Stochastic evolution equations with fractional Brownian motion

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Abstract

In this paper different types of stochastic evolution equations driven by infinite-dimensional fractional Brownian motion are studied. We consider first the case of the linear additive noise; a necessary and sufficient condition for the existence and uniqueness of the solution is established; separate proofs are required for the cases of Hurst parameter above and below 1/2. Moreover, we present a characterization of almost-sure moduli of continuity for the solution via a sharp theory of Gaussian regularity. Then we prove an existence and uniqueness result for the solution in the case of the linear equation with multiplicative noise and we derive a fractional stochastic Feynman-Kac formula.

Key words and phrases: fractional Brownian motion, stochastic partial differential equation, Feynman-Kac formula, Gaussian regularity, almost-sure modulus of continuity, Hurst parameter.

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1 Introduction

The recent development of stochastic calculus with respect to fractional Brownian motion (fBm) has led to various interesting mathematical applications, and in particular, several types of stochastic differential equations driven by fBm have been considered in finite dimensions (see among others [14], [13] or [5]). The question of infinite dimensional equations has emerged very recently (see [11], [12]). The purpose of this article is to provide a detailed study of the existence and regularity properties of the stochastic evolution equations with linear additive and linear multiplicative fractional Brownian noise. Before providing a complete summary of the contents of this article, we comment on the fact that, as in the few published works ([11], [12]) on infinite-dimensional fBm-driven equations, we study only equations in which noise enters linearly. Moreover, we believe our article, together with the preprint [17], contains the first instance in which multiplicative noise is considered; our Feynman-Kac formula in the linear multiplicative case is new. The difficulty with non-linear fBm-driven equations is notorious: the Picard iteration technique involves Malliavin derivatives in such a way that the equations for estimating these derivatives cannot be closed. The preprint [17] treats an equation with fBm multiplied by a nonlinear term; however the noise term has a trace-class correlation, and moreover they treat only the case $H > 1/2$, which allows one to solve the equation using stochastic integrals understood in a pathwise way, not in the Skorohod sense. The general non-linearity issue remains unsolved.

Let $B^H = (B_t^H)_{t\in[0,1]}$ be a fractional Brownian motion on a real and separable Hilbert space U. That is, B^H is a U-valued centered Gaussian process, starting from zero, defined by its covariance

$$E(B^H(t)B^H(s)) = R(s,t)Q, \qquad \text{for every } s,t \in [0,1]$$

where Q is a self-adjoint and positive operator from U to U and R is the standard covariance structure of one-dimensional fractional Brownian motion (as in (2)). We consider the following stochastic differential equation

$$X(dt) = AX(t)dt + F(X(t))\Phi dB^H(t) \tag{1}$$

and we study the existence, uniqueness, and regularity properties of the solution in several particular cases. The goal is to formulate necessary and sufficient conditions for these properties as conditions on the equations' input parameters A, Φ, and Q. It is always possible, and usually convenient, to assume that B^H is *cylindrical*, i.e. that Q is the identity operator. We will also translate the conditions for regularity as necessary and sufficient conditions on the almost-sure regularity of B^H itself.

In Section 3 we let $F(u) \equiv 1$ and A a linear operator from another Hilbert space V to V with $\Phi \in \mathcal{L}(U;V)$ a deterministic linear operator not depending on t. We give a necessary and sufficient condition for the existence of the solution. The stochastic integral appearing in (1) is a Wiener integral over Hilbert spaces. Our context is more general than the one studied in [12], or in [11], since we consider both cases $H > \frac{1}{2}$ and $H < \frac{1}{2}$. Our study goes further since we prove the sufficiency *and* the necessity of the condition for the existence of the solution. Section 4 contains a study of the space-time regularity of the solution using the so-called factorization method.

