j+1}}-W^{\alpha,\beta}_{s_i,t_{j+1}}-W^{\alpha,\beta}_{s_{i+1},t_j}+W^{\alpha,\beta}_{s_i,t_j}\right) \\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(f'\left(W^{\alpha,\beta}_{s_i,t_{j+1}}\right)-f'\left(W^{\alpha,\beta}_{s_i,t_j}\right)\right)\left(W^{\alpha,\beta}_{s_{i+1},t_{j+1}}-W^{\alpha,\beta}_{s_i,t_{j+1}}\right) \\
&= A+B.
\end{aligned}
$$

3.2.1 The estimation of the term A

Note that

$$\left(W^{\alpha,\beta}_{s_{i+1},t_{j+1}}-W^{\alpha,\beta}_{s_i,t_{j+1}}-W^{\alpha,\beta}_{s_{i+1},t_j}+W^{\alpha,\beta}_{s_i,t_j}\right)=\delta\left(1_{\Delta_i\times\Delta_j}(\cdot)\right)$$

and by the properties of the Skorohod integral we obtain

$$\begin{aligned}A &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \delta\left[f'\left(W^{\alpha,\beta}_{s_i,t_{j+1}}\right) 1_{\Delta_i\times\Delta_j}(\cdot)\right] \\ &+ \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f''\left(W^{\alpha,\beta}_{s_i,t_{j+1}}\right) \langle 1_{[0,s_i]\times[0,t_{j+1}]}, 1_{\Delta_i\times\Delta_j}\rangle_{\mathcal{H}^{\alpha,\beta}} \\ &= A_1 + A_2.\end{aligned}$$

The convergence of A_1 follows from Lemma 1 point 2, assuming condition (H) for f', f'' and f''', yielding

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \delta\left[f'\left(W^{\alpha,\beta}_{s_i,t_{j+1}}\right) 1_{\Delta_i\times\Delta_j}(\cdot)\right] \to_{L^2(\Omega)} \int_0^t\int_0^s f'(W^{\alpha,\beta}_{u,v})dW^{\alpha,\beta}_{u,v}. \tag{35}$$

Concerning the term A_2, we can write

$$\begin{aligned}A_2 &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f''\left(W^{\alpha,\beta}_{s_i,t_{j+1}}\right) \langle 1_{[0,s_i]}, 1_{\Delta_i}\rangle_{\mathcal{H}^{\alpha}} \langle 1_{[0,t_{j+1}]}, 1_{\Delta_j}\rangle_{\mathcal{H}^{\beta}} \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f''\left(W^{\alpha,\beta}_{s_i,t_{j+1}}\right) \frac{1}{4}\left(s_{i+1}^{2\alpha} - s_i^{2\alpha} - (s_{i+1}-s_i)^{2\alpha}\right)\left(t_{j+1}^{2\beta} - t_j^{2\beta} - (t_{j+1}-t_j)^{2\beta}\right)\end{aligned}$$

and as above this converges, due to Lemma 2, to

$$\int_0^t f''\left(W^{\alpha,\beta}_{u,v}\right) dR^{\alpha}_u dR^{\beta}_v = \alpha\beta \int_0^t f''\left(W^{\alpha,\beta}_{u,v}\right) u^{2\alpha-1}v^{2\beta-1}dudv \tag{36}$$

where $R^{\alpha}_u = R^{\alpha}(u,u) = u^{2\alpha}$ and $R^{\beta}_v = R^{\beta}(v,v) = v^{2\beta}$. due to Lemma 2.

3.2.2 The estimation of the term B

Since

$$f'\left(W^{\alpha,\beta}_{s_i,t_{j+1}}\right) - f'\left(W^{\alpha,\beta}_{s_i,t_j}\right) = f''\left(W^{\bar{\alpha},\beta}_{s_i,t_j}\right)\left(W^{\alpha,\beta}_{s_i,t_{j+1}} - W^{\alpha,\beta}_{s_i,t_j}\right)$$

where $W^{\bar{\alpha},\beta}_{s_i,t_j}$ is a point located between $W^{\alpha,\beta}_{s_i,t_j}$ and $W^{\alpha,\beta}_{s_i,t_{j+1}}$, the term B becomes

$$B = \sum_{i,j} f''\left(W^{\bar{\alpha},\beta}_{s_i,t_j}\right)\left(W^{\alpha,\beta}_{s_i,t_{j+1}} - W^{\alpha,\beta}_{s_i,t_j}\right)\left(W^{\alpha,\beta}_{s_{i+1},t_j} - W^{\alpha,\beta}_{s_i,t_j}\right)$$

and using the argument as above, assuming Condition (H) for f''', we can prove that the limit of B is the same as the limit of the term

$$B' = \sum_{i,j} f''\left(W^{\alpha,\beta}_{s_i,t_j}\right)\left(W^{\alpha,\beta}_{s_i,t_{j+1}} - W^{\alpha,\beta}_{s_i,t_j}\right)\left(W^{\alpha,\beta}_{s_{i+1},t_j} - W^{\alpha,\beta}_{s_i,t_j}\right).$$

It holds that

$$B' = \sum_{i,j} f''\left(W^{\alpha,\beta}_{s_i,t_j}\right) \delta\left[1_{[0,s_i]\times\Delta_j}\left(W^{\alpha,\beta}_{s_{i+1},t_j} - W^{\alpha,\beta}_{s_i,t_j}\right)\right]$$

$$+ \sum_{i,j} f''\left(W^{\alpha,\beta}_{s_i,t_j}\right) \langle 1_{[0,s_i]}, 1_{\Delta_i}\rangle_{\mathcal{H}^\alpha} \langle 1_{[0,t_{j+1}]}, 1_{\Delta_j}\rangle_{\mathcal{H}^\beta} = B_1 + B_2.$$

The second term is similar with the term A_2 studied before and we have its convergence to

$$\int_0^t f''\left(W^{\alpha,\beta}_{u,v}\right) dR^\alpha_u dR^\beta_v = \alpha\beta \int_0^t f''\left(W^{\alpha,\beta}_{u,v}\right) u^{2\alpha-1} v^{2\beta-1} du dv. \tag{37}$$

For the term B_1 we can write

$$\begin{aligned}
B_1 &= \sum_{i,j} f''\left(W^{\alpha,\beta}_{s_i,t_j}\right) \delta_2^{s_i}\left[1_{\Delta_j}(\cdot)\left(W^{\alpha,\beta}_{s_{i+1},t_j} - W^{\alpha,\beta}_{s_i,t_j}\right)\right] \\
&= \sum_{i,j} \delta_2^{s_i}\left[f''\left(W^{\alpha,\beta}_{s_i,t_j}\right) 1_{\Delta_j}(\cdot)\left(W^{\alpha,\beta}_{s_{i+1},t_j} - W^{\alpha,\beta}_{s_i,t_j}\right)\right] \\
&+ \sum_{i,j} f'''\left(W^{\alpha,\beta}_{s_i,t_j}\right)\left(W^{\alpha,\beta}_{s_{i+1},t_j} - W^{\alpha,\beta}_{s_i,t_j}\right) s_i^{2\alpha} \frac{1}{2}\left(t_{j+1}^{2\beta} - t_j^{2\beta} - (t_{j+1}-t_j)^{2\beta}\right) \\
&= \sum_{i,j} \delta_2^{s_i}\left[f''\left(W^{\alpha,\beta}_{s_i,t_j}\right) \delta_1^{t_j}\left(1_{\Delta_i}(\star)\right) 1_{\Delta_j}(\cdot)\left(W^{\alpha,\beta}_{s_{i+1},t_j} - W^{\alpha,\beta}_{s_i,t_j}\right)\right] \\
&+ \sum_{i,j} f'''\left(W^{\alpha,\beta}_{s_i,t_j}\right) \delta_1^{t_j}\left(1_{\Delta_i}(\star)\right) s_i^{2\alpha} \frac{1}{2}\left(t_{j+1}^{2\beta} - t_j^{2\beta} - (t_{j+1}-t_j)^{2\beta}\right)
\end{aligned}$$

and by applying the integration by parts several times for the Skorohod integral with respect to the corresponding fractional Brownian motion, we obtain

$$\begin{aligned}
B_1 &= \sum_{i,j} \delta_2^{s_i}\delta_1^{t_j}\left[f''\left(W^{\alpha,\beta}_{s_i,t_j}\right) 1_{\Delta_j}(\cdot) 1_{\Delta_i}(\star)\right] \\
&+ \sum_{i,j} \delta_2^{s_i}\left[f'''\left(W^{\alpha,\beta}_{s_i,t_j}\right) 1_{\Delta_j}(\cdot)\right] \frac{1}{2}\left(s_{i+1}^{2\alpha} - s_i^{2\alpha} - (s_{i+1}-s_i)^{2\alpha}\right) t_j^{2\beta} \\
&+ \sum_{i,j} \delta_1^{t_j}\left[f'''\left(W^{\alpha,\beta}_{s_i,t_j}\right) 1_{\Delta_i}(\star) s_i^{2\alpha} \frac{1}{2}\left(t_{j+1}^{2\beta} - t_j^{2\beta} - (t_{j+1}-t_j)^{2\beta}\right)\right] \\
&+ \sum_{i,j} f^{(iv)}(W^{\alpha,\beta}_{s_i,t_j}) s_i^{2\alpha} \frac{1}{2}\left(t_{j+1}^{2\beta} - t_j^{2\beta} - (t_{j+1}-t_j)^{2\beta}\right) \frac{1}{2}\left(s_{i+1}^{2\alpha} - s_i^{2\alpha} - (s_{i+1}-s_i)^{2\alpha}\right) t_j^{2\beta} \\
&= B_{11} + B_{12} + B_{13} + B_{14}.
\end{aligned}$$

For the summand B_{12} we have

$$\begin{aligned}
B_{12} &= \frac{1}{2}\sum_{i,j} \delta_2^{s_i}\left[f'''\left(W^{\alpha,\beta}_{s_i,t_j}\right) t_j^{2\beta} 1_{\Delta_j}(\cdot)\right]\left(s_{i+1}^{2\alpha} - s_i^{2\alpha}\right) \\
&+ \frac{1}{2}\sum_{i,j} \delta_2^{s_i}\left[f'''\left(W^{\alpha,\beta}_{s_i,t_j}\right) t_j^{2\beta} 1_{\Delta_j}(\cdot)\right](s_{i+1}-s_i)^{2\alpha}.
\end{aligned}$$

By Lemma 1 point 1, assuming condition (H) for $f'''$and $f^{(iv)}$, it holds that, for every s_i

$$\sum_{j=0}^{m-1} \delta_2^{s_i} \left[f''' \left(W^{\alpha,\beta}_{s_i,t_j} \right) t_j^{2\beta} 1_{\Delta_j}(\cdot) \right] \to_{L^2(\Omega)} \int_0^t f''' \left(W^{\alpha,\beta}_{s_i,v} \right) v^{2\beta} d_v W^{\alpha,\beta}_{s_i,v}$$

and this fact together with Lemma 2 implies that the first part of B_{12} goes to the integral $\alpha \int_0^s \int_0^t f''' \left(W^{\alpha,\beta}_{u,v} \right) u^{2\alpha-1} du v^{2\beta} d_v W^{\alpha,\beta}_{u,v}$. Using Condition (H) on f''', since $2\alpha > 1$ it is easy to observe that the last part of B_{12} converges to zero and therefore we have the convergence

$$B_{12} \to \alpha \int_0^s \int_0^t f''' \left(W^{\alpha,\beta}_{u,v} \right) u^{2\alpha-1} du v^{2\beta} d_v W^{\alpha,\beta}_{u,v}. \tag{38}$$

In the same way

$$B_{13} \to \beta \int_0^s \int_0^t f''' \left(W^{\alpha,\beta}_{u,v} \right) u^{2\alpha} v^{2\beta-1} d_u W^{\alpha,\beta}_{u,v} dv. \tag{39}$$

Lemma 2 also gives that $B_{1,4}$ converges to

$$\frac{1}{4} \int_0^s \int_0^t f^{(iv)}(W^{\alpha,\beta}_{u,v}) u^{2\alpha} v^{2\beta} dR^{\alpha}(u) dR^{\beta}(v) = \alpha\beta \int_0^s \int_0^t f^{(iv)}(W^{\alpha,\beta}_{u,v}) u^{4\alpha-1} v^{4\beta-1} du dv. \tag{40}$$

Now, note that

$$B_{11} = \sum_{i,j} \delta^{(2)} \left[f'' \left(W^{\alpha,\beta}_{s_i,t_j} \right) 1_{[0,s_i]\times\Delta_j}(\cdot) 1_{\Delta_i \times [0,t_j]}(\star) \right]$$

where $\delta^{(2)}$ denotes again the double Skorohod integral, and specifically,

$$\delta^{(2)} \left[f'' \left(W^{\alpha,\beta}_{s_i,t_j} \right) 1_{[0,s_i]\times\Delta_j}(\cdot) 1_{\Delta_i \times [0,t_j]}(\star) \right] = \int_0^{s_i} \int_{t_j}^{t_{j+1}} \int_{s_i}^{s_{i+1}} \int_0^{t_j} f''(W^{\alpha,\beta}_{s_i,t_j}) dW^{\alpha,\beta}_{u_1,v_1} dW^{\alpha,\beta}_{u_2,v_2}.$$