Section 5 proposes a detailed theory of spatial regularity when A is the Laplacian and $U = L^2(S^1)$, S^1 being the circle. Our regularity objective was to completely characterize the solution's almost-sure uniform spatial modulus of continuity. We have achieved this almost to its fullest extend, going beyond the scale of Hölder continuity, and proposing both intrinsic and distributional characterizations. The first and third authors had been working on such a characterization for stochastic heat equations with $H = 1/2$ (Brownian case). Their intuition gained in [24] in the Hölder scale had led them to formulate conjectures in [25] on how to extend their characterizations beyond this scale, and what strategy to use. Here we show that their conjecture on a sharp regularity characterization was incorrect (see Remark 9) and that the strategy can be simplified (see Remark 10). Moreover we show that the case of fBm is not more difficult than the Brownian case, while the results do depend heavily on the value of H. Our results herein were made possible by the sharp calculations performed in Section 3, and by establishing new sharp Gaussian regularity results in Section 5.1, which are of independent interest. For the sake of conciseness, we have chosen not to study the issue of sharp time regularity. However, the same Gaussian tools could be used for this problem, which we will tackle in a more general setting in a separate publication.

In Section 6, for $H > 1/2$, we let $F(u) = u$ in (1) and we prove existence and uniqueness of the evolution solution, our main tool being a straightforward infinite-dimensional extension of the definition of multiple Wiener-Itô integrals with respect to the fractional Brownian motion introduced in [7] and [19]. Now, the stochastic integral in (1) is a stochastic integral in the Skorohod sense. Finally, we derive a Feynman-Kac formula for the solution of the stochastic differential equation in the case $F(u) = u$. This is done by identifying the formula with the fractional chaos expansion given by the Picard iterations. A Feynman-Kac formula is a significant application, as it opens the path to studying the long-term behavior of the solution in the spirit of the parabolic Anderson model (see [3], [4]); one notices clearly from the Feynman-Kac formula (55) that the standard *Lyapunov exponent* scale is not the correct one if $H \neq 1/2$. Rather, one expect that $t^{-2H} \log u(t, x)$ will have a non-trivial limit as $t \to \infty$. We plan to show this in a separate publication.

2 Preliminaries

2.1 Malliavin Calculus for one-dimensional fractional Brownian motion

Consider $T = [0, \tau]$ a time interval with arbitrary fixed horizon τ, and let $(B_t^H)_{t \in T}$ the one-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$. This means by definition that B^H is a centered Gaussian process with covariance

$$R(t, s) = E(B_s^H B_t^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \tag{2}$$

Note that $B^{1/2}$ is standard Brownian motion. Moreover B^H has the following Wiener integral representation:

$$B_t^H = \int_0^t K^H(t, s) dW_s, \tag{3}$$

where $W = \{W_t : t \in T\}$ is a Wiener process, and $K^H(t, s)$ is the kernel given by

$$K^H(t, s) = c_H (t - s)^{H - \frac{1}{2}} + s^{H - \frac{1}{2}} F\left(\frac{t}{s}\right) \tag{4}$$

c_H being a constant and

$$F(z) = c_H \left(\frac{1}{2} - H\right) \int_0^{z-1} r^{H - \frac{3}{2}} \left(1 - (1 + r)^{H - \frac{1}{2}}\right) dr. \tag{5}$$

From (4) we obtain

$$\frac{\partial K^H}{\partial t}(t, s) = c_H (H - \frac{1}{2})(t - s)^{H - \frac{3}{2}} \left(\frac{s}{t}\right)^{\frac{1}{2} - H} \tag{6}$$

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of set of indicator functions $\{1_{[0,t]}, t \in T\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s)$$

The mapping $1_{[0,t]} \to B_t$ provides an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $B(\phi)$ the image of $\phi \in \mathcal{H}$ by the previous isometry. Consequently, the $\mathcal{H}$–indexed process $(B(\phi))_{\phi \in \mathcal{H}}$ is a centered Gaussian process such that $E(B(\phi)B(h)) = \langle \phi, h \rangle_{\mathcal{H}}$; hence one can develop a Malliavin calculus with respect to B^H.