We now show

Lemma 5 *Using the same partitions as in Lemma 3, define the process*

$$N^{\pi}_{(u_1,v_1);(u_2,v_2)} = \sum_{i,j} f'' \left(W^{\alpha,\beta}_{s_i,t_j} \right) 1_{\Delta_i \times [0,t_j]}((u_1,v_1)) 1_{[0,s_i]\times\Delta_j}((u_2,v_2)). \tag{41}$$

Then assuming condition (H) for f'' the sequence $(N^{\pi})_{\pi}$ converges in $L^2 \left(\Omega; \left(\mathcal{H}^{\alpha,\beta} \right)^{\otimes 2} \right)$ to the function N defined in the statement of Theorem 1.

Proof: The proof uses Condition (H) combined with the proof of Lemma 3. We only need to establish the convergence in $\left(\mathcal{H}^{\alpha,\beta} \right)^{\otimes 2}$ for fixed randomness $\omega \in \Omega$. Indeed, the convergence in $L^2 \left(\Omega; \left(\mathcal{H}^{\alpha,\beta} \right)^{\otimes 2} \right)$ follows by dominated convergence under condition (H) for f'', as long as $a < 2^{-1} Var\left(\max_{[0,1]^2} \left|W^{\alpha,\beta}\right|\right)^{-1}$ for instance. Now, using the almost-everywhere convergence established in proving Lemma 3, and noticing that we can rewrite N^{π} as

$$N^{\pi}_{(u_1,v_1);(u_2,v_2)} = \left(\sum_{i,j} f''\left(W^{\alpha,\beta}_{s_i,t_j}\right) 1_{\Delta_i}(u_1) 1_{\Delta_j}(v_2)\right) \cdot \left(\sum_{i,j} 1_{\Delta_i \times [0,t_j]}((u_1,v_1)) 1_{[0,s_i]\times\Delta_j}((u_2,v_2))\right),$$

we only need to show that for almost every $\omega \in \Omega$, for fixed u_1, v_2, the quantity

$$\sum_{i,j} f''\left(W^{\alpha,\beta}_{s_i,t_j}(\omega)\right) 1_{\Delta_i}(u_1) 1_{\Delta_j}(v_2)$$

converges to $f''\left(W^{\alpha,\beta}_{u_1,v_2}(\omega)\right)$. Since $W^{\alpha,\beta}$ is almost-surely continuous on $[0,1]^2$, this convergence is trivial. The lemma is proved. □

To guarantee that Lemma 4 can be invoked to conclude that the term

$$\delta^{(2)}\left(N^{\pi}\right) := \delta^{(2)}\left[\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f''\left(W^{\alpha,\beta}_{s_i,t_j}\right) 1_{[0,s_i]\times\Delta_j}(\cdot) 1_{\Delta_i\times[0,t_j]}(\star)\right] \tag{42}$$

is convergent in $L^2(\Omega)$ to a Skorohod integral, since we already have the convergence of the integrands in $L^2\left(\Omega; \left(\mathcal{H}^{\alpha,\beta}\right)^{\otimes 2}\right)$ from Lemma 5, we only need to guarantee that the Skorohod integrals in (42) converge to a random variable in $L^2(\Omega)$. However this is trivial since the term in (42) is the difference of all the other terms that have been estimated in this proof, and have been proved to converge in $L^2(\Omega)$ to $f(W^{\alpha,\beta}_{s,t})$ minus the sum of all terms on the right hand side of (32) except for the one denoted by

$$\int_0^t \int_0^s f''\left(W^{\alpha,\beta}_{u,v}\right) d\tilde{M}_{u,v}.$$

Therefore, the Skorohod integral in (42) converges to the Skorohod integral $\delta^{(2)}(N)$, and (32) is established, so Theorem 1 is proved. □

4 Interpretation of the integral with respect to $\tilde{M}$

As we mentioned before in the statement of the Itô formula (Theorem 1), the notation $\tilde{M}$ is defined using a double Skorohod integral. However, the rationale for using a specific differential notation for $\tilde{M}$ is the following. We can write

$$\begin{aligned}\tilde{M}(s,t) &= \int_{u_1,v_1}\int_{u_2,v_2} dW^{\alpha,\beta}_{u_1,v_1} dW^{\alpha,\beta}_{u_2,v_2} 1_{[0,s]}(u_1)\,1_{[0,t]}(v_2)\,1_{[0,u_1]}(u_2)\,1_{[0,v_2]}(v_1) \\ &= \int_{u_1=0}^{s}\int_{v_2=0}^{t}\int_{u_2=0}^{u_1}\int_{v_1=0}^{v_2} dW^{\alpha,\beta}_{u_1,v_1} dW^{\alpha,\beta}_{u_2,v_2} \qquad (43) \\ &= \int_{u_1=0}^{s}\int_{v_2=0}^{t}\left(\int_{v_1=0}^{v_2} dW^{\alpha,\beta}_{u_1,v_1}\right)\left(\int_{u_2=0}^{u_1} dW^{\alpha,\beta}_{u_2,v_2}\right) \qquad (44) \\ &= \int_{u_1=0}^{s}\int_{v_2=0}^{t} d_{u_1}W^{\alpha,\beta}_{u_1,v_2}\cdot d_{v_2}W^{\alpha,\beta}_{u_1,v_2} \qquad (45)\end{aligned}$$

which shows that $d\tilde{M}(s,t)$ should be interpreted as

$$d_s W^{\alpha,\beta}_{s,t}\cdot d_t W^{\alpha,\beta}_{s,t}. \qquad (46)$$

To make the string of equalities above rigorous, the Fubini-type property used to go from (43) to (44) must be justified, and the double stochastic integral in (45) must be properly defined. This can be done without needing to invoke a new Skorohod integration theory, by simply noting that the integral in (45) can be defined as a random variable in the second Gaussian chaos of $W^{\alpha,\beta}$ with an appropriate distribution, and checking that it equals $\tilde{M}(s,t)$. We omit these details.

Defining stochastic integration with respect to $\tilde{M}$ is also trivial if the integrands are deterministic. However, in connection with the general Itô formula (Theorem 1), we are much more interested in random integrands. A proper theory of general Skorohod integration with respect to $\tilde{M}$ can be given. We will not present this theory here, since it constitutes only a tangent to the Itô formula that is the subject of this article, and especially since it is not clear that it brings any more computational information than formulas (33) and (34) which can be taken as a definition of general Skorohod integration against $\tilde{M}$. What we will do instead is to present briefly a somewhat formal calculation that shows exactly how formula (33) is related to integration against $\tilde{M}$. The main point in this calculation is that, although the integrands are non-deterministic, the familiar Malliavin-derivative-type corrections that come from pulling random non-time-dependent terms out of Skorohod integrals do not appear in the final formula. We recall the general formula for Skorohod integration with respect to an arbitrary Gaussian field, and a fixed L^2 random variable F:

$$F\delta(u) = \delta(uF) + \langle u; DF\rangle_{\mathcal{H}}$$

where $\langle\cdot;\cdot\rangle_{\mathcal{H}}$ is the inner product in the canonical Hilbert space of the underlying Gaussian field.

We also recall that the stochastic differential of $W^{\alpha,\beta}$ can be formally understood as follows, using a standard Brownian sheet W:

$$dW^{\alpha,\beta}_{u,v} = du\,dv\int_{a=0}^{u}\int_{b=0}^{v}\frac{\partial K^{\alpha}}{\partial u}(u,a)\,\frac{\partial K^{\beta}}{\partial v}(v,b)\,dW_{a,b}.$$

Therefore we have

$$\begin{aligned} I &:= \int_{u'=0}^{u}\int_{v'=0}^{v} f\left(u,v\right) dW_{u',v}^{\alpha,\beta} dW_{u,v'}^{\alpha,\beta} \\ &= \int_{u'=0}^{u}\int_{v'=0}^{v} f\left(u,v\right) \\ &\cdot du'dv \int_{a=0}^{u'}\int_{b=0}^{v} \frac{\partial K^{\alpha}}{\partial u}\left(u',a\right)\frac{\partial K^{\beta}}{\partial v}\left(v,b\right) dW_{a,b} \\ &\cdot dudv' \int_{a'=0}^{u}\int_{b'=0}^{v'} \frac{\partial K^{\alpha}}{\partial u}\left(u,a'\right)\frac{\partial K^{\beta}}{\partial v}\left(v',b'\right) dW_{a',b'}. \end{aligned}$$

Now we pull the constant $L^2\left(\Omega\right)$-random variable $f\left(u,v\right)$ into the integral with respect to $dW_{a,b}$, modulo a correction term, then use Fubini to pull the Riemann integral with respect to u' all the way in (which requires no correction), yielding

$$\begin{aligned} I = &\int_{v'=0}^{v} \cdot dv \left[\int_{a=0}^{u}\int_{b=0}^{v}\int_{u'=a}^{u} du' \frac{\partial K^{\alpha}}{\partial u}\left(u',a\right)\frac{\partial K^{\beta}}{\partial v}\left(v,b\right) f\left(u,v\right) dW_{a,b} - \text{correction term}\right] \\ &\cdot dudv' \int_{a'=0}^{u}\int_{b'=0}^{v'} \frac{\partial K^{\alpha}}{\partial u}\left(u,a'\right)\frac{\partial K^{\beta}}{\partial v}\left(v',b'\right) dW_{a',b'} \\ &= dudv \int_{v'=0}^{v} \left[\int_{a=0}^{u}\int_{b=0}^{v} \left(K^{\alpha}\left(u,a\right) - K^{\alpha}\left(a,a\right)\right)\frac{\partial K^{\beta}}{\partial v}\left(v,b\right) f\left(u,v\right) dW_{a,b} - \text{correction term}\right] \\ &\cdot dv' \int_{a'=0}^{u}\int_{b'=0}^{v'} \frac{\partial K^{\alpha}}{\partial u}\left(u,a'\right)\frac{\partial K^{\beta}}{\partial v}\left(v',b'\right) dW_{a',b'} \\ &= dudv \left[\int_{a=0}^{u}\int_{b=0}^{v} K^{\alpha}\left(u,a\right)\frac{\partial K^{\beta}}{\partial v}\left(v,b\right) dW_{a,b}\right] \\ &\cdot \int_{v'=0}^{v} dv' f\left(u,v\right) \int_{a'=0}^{u}\int_{b'=0}^{v'} \frac{\partial K^{\alpha}}{\partial u}\left(u,a'\right)\frac{\partial K^{\beta}}{\partial v}\left(v',b'\right) dW_{a',b'} \end{aligned}$$

Now we perform the same operation on f and the integral with respect to v', yielding

$$\begin{aligned} I = &dudv \left[\int_{a=0}^{u}\int_{b=0}^{v} K^{\alpha}\left(u,a\right)\frac{\partial K^{\beta}}{\partial v}\left(v,b\right) dW_{a,b}\right] \\ &\cdot \int_{v'=0}^{v} dv' \left[\int_{a'=0}^{u}\int_{b'=0}^{v'} \frac{\partial K^{\alpha}}{\partial u}\left(u,a'\right)\frac{\partial K^{\beta}}{\partial v}\left(v',b'\right) f\left(u,v\right) dW_{a',b'} - \text{correction term}\right] \\ &= dudv \left[\int_{a=0}^{u}\int_{b=0}^{v} K^{\alpha}\left(u,a\right)\frac{\partial K^{\beta}}{\partial v}\left(v,b\right) dW_{a,b}\right] \\ &\cdot \left[\int_{a'=0}^{u}\int_{b'=0}^{v} \frac{\partial K^{\alpha}}{\partial u}\left(u,a'\right) K^{\beta}\left(v,b'\right) f\left(u,v\right) dW_{a',b'} - \text{correction term}\right] \end{aligned}$$

$$= dudv \left[\int_{a=0}^{u} \int_{b=0}^{v} K^{\alpha}(u,a) \frac{\partial K^{\beta}}{\partial v}(v,b)\, dW_{a,b} \right]$$
$$\cdot f(u,v) \left[\int_{a'=0}^{u} \int_{b'=0}^{v} \frac{\partial K^{\alpha}}{\partial u}(u,a')\, K^{\beta}(v,b')\, dW_{a',b'} \right]$$
$$= f(u,v) \left[dv \int_{b=0}^{v} \frac{\partial K^{\beta}}{\partial v}(v,b)\, d_b W_{u,b}^{\alpha,1/2} \right] \left[du \int_{a'=0}^{u} \frac{\partial K^{\alpha}}{\partial u}(u,a')\, d_{a'} W_{a',v}^{1/2,\beta} \right]$$
$$= f(u,v)\, d_v W_{u,v}^{\alpha,\beta} \cdot d_u W_{u,v}^{\alpha,\beta}.$$