We will denote by $\mathcal{S}$ the set of smooth random variables F of the form

$$F = f\left((B(\phi_1), \cdots, B(\phi_n)\right) \ n \geq 1 \ , \ \phi_i \in \mathcal{H}$$

with $f \in C_b^\infty(R^n)$ (f and all its derivatives are bounded). The Malliavin derivative is defined as

$$D^B F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \left(B(\phi_1), \cdots, B(\phi_n)\right) \phi_i$$

if $F \in \mathcal{S}$. The operator D^B is a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$ and we will consider its extension to the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2,H}^2 = E|F|^2 + E\|DF\|_{\mathcal{H}}^2$$

We denote by $D_H^{1,2}$ the closure of the set of smooth random variables $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,2,H}$. The adjoint of D^B is denoted by δ^B and it is called the Skorohod (or the divergence) integral. The operator δ^B is well-defined by the duality relationship

$$E(F\delta^B(u)) = E\langle D^B F, u \rangle_{\mathcal{H}}$$

and its domain $Dom(\delta)$ is the class of processes $u \in L^2(\Omega; \mathcal{H})$ for which there is a constant C such that

$$|E\langle D^B F, u \rangle_{\mathcal{H}}| \leq C\|F\|_2$$

for all $F \in \mathcal{S}$. By $L_H^{1,2}$ we denote the set $L^2(\mathcal{H}; D_H^{1,2})$ endowed with the norm

$$\|u\|_{1,2,H}^2 = \|u\|_{L^2(\Omega;\mathcal{H})}^2 + \|D^B u\|_{L^2(\mathcal{H}^{\otimes 2} \times \Omega)}^2$$

and we recall that $L_H^{1,2}$ is a subset of $Dom(\delta^B)$.

Let us consider the operator K^* in $L^2(T)$

$$(K^*\varphi)(s) = K(\tau, s)\varphi(s) + \int_s^\tau \left(\varphi(r) - \varphi(s)\right) \frac{\partial K}{\partial r}(r,s) dr \tag{7}$$

When $H > \frac{1}{2}$, the operator K^* has the simpler expression

$$(K^*\varphi)(s) = \int_s^\tau \varphi(r) \frac{\partial K}{\partial r}(r,s) dr$$

We refer to [2] for the proof of the fact that K^* is a isometry between $\mathcal{H}$ and $L^2(T)$ and, as a consequence, we will have the following relationship between the Skorohod integral with respect to fBm and the Skorohod integral with respect to the Wiener process W

$$\delta^B(u) = \delta\left((K^* u)\right), \text{ if } u \in Dom(\delta^B) \tag{8}$$

We also recall that, if $H > \frac{1}{2}$, for $\phi, \chi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \phi, \chi \rangle_{\mathcal{H}} = H(2H-1) \int_0^{\tau} \int_0^{\tau} \phi(s)\chi(t)|t-s|^{2H-2} ds dt \tag{9}$$

Note that in the general theory of Skorohod integration with respect to fBm with values in a Hilbert space V, a relation such as (8) requires careful justification of the existence of its right-hand side (see [18], Section 5.1). But we will work only with Wiener integrals over Hilbert spaces; in this case we note that, if $u \in L^2(T;V)$ is a deterministic function, then relation (8) holds, the Wiener integral on the right-hand side being well defined in $L^2(\Omega;V)$ if K^*u belongs to $L^2(T \times V)$.

2.2 Infinite dimensional fractional Brownian motion and stochastic integration

Let U a real and separable Hilbert space. We consider Q a self-adjoint and positive operator on U ($Q = Q^* > 0$). It is typical and usually convenient to assume moreover that Q is nuclear ($Q \in L_1(U)$). In this case it is well-known that Q admits a sequence $(\lambda_n)_{n\geq 0}$ of eigenvalues with $0 < \lambda_n \searrow 0$ and $\sum_{n\geq 0} \lambda_n < \infty$. Moreover, the corresponding eigenvectors form an orthonormal basis in U. We define the infinite dimensional fBm on U with covariance Q as

$$B^H(t) = B^H_Q(t) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t) \tag{10}$$

where β^H_n are real, independent fBm's. This process is a U-valued Gaussian process, it starts from 0, has zero mean and covariance

$$E(B^H_Q(t)B^H_Q(s)) = R(s,t)Q, \text{ for every } s,t \in T \tag{11}$$

(see [11], [26], [12]).

We will encounter below cases in which the assumption that Q is nuclear is not convenient. For example one may wish to consider the case of a genuine *cylindrical* fractional Brownian motion on U by setting $\lambda_n \equiv 1$, i.e.

$$B^H(t) = \sum_{n=0}^{\infty} e_n \beta^H_n(t).$$

More generally we state the following.