Modulo the definition of Skorohod integration with respect to $d_v W_{u,v}^{\alpha,\beta} \cdot d_u W_{u,v}^{\alpha,\beta}$, this calculation justifies the formula

$$\int_0^t \int_0^s f(u,v) d\tilde{M}_{u,v}$$
$$:= \int_{[0,1]^2} \int_{[0,1]^2} dW_{u_1,v_1}^{\alpha,\beta} dW_{u_2,v_2}^{\alpha,\beta} f(u_1,v_2)\, 1_{[0,s]}(u_1)\, 1_{[0,t]}(v_2)\, 1_{[0,u_1]}(u_2)\, 1_{[0,v_2]}(v_1)$$
$$= \int_0^t \int_0^s f(u,v)\, d_v W_{u,v}^{\alpha,\beta} \cdot d_u W_{u,v}^{\alpha,\beta},$$

which can be summarized by

$$d\tilde{M}_{u,v} = d_v W_{u,v}^{\alpha,\beta} \cdot d_u W_{u,v}^{\alpha,\beta}$$

as announced in (46). □

5 Application to the local time of the fractional Brownian sheet

As an application of the Itô formula, we will establish in this section the existence and the stochastic integral representation of the local time of the fractional Brownian sheet.

5.1 Known results and motivation

We start with a short summary of the results on the local times for one and multiparameter fractional Brownian motion.

- The local time λ_t^a of the one parameter fractional Brownian motion B^H was introduced in [5] as the density of the occupation measure $\Gamma \to \int_0^t 1_\Gamma(B_s^H)ds$. The author proved that λ_t^a has a jointly continuous version in the variables a and t. Moreover λ_t^a has Hölder-continuous paths of order $\delta < 1 - H$ in time and of order $\gamma < \frac{1-H}{2H}$ in the space variable a provided that $H \geq \frac{1}{3}$. As a density of an occupation measure, the local time is increasing in t and the measure $L^a(dt)$ is concentrated on the level set $\{s : B_s^H = a\}$. A chaos expansion of λ_t^a is given in [14], along with an L^2 estimate.

- Another version of the local time of the fractional Brownian motion was introduced in [8] as the density of the occupation measure $m_t(\Gamma) = 2H\int_0^t 1_\Gamma(B_s^H)s^{2H-1}ds$. If $H \geq \frac{1}{3}$, this local time satisfy the Tanaka formula

$$|B_t^H - a| = |-a| + \int_0^t sign(B_s^a)dB_s^H + L_t^a$$

where the stochastic integral is defined in the Skorohod sense. This formula was recently extended by [7] to $H \in (0,1)$. The regularity properties of λ_t^a can be transferred to L_t^a by integrating by parts.

- In [12] the local time of the d-dimensional fractional sheet with N-parameters was studied. This local time can be formally defined as $L^a_{\underline{t}} = \int_{[0,\underline{t}]} \delta_a(W^H_{\underline{s}})d\underline{s}$, where $\underline{t} = (t_1, \ldots, t_N) \in T^N$, $H = (H_1, \ldots, H_N)$ and δ_a denotes the Dirac function. The smoothness of the local time in the Sobolev-Watanabe spaces and its asymptotic behavior were studied.

- Recently, in [18], using the techniques of the Fourier analysis the authors proved the existence of the local time of the N-parameters d-dimensional fractional Brownian sheet satisfying the occupation density formula

$$\int_A f\left(W^H_{\underline{t}}\right) d\,\underline{t} = \int_{R^d} f(x)L(x,A)dx$$

for any Borel set $A \subset R^N$ and for any measurable function $f : R^d \to R$. The existence of a jointly continuous version of the local time is proved. Obviously, this local time coincides with the one used in [12].

In this section we define the local time $L^a_{s,t}$, $a \in R$ of the fractional Brownian sheet $\left(W^{\alpha,\beta}_{s,t}\right)_{s,t\in[0,1]}$ as the density of the occupation measure

$$m_{s,t}(\Gamma) = \alpha\beta \int_0^t \int_0^s 1_\Gamma(W^{\alpha,\beta}_{u,v})u^{4\alpha-1}v^{4\beta-1}dudv. \tag{47}$$

Note that in the case $\alpha = \beta = \frac{1}{2}$ this local time coincide with the local time introduced in [13] and [16] for the standard Wiener process with two parameters.

Remark 1 *Using the integration by parts we can transfer the joint continuity of the local time defined in [18] to $L^a_{s,t}$. The details are left to the reader.*

5.2 Stochastic representation of local time

Let

$$p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}}e^{-x^2/2\varepsilon}$$

be the heat kernel with variance $\varepsilon > 0$. The following results gives an approximation in L^2 of the local time of the fractional Brownian sheet. It can be deduced from [12], where the authors used the Wiener chaos decomposition of the local time.

Proposition 2 *Let $a \in R$ and $s, t \in [0,1]$ and define*

$$L^{a,\varepsilon}_{s,t} = \alpha\beta \int_0^t \int_0^s p_\varepsilon(W^{\alpha,\beta}_{u,v}) u^{4\alpha-1} v^{4\beta-1} du dv.$$

Then the random variable $L^{a,\varepsilon}_{s,t}$ converges to $L^a_{s,t}$ in $L^2(\Omega)$ as ε tends to zero.

The next result gives the stochastic integral representation of the local time.

Theorem 2 *For any $a \in R$ and $s, t \in [0,1]$, we have*

$$\begin{aligned} L^a_{s,t} &= \frac{1}{6}\left|W^{\alpha,\beta}_{s,t} - a\right| \left(W^{\alpha,\beta}_{s,t} - a\right)^2 - \frac{1}{2}\int_0^t \int_0^s \left|W^{\alpha,\beta}_{u,v} - a\right| \left(W^{\alpha,\beta}_{u,v} - a\right) dW^{\alpha,\beta}_{u,v} \\ &- 2\alpha\beta \int_0^t \int_0^s \left|W^{\alpha,\beta}_{u,v} - a\right| u^{2\alpha-1} v^{2\beta-1} du dv \\ &- \int_0^t \int_0^s \left|W^{\alpha,\beta}_{u,v} - a\right| d\tilde{M}_{u,v} \\ &- \alpha \int_0^t \int_0^s sign(W^{\alpha,\beta}_{u,v} - a) u^{2\alpha-1} v^{2\beta} d_v W^{\alpha,\beta}_{u,v} du \\ &- \beta \int_0^t \int_0^s sign(W^{\alpha,\beta}_{u,v} - a) u^{2\alpha} v^{2\beta-1} d_u W^{\alpha,\beta}_{u,v} dv. \end{aligned}$$

Proof: Let us introduce the functions

$$g^{IV}_\varepsilon(x) = p_\varepsilon(x), \quad g'''_\varepsilon(x) = 2\int_0^x p_\varepsilon(y)dy - 1,$$

$$g''_\varepsilon(x) = \int_0^x g'''_\varepsilon(y)dy, \quad g'_\varepsilon(x) = \int_0^x g''_\varepsilon(y)dy, \quad g_\varepsilon(x) = \int_0^x g'_\varepsilon(y)dy.$$

Clearly, as ε tends to zero, we have

$$g'''_\varepsilon(x) \to g'''(x) = sign(x), \quad g''_\varepsilon(x) \to g''(x) = |x|,$$

$$g'_\varepsilon(x) \to g'(x) = \frac{1}{2}x|x|, \quad g_\varepsilon(x) \to g(x) = \frac{1}{6}x^2|x|.$$

Let's write Itô's formula for the function $g_\varepsilon(x-a)$, where a is a real number. We get

$$\begin{aligned}
& g_\varepsilon(W^{\alpha,\beta}_{s,t}-a) \\
& = g_\varepsilon(0) + \int_0^t\int_0^s g'_\varepsilon(W^{\alpha,\beta}_{u,v}-a)dW^{\alpha,\beta}_{u,v} \\
& + 2\alpha\beta\int_0^t\int_0^s g''_\varepsilon(W^{\alpha,\beta}_{u,v}-a)u^{2\alpha-1}v^{2\beta-1}dudv + \int_0^t\int_0^s g''_\varepsilon(W^{\alpha,\beta}_{u,v}-a)d\tilde{M}_{u,v} \\
& + \alpha\int_0^t\int_0^s g'''_\varepsilon(W^{\alpha,\beta}_{u,v}-a)u^{2\alpha-1}v^{2\beta}dud_vW^{\alpha,\beta}_{u,v} + \beta\int_0^t\int_0^s g'''_\varepsilon(W^{\alpha,\beta}_{u,v}-a)u^{2\alpha}v^{2\beta-1}dvd_uW^{\alpha,\beta}_{u,v} \\
& + \alpha\beta\int_0^t\int_0^s g^{IV}_\varepsilon(W^{\alpha,\beta}_{u,v}-a)u^{4\alpha-1}v^{4\beta-1}dudv.
\end{aligned}$$

We will decompose the proof into several steps.

Step 1. Given the existence of moments of all orders for the supremum of continuous Gaussian fields, dominated convergence implies the following convergences hold in $L^2(\Omega)$:

$$g_\varepsilon(W^{\alpha,\beta}_{s,t}-a) \to \frac{1}{6}(W^{\alpha,\beta}_{s,t}-a)^2|W^{\alpha,\beta}_{s,t}-a|,$$

$$2\alpha\beta\int_0^t\int_0^s g''_\varepsilon(W^{\alpha,\beta}_{u,v}-a)u^{2\alpha-1}v^{2\beta-1}dudv \to 2\alpha\beta\int_0^t\int_0^s \left|W^{\alpha,\beta}_{u,v}-a\right| u^{2\alpha-1}v^{2\beta-1}dudv,$$

and by Proposition 2,

$$\alpha\beta\int_0^t\int_0^s p_\varepsilon(W^{\alpha,\beta}_{u,v})u^{4\alpha-1}v^{4\beta-1}dudv \to L^a_{s,t}.$$

Step 2. We will show that the integrand of the stochastic integral with respect to the fractional Brownian sheet is convergent. More precisely, consider the process

$$g'_\varepsilon(W^{\alpha,\beta}_{u,v})1_{[0,t]\times[0,s]}(u,v)$$

and let us show its convergence in $L^2\left(\Omega;\mathcal{H}^{\alpha,\beta}\right)$ as $\varepsilon\to 0$ to

$$g'(W^{\alpha,\beta}_{u,v})1_{[0,t]\times[0,s]}(u,v)$$

where $g'(x)=\frac{1}{2}x|x|$. Equivalently, we will show that $K^{*,2}(g'(W^{\alpha,\beta})1_{[0,t]\times[0,s]})(u,v)$ converges to $K^{*,2}(g'(W^{\alpha,\beta})1_{[0,t]\times[0,s]})(u,v)$ in $L^2([0,1]^2\times\Omega)$. We have, by Jensen's inequality

$$\begin{aligned}
& \mathbf{E}\int_0^t\int_0^s \left|\int_u^t\int_v^s \left(g'_\varepsilon(W^{\alpha,\beta}_{a,b})-g'(W^{\alpha,\beta}_{a,b})\right)\frac{\partial K^\alpha}{\partial a}(a,u)\frac{\partial K^\beta}{\partial b}(b,v)dadb\right|^2 dudv \\
& \le c\mathbf{E}\int_0^t\int_0^s \left(\int_u^t\int_v^s \left|g'_\varepsilon(W^{\alpha,\beta}_{a,b})-g'(W^{\alpha,\beta}_{a,b})\right|^2 (a-u)^{H-\frac{3}{2}}(b-v)^{H-\frac{3}{2}}dadb\right)dudv \\
& \le c\mathbf{E}\int_0^t\int_0^s \left|g'_\varepsilon(W^{\alpha,\beta}_{a,b})-g'(W^{\alpha,\beta}_{a,b})\right|^2 dadb
\end{aligned}$$

which goes to zero as $\varepsilon \to 0$.

Step 3. Lemma 5 together with the proof of Theorem 1 shows that the limit in $L^2(\Omega)$

$$\lim_{|\pi|\to 0} \delta^{(2)} \sum_{i,j} \left[f'' \left(W^{\alpha,\beta}_{s_i,t_j} \right) 1_{[0,s_i]\times\Delta_j}(\cdot) 1_{\Delta_i\times[0,t_j]}(\star) \right]$$

exists for every f satisfying Condition (H). Consequently, for every $\varepsilon > 0$, denote by A_ε the following random variable in $L^2(\Omega)$:

$$A_\varepsilon = \lim_{|\pi|\to 0} \delta^{(2)} \sum_{i,j} \left[g''_\varepsilon \left(W^{\alpha,\beta}_{s_i,t_j} \right) 1_{[0,s_i]\times\Delta_j}(\cdot) 1_{\Delta_i\times[0,t_j]}(\star) \right].$$

Since g''_ε converges to $g(x) = \frac{1}{2}|x|x$, we obtain by Lemma 4 (with the observation made at the end of the proof of Proposition 1) the convergence in $L^2(\Omega)$ of A_ε, as ε goes to zero, to the random variable $\lim_{|\pi|\to 0} \delta^{(2)} \sum_{i,j} \left[g \left(W^{\alpha,\beta}_{s_i,t_j} \right) 1_{[0,s_i]\times\Delta_j}(\cdot) 1_{\Delta_i\times[0,t_j]}(\star) \right]$ which is in $L^2(\Omega)$.