Remark 1 *Following the standard approach as in [6] for $H = 1/2$, it is possible to define a generalized fractional Brownian motion on U (e.g. in the sense of generalized functions if U is a space of functions) by the right-hand side of formula (10) for any fixed complete orthonormal system $(e_n)_n$ in U, and any fixed sequence of positive numbers $(\lambda_n)_n$, even if $\sum_{n\geq 0} \lambda_n = \infty$. Although for any fixed t the series (10) is not convergent in $L^2(\Omega \times U)$, we consider a Hilbert space U_1 such that $U \subset U_1$ and such that this inclusion is a Hilbert-Schmidt operator. In this way, $B^H(t)$ given by (10) is a well-defined U_1-valued Gaussian stochastic process.*

Let now V be another real separable Hilbert space, B^H_Q the process defined above, defined as a U_1-valued process if necessary (see Remark 1), and $(\Phi_s)_{s\in T}$ a deterministic function with values in $\mathcal{L}_2(U;V)$, the space of Hilbert-Schmidt operators from U to V. The stochastic integral of Φ with respect to B^H is defined by

$$\int_0^t \Phi_s dB^H(s) = \sum_{n=0}^{\infty} \int_0^t \Phi_s e_n d\beta_n^H(s) = \sum_{n=0}^{\infty} \int_0^t (K^*(\Phi e_n))_s d\beta_n(s) \tag{12}$$

where β_n is the standard Brownian motion used to represent β_n^H as in (3), and the above sum is finite when

$$\sum_n \|K^*(\Phi e_n)\|_{L^2(T\times V)}^2 = \sum_n |\|\Phi e_n\|_{\mathcal{H}}|_V^2 < \infty.$$

In this case the integral (12) is well defined as a V-valued Gaussian random variable. However, as we are about to see, the linear additive equation in its evolution form can have a solution even if $\int_0^t \Phi_s dB^H(s)$ is not properly defined as a V-valued Gaussian random variable. A remark similar to Remark 1 applies in order to define this stochastic integral in a larger Hilbert space than V. In particular, there is no reason to assume that $\Phi \in \mathcal{L}_2(U,V)$.

3 Linear stochastic evolution equations with fractional Brownian motion

We will work in this section with a cylindrical fBm B^H on a real separable Hilbert space U, Φ a linear operator in $\mathcal{L}(U,V)$ that is not necessarily Hilbert-Schmidt, and $A : Dom(A) \subset V \to V$ the infinitesimal generator of the strongly continuous semigroup $(e^{tA})_{t\in T}$. We study the equation

$$dX(t) = AX(t)dt + \Phi dB^H(t),\ X(0) = x \in V \tag{13}$$

As previously noted, the stochastic integral $\int_0^t \Phi dB^H(s)$ is only well-defined as a V-valued random variable if $\Phi \in \mathcal{L}_2(U,V)$ since

$$E\left|\int_0^t \Phi dB^H(s)\right|_V^2 = \sum_n E\left|\int_0^t \Phi e_n d\beta_n^H(s)\right|_V^2 = \sum_n E\left|\int_0^t d\beta_n^H(s)\right|^2 |\Phi e_n|_V^2 = t^{2H}\|\Phi\|_{HS}^2$$

where here and in the sequel, $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm.

However, the operator A may be irregular enough that no strong solution to (13) exists even if $\int_0^t \Phi dB^H(s)$ exists. We then consider the so-called *mild* form (a.k.a. *evolution* form) of the equation, whose unique solution, if it exists, can be written in the evolution form

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}\Phi dB^H(s),\ t \in T \tag{14}$$

Our aim is to find necessary and sufficient conditions on A and Φ that this solution exists in $L^2(\Omega)$. For this goal, we will see that it is no longer necessary to even assume that $\int_0^t \Phi dB^H(s)$ exists; in contrast, we only need to guarantee the existence of the stochastic integral in (14). This is the reason for dropping the hypothesis that Φ is Hilbert-Schmidt.

Note that, in the case where V is a space of functions, the so-called *weak* form of (13), using test functions, is another alternative formulation which is morally equivalent to the mild form. We will use this form below in Proposition 1 to formulate a slightly stronger existence result than is possible with the mild form. Proposition 1 excluded, this article deals only with the mild form.