Step 4. We consider now the term

$$\beta \int_0^t \int_0^s g'''_\varepsilon (W^{\alpha,\beta}_{u,v} - a) u^{2\alpha} v^{2\beta-1} dv d_u W^{\alpha,\beta}_{u,v}$$

and we will prove its convergence to

$$\beta \int_0^t \int_0^s sign(W^{\alpha,\beta}_{u,v} - a) u^{2\alpha} v^{2\beta-1} dv d_u W^{\alpha,\beta}_{u,v}.$$

An argument from the one-parameter case can be used to conclude this fact. The proof of this step follows from the next lemma and the dominated convergence theorem.

Lemma 6 *Consider B^α a one-dimensional fBm with $\alpha \in (\frac{1}{2}, 1)$. Then, for every $s \in T$, $\int_0^s g'''_\varepsilon(B^\alpha_u - a) u^{2\alpha} dW^\alpha_u$ converges in $L^2(\Omega)$ to $\int_0^s sign(B^\alpha_u - a) u^{2\alpha} dW^\alpha_u$.*

Proof: Let us write Itô's formula with $h_\varepsilon(t,x) = t^{2\alpha} g''_\varepsilon(x)$ (this is a straightforward application of Theorem 1 of [1]). We get

$$s^{2\alpha} g''_\varepsilon(B^\alpha_s - a) = 2\alpha \int_0^s g''_\varepsilon(B^\alpha_u - a) u^{2\alpha-1} du + \int_0^s g'''_\varepsilon(B^\alpha_u - a) u^{2\alpha} dB^\alpha_u$$
$$+ 2\alpha \int_0^s p_\varepsilon(B^\alpha_u - a) u^{4\alpha-1} du$$

The following convergences in $L^2(\Omega)$ can be proved exactly as in [8]:

$$s^{2\alpha} g''_\varepsilon(B^\alpha_s - a) \to s^{2\alpha} |B^\alpha_s - a|$$

$$2\alpha \int_0^s g_\varepsilon''(B_u^\alpha - a) u^{2\alpha-1} du \to 2\alpha \int_0^s |B_u^\alpha - a| \, u^{2\alpha-1} du$$

$$2\alpha \int_0^s p_\varepsilon(B_u^\alpha - a) u^{4\alpha-1} du \to \int_0^s L_u^a u^{2\alpha} du$$

(here L_u^a is the local time of the one-parameter fBm defined in [8] and presented at the beginning of this section), and

$$1_{[0,s]}(u) g_\varepsilon'''(B_u^\alpha - a) u^{4\alpha-1} \to 1_{[0,s]}(u) sign(B_u^\alpha - a) u^{2\alpha},$$

the last convergence being in $L^2(\Omega; \mathcal{H})$. That gives the conclusion using Lemma 4. □

Lemma 4 again and Steps 1 to 4 above finish the proof of the theorem. □

Remark 2 *Using standard arguments, additional versions of the stochastic integral representation of the local time of the fractional Brownian sheet can be obtained by using, instead of the functions g, g', g'' and g''' appearing in the proof of Theorem 2, the functions j, j', j'' and j''', where*

$$j(x) = \frac{1}{6}[(x-a)^+]^3, \; j'(x) = \frac{1}{2}[(x-a)^+]^2$$

and

$$j''(x) = (x-a)^+ \;, \; j'''(x) = 1_{(a,\infty)}(x).$$

More precisely, we have

$$\begin{aligned}\frac{1}{2} L_{s,t}^a = \; & j(W_{s,t}^{\alpha,\beta}) - \int_0^t \int_0^s j'\left(W_{u,v}^{\alpha,\beta} - a\right) dW_{u,v}^{\alpha,\beta} \\ & - 2\alpha\beta \int_0^t \int_0^s j''\left(W_{u,v}^{\alpha,\beta} - a\right) u^{2\alpha-1} v^{2\beta-1} du dv \\ & - \int_0^t \int_0^s j''\left(W_{u,v}^{\alpha,\beta} - a\right) d\tilde{M}_{u,v} \\ & - \alpha \int_0^t \int_0^s j'''\left(W_{u,v}^{\alpha,\beta} - a\right) u^{2\alpha-1} v^{2\beta} d_v W_{u,v}^{\alpha,\beta} du \\ & - \beta \int_0^t \int_0^s j'''\left(W_{u,v}^{\alpha,\beta} - a\right) u^{2\alpha} v^{2\beta-1} d_u W_{u,v}^{\alpha,\beta} dv.\end{aligned}$$

Negative part functions can be also used to express the local time.

Our last result is an occupation time formula for the local time of the fractional Brownian sheet, for which the previous remark is useful.

Proposition 3 *For any Borel function $f : R \to R$ and and for any $a \in R$, $s, t \in [0,1]$, it holds that*

$$2\alpha\beta \int_0^t \int_0^s f(W_{u,v}^{\alpha,\beta}) u^{4\alpha-1} v^{4\beta-1} du dv = \int_R f(a) L_{s,t}^a da. \tag{48}$$

Proof: We use the same idea as in [16]. Let us introduce the function $h_x^{IV}(y) = 1_{(0,x)}(y)$ and its anti-derivatives. Notice that

$$h_x^{'''}(y) = \int_0^x 1_{(y>x')}dx', \quad h_x^{''}(y) = \int_0^x (y-x')^+ dx',$$

and

$$h_x^{'}(y) = \int_0^x \frac{1}{2}[(y-x')^+]^2 dx', \quad h_x(y) = \int_0^x \frac{1}{6}[(y-x')^+]^3 dx'.$$