We assume throughout that A is a self-adjoint operator on V. In this situation, it is well known that (see [21], Section 8.3 for a classical account on this topic) there exists a uniquely defined projection-valued measure dP_λ on the real line such that, for every $\phi \in V$, $d\langle \phi, P_\lambda \phi \rangle$ is a Borel measure on R and for every $\phi \in Dom(A)$, we have

$$\langle \phi, A\phi \rangle = \int_R \lambda d\langle \phi, P_\lambda \phi \rangle.$$

Furthermore, for any real-valued Borel function g on R, we can define a self-adjoint operator $g(A)$ by setting

$$\langle \phi, g(A)\phi \rangle = \int_R g(\lambda) d\langle \phi, P_\lambda \phi \rangle \tag{15}$$

for $\phi \in D_g$ with

$$D_g = \{x; \int_R |g(\lambda)|^2 \, d\langle x, P_\lambda x \rangle < \infty\}.$$

The statement of our main existence and uniqueness theorem follows.

Theorem 1 *Let B^H be a cylindrical fBm in a Hilbert space U and let $A : Dom(A) \subset V \to V$ be a self-adjoint operator on a Hilbert space V. Assume that A is a negative operator, and more specifically that there exists some $l > 0$ such that dP_λ is supported on $(-\infty, -l]$. Then for any fixed $\Phi \in \mathcal{L}_2(U, V)$, there exists a unique mild solution $(X(t))_{t \in T}$ of (13) belonging to $L^2(\Omega; V)$ if and only if $\Phi^* G_H(-A)\Phi$ is a trace class operator, where*

$$G_H(\lambda) = (\max(\lambda, 1))^{-2H}. \tag{16}$$

This theorem is valid for both $H < 1/2$ and $H > 1/2$. However, separate proofs are required in each case: Theorems 2 and 3. The most technical calculations, albeit interesting in their own right, are given in the Appendix in order to increase the article's readability.

Remark 2 *Theorem 1 holds for those operators A satisfying only a "spectral gap" condition, i.e. such that dP_λ is supported on $(-\infty, -l]$ except for an atom at $\{0\}$, as long as one assumes that the kernel of A is finite-dimensional. To check this one only needs to include the terms corresponding to $\lambda = 0$ in the proofs of Theorems 2 and 3.*

Remark 3 *When $Supp(P_\lambda) \subset (-\infty, -l)$, with $l > 0$, we can replace $G_H(-A)$ in Theorem 1 by $(-A)^{-2H}$. Seeing this is obvious, for example, in the proof of the case $H > 1/2$ (see Lemma 1 below, and its usage). When A is non-positive with a spectral gap, one can instead replace by $G_H(-A)$ by $(-A+I)^{-2H}$ for example. The spectral gap situation occurs for example in the case of the Laplace-Beltrami operator on compact Lie groups; in this situation, with $H = 1/2$, the trace condition with $(-A+I)^{-2H}$ was proved to be optimal in [24]. This condition is equivalent to conditions presented in work done in [20] for both the stochastic heat and wave equations in Euclidean space $\mathbf{R}^d$ with $d \geq 2$; therein, the authors even treat non-linear equations under a non-degeneracy assumption on the nonlinearity function F (F bounded above and below by positive numbers). Proposition 1 below shows that we can have existence of a weak solution to (13) even if P_λ charges all of $(-\infty, a)$ for some $a \geq 0$. In this case, using $(-A)^{-2H}$, or even $(-A+I)^{-2H}$, instead of $G_H(-A)$ for a trace condition for existence is too strong to be necessary.*

3.1 A fundamental example: the Laplacian on the circle

Before proving the theorem we discuss its consequences for the fundamental example in which the operator A is the Laplacian Δ on the circle. This means that with $e_n(x) = (2\pi)^{-1}\cos nx$ and $f_n(x) = (2\pi)^{-1}\sin nx$ for each $n \in N$, the set of functions $\{e_n, f_n : n \in N\}$ is not only an orthogonal basis for $U = L^2(S^1, dx)$ where dx is the normalized Lebesgue measure on $[-\pi, \pi)$, this set is exactly the set of eigenfunctions of Δ. An infinite-dimensional fractional Brownian motion B^H in $L^2(S^1)$ can be defined by

$$B^H(t,x) = \sum_{n=0}^{\infty} \sqrt{q_n} e_n(x)\, \beta_n^H(t) + \sum_{n=1}^{\infty} \sqrt{q_n} f_n(x)\, \bar{\beta}_n^H(t).$$

where $\{\beta_n^H, \bar{\beta}_n^H : n \in N\}$ is a family of IID standard fractional Brownian motions with common parameter H. If $\sum q_n < \infty$ then B^H is a bonafide $L^2(S^1)$-valued process. Otherwise we can consider that it is a generalized-function-valued process in $L^2(S^1)$, as in remark 1. Note that B^H defined in this way is a Gaussian field on $T \times S^1$ that is fBm in time for fixed x and that is homogeneous in space for fixed t. The spatial covariance function calculates to

$$Q(x-y) = E\left[B^H(1,x)\, B^H(1,y)\right] = \sum_{n=0}^{\infty} q_n \cos(n(x-y))$$

To apply Theorem 1, we only need to represent B^H as $\Phi \tilde{B}^H$ where $\tilde{B}^H$ is cylindrical on $L^2(S^1)$. This is obviously achieved using $\Phi e_n = \sqrt{q_n} e_n$, yielding the following immediate Corollary.

Corollary 1 *Let B^H be the fBm on $L^2(S^1)$ with $H \in (0,1)$ and the assumptions above. Then there exists a square integrable solution of (14) if and only if*

$$\sum_{n=1}^{\infty} q_n n^{-4H} < \infty. \tag{17}$$

This corollary clearly shows that many generalized-function-valued fBm's on $L^2(S^1)$ yield a solution. More precisely, if we define a fractional "antiderivative" of order $2H$ of B^H by $Y = (I-\Delta)_x^{-H} B$, we have existence if and only if Y is a bonafide $L^2(S^1)$-valued process. The following examples may be enlightening, in view of the well-known results for standard Brownian motion.

- Let B^H be fBm in time and white-noise in space, i.e. let $q_n \equiv 1$. Then equation (13) has a unique mild solution in $L^2(S^1)$ if and only if $H > 1/4$.
- More generally consider the equation (13) with space-time fractional noise as a generalization of the well-known space-time white noise. This would mean that B^H is the space derivative of a field Z that is fBm in time *and* in space. Call H' the Hurst parameter of Z in space. To translate this on the behavior of the q_n's we can say that, by analogy with the standard white-noise, and at least up to universal multiplicative constants, we can take $\sqrt{q_n} = n^{1/2-H'}$. Section 5.1 can be consulted for a justification of this argument. Then equation (13) has a unique mild solution in $L^2(S^1)$ if and only if $H' > 1-2H$. Thus if B^H is fractional Brownian in time with $H \geq 1/2$, existence holds for any fractional noise behavior in space, while if B^H is fractional Brownian in time with $H < 1/2$, existence holds if and only if the fractional noise behavior in space exceeds $1-2H$.

- In particular, for dB^H that is space-time fractional noise with the same parameter H in time and space, existence holds if and only if $H > 1/3$.

Remark 4 *The thresholds obtained in the three situations above for the circle should also hold for in any non-degenerate one-dimensional situation. This can be easily established for the Laplace-Beltrami on a smooth compact one-dimensional manifold. We also believe it should hold in non-compact situations such as for the Laplacian on R.*

3.2 The case $H > \frac{1}{2}$

Theorem 2 *Assume $H \in (1/2, 1)$. Then the result of Theorem 1 holds.*

Proof: Let us estimate the mean square of the Wiener integral of (14). For every $t \in T$, it holds ($C(H)$ denoting a generic constant throughout this proof)

$$
\begin{aligned}
I_t = E\left|\int_0^t e^{(t-s)A}\Phi dB^H(s)\right|_V^2 &= E\left|\sum_n \int_0^t e^{(t-s)A}\Phi e_n d\beta_n^H(s)\right|_V^2 \\
&= \sum_n C(H)\int_0^t\int_0^t \langle e^{(t-u)A}\Phi e_n, e^{(t-v)A}\Phi e_n\rangle_V |u-v|^{2H-2}dudv \\
&= C(H)\sum_n \int_0^t\int_0^t \langle e^{(2t-u-v)A}\Phi e_n, \Phi e_n\rangle_V |u-v|^{2H-2}dudv \\
&= 2C(H)\sum_n \int_0^t \left(\int_0^u \langle e^{(2t-2u+v)A}\Phi e_n, \Phi e_n\rangle_V v^{2H-2}dv\right)du. \qquad (18)
\end{aligned}
$$