On the one hand, we can write Itô's formula for $h_x(y)$ (although the fourth derivative is not continuous, is it not difficult to see that Itô's formula still holds, by approximating h_x by smooth functions) and we get

$$\begin{aligned}
&\alpha\beta \int_0^t \int_0^s 1_{(0,x)}(W_{u,v}^{\alpha,\beta}) u^{4\alpha-1} v^{4\beta-1} dudv \\
&= h_x(W_{s,t}^{\alpha,\beta}) - h_x(0) - \int_0^t \int_0^s h_x^{'}(W_{u,v}^{\alpha,\beta}) dW_{u,v}^{\alpha,\beta} \\
&- 2\alpha\beta \int_0^t \int_0^s h_x^{''}(W_{u,v}^{\alpha,\beta}) u^{2\alpha-1} v^{2\beta-1} dudv - \int_0^t \int_0^s h_x^{''}(W_{u,v}^{\alpha,\beta}) d\tilde{M}_{u,v} \\
&- \alpha \int_0^t \int_0^s h_x^{'''}(W_{u,v}^{\alpha,\beta}) u^{2\alpha-1} v^{2\beta} du d_v W_{u,v}^{\alpha,\beta} - \beta \int_0^t \int_0^s h_x^{'''}(W_{u,v}^{\alpha,\beta}) u^{2\alpha} v^{2\beta-1} d_u W_{u,v}^{\alpha,\beta} dv.
\end{aligned}$$

On the third hand, using the integral expression of the local time $L_{s,t}^y$ given in Remark 2, multiplying both sides by $1_{(0,x)}(y)$, integrating over the real line and changing the order of integration (see exercise 3.2.8 of [17] for the Fubini anticipating theorem), we will obtain that

$$\frac{1}{2}\int_R 1_{(0,x)}(y) L_{s,t}^y dy = \alpha\beta \int_0^t \int_0^s 1_{(0,x)}(W_{u,v}^{\alpha,\beta}) u^{4\alpha-1} v^{4\beta-1} dudv.$$

The formula (48) is proved for $f(y) = 1_{(0,x)}(y)$. A monotone class argument is sufficient to conclude the proof. □

5.3 Relation with the variation of $\tilde{M}$

When $\alpha = \beta = \frac{1}{2}$ the process $\tilde{M}$ is a martingale and in this case (see [16]) the formula (48) can be written as

$$\frac{1}{2}\int_0^t \int_0^s f(W_{u,v}^{\alpha,\beta}) d\langle \tilde{M} \rangle_{u,v} = \int_R f(a) L_{s,t}^a da. \tag{49}$$

This can be seen immediately by formally taking the square of the differential $d\tilde{M}_{u,v} = d_u\tilde{M}_{u,v} \cdot d_v\tilde{M}_{u,v}$ which yields $\left(d\tilde{M}_{u,v}\right)^2 = uvdudv$.

If α or β are not $\frac{1}{2}$, then $\tilde{M}$ is not a semimartingale. To investigate whether a formula such as (49) still holds, the results of Section 4 can be invoked. They show that the quadratic variation of $\tilde{M}$ is obviously not the right quantity to look at, since it is 0.

As with one-parameter fBm, there is a specific non-quadratic variation of $\tilde{M}$ that will be non-zero and non-infinite. One can easily prove that, for small $h, k > 0$, and $p, q > 1$

$$\mathbf{E}\left[\left(W^{\alpha,\beta}_{u+h,v} - W^{\alpha,\beta}_{u,v}\right)^p \left(W^{\alpha,\beta}_{u,v+k} - W^{\alpha,\beta}_{u,v}\right)^q\right] = v^{\beta p} h^{\alpha p} u^{\alpha q} k^{\beta q} + o\left(\max\left(h^{\alpha p}, k^{\beta q}\right)\right). \quad (50)$$

In order for the increment

$$d\tilde{M}_{u,v} = d_v W^{\alpha,\beta}_{u,v} \cdot d_u W^{\alpha,\beta}_{u,v} = \left(W^{\alpha,\beta}_{u+du,v} - W^{\alpha,\beta}_{u,v}\right)^p \left(W^{\alpha,\beta}_{u,v+dv} - W^{\alpha,\beta}_{u,v}\right)$$

to yield the differential of a non-trivial bounded variation process, one must choose the powers p and q above in order to get the product $dudv$. Therefore we define the (p, q)-variation of $\tilde{M}$ by the formula

$$d\left\langle \tilde{M} \right\rangle^{(p,q)}_{u,v} = d\left\langle W^{\alpha,\beta}_{\cdot,v} \right\rangle^{(p)}_u d\left\langle W^{\alpha,\beta}_{u,\cdot} \right\rangle^{(q)}_v \quad (51)$$

where for any number $r > 1$, the r-variation of a one-parameter process X is defined as usual using partitions $\pi = \{t_1, \cdots, t_n\}$ of $[0, t]$ by

$$\langle X \rangle^{(r)}(t) = \lim_{|\pi| \to 0} \sum_{i=1}^{n} \left|X_{t_i} - X_{t_{i-1}}\right|^r. \quad (52)$$

Formula (50) proves that $\left\langle \tilde{M} \right\rangle^{\left(\alpha^{-1},\beta^{-1}\right)}_{u,v}$ exists and is non-zero, and in fact

$$d\left\langle \tilde{M} \right\rangle^{\left(\alpha^{-1},\beta^{-1}\right)}_{u,v} = u^{\alpha/\beta} v^{\beta/\alpha} dudv.$$

Despite the somewhat formal nature of the above development, the definition of $\left\langle \tilde{M} \right\rangle^{\left(\alpha^{-1},\beta^{-1}\right)}_{u,v}$ can be made rigorous because of the tensor-type nature of $\tilde{M}$, and the last formula can be established rigorously by only referring to the definition of $\tilde{M}$ as a double Skorohod integral in the statement of Proposition 1 or Theorem 1. By comparing with formula (48) in Proposition 3, we can state the following, which answers the issue of the validity of formula (49) negatively, leaving a rigorous proof, and a rigorous definition of $\left\langle \tilde{M} \right\rangle$, to the reader.

Proposition 4 *The (p, q)-variation of the fractional Brownian sheet $(u, v) \mapsto \left\langle \tilde{M} \right\rangle^{(p,q)}_{u,v}$, defined in equations (51) and (52), is a non-trivial bounded-variation process on $[0, 1]^2$ if and only if $p = 1/\alpha$ and $q = 1/\beta$. In that case*

$$\left\langle \tilde{M} \right\rangle^{\left(\alpha^{-1},\beta^{-1}\right)}_{s,t} = \int_0^s \int_0^t dudv u^{\alpha/\beta} v^{\beta/\alpha}.$$

The formula (49) does not hold unless $\alpha = \beta = 1/2$.

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