Consider now the measure $d\mu_n(\lambda)$ defined as

$$
d\mu_n(\lambda) = d\langle \Phi e_n, P_\lambda \Phi e_n\rangle_V \qquad (19)
$$

where P_λ is the spectral measure of the operator $-A$. We have

$$
\langle e^{(2t-2u+v)A}\Phi e_n, \Phi e_n\rangle_V = \int_R e^{(2t-2u+v)\lambda}d\mu_n(\lambda) = \int_0^\infty e^{-(2t-2u+v)\lambda}d\mu_n(\lambda)
$$

because, since $A \le 0$, P_λ vanishes for $\lambda > 0$. The expression (18) becomes, using Fubini theorem

$$
\begin{aligned}
I_t &= C(H)\sum_n \int_0^t\int_0^u v^{2H-2}\left(\int_0^\infty e^{-(2t-2u+v)\lambda}d\mu_n(\lambda)\right)dvdu \\
&= C(H)\sum_n \int_0^\infty e^{-2t\lambda}\int_0^t e^{2u\lambda}\left(\int_0^u v^{2H-2}e^{-v\lambda}dv\right)dud\mu_n(\lambda)
\end{aligned}
$$

and doing the change of variables $v\lambda = v'$ in the integral with respect to dv, and integrating by parts with respect to u, we get

$$
\begin{aligned}
I_t &= C(H)\sum_n \int_0^\infty e^{-2t\lambda}\lambda^{1-2H}\int_0^t e^{2u\lambda}\left(\int_0^{\lambda u} v^{2H-2}e^{-v}dv\right)dud\mu_n(\lambda) \\
&= C(H)\sum_n \int_0^\infty \lambda^{-2H}\left(\int_0^{\lambda t} v^{2H-2}e^{-v}\left[\frac{e^{2\lambda t}-e^{2v}}{e^{2\lambda t}}\right]dv\right)d\mu_n(\lambda) \qquad (20)
\end{aligned}
$$

Denote by

$$
A(\lambda) = \left(\int_0^\lambda v^{2H-2}e^{-v}\left[1-e^{-2(\lambda-v)}\right]dv\right). \qquad (21)
$$

At this point we need the following technical lemma whose proof is given in the Appendix.

Lemma 1 *There exist positive constants* $c(H)$ *and* $C(H)$ *depending only on* H *such that*

(i) *If* $\lambda > 1$, $c(H) \leq A(\lambda) \leq C(H)$, *and*

(ii) *if* $\lambda \leq 1$, $c(H) \leq A(\lambda)\lambda^{-2H} \leq C(H)$.

Using the notation $A \asymp B$ for two quantities whose ratio is bounded above and below by positive constants (in which case we say the quantities are *commensurate*), putting the two estimations of $A(\lambda)$ together we obtain

$$
\begin{aligned}
I_1 &\asymp \sum_n \int_0^1 d\mu_n(\lambda) + \int_1^\infty \lambda^{-2H} d\mu_n(\lambda) \\
&\asymp \sum_n \int_0^\infty (\max(\lambda; 1))^{-2H} d\mu_n(\lambda),
\end{aligned}
$$

where the constants needed in the $\asymp$ relations depend only on H. This yields the theorem for $t = 1$. The case of general t is not more complex: one only needs to multiply the right-hand side in the last equation by t^{2H}; the proof of this fact does not follow, strictly speaking, from the scaling property of fBm, since this scaling does not hold for I_t; we leave the details to the reader. □

3.3 The case $H < \frac{1}{2}$

Theorem 3 *Let* $H \in (0, \frac{1}{2})$, *and let* P_λ *denote the spectral measure of* $-A$. *If there exists a positive constant* l *such that*

$$
\mathit{Supp}\,(P_\lambda) \subset (l; \infty), \tag{22}
$$

then Theorem 1 holds.

Proof. We let P_λ denote the spectral measure of $-A$, and μ_n the corresponding scalar measures as before. Denoting $I_t = E\left|X(t) - e^{tA}x\right|_V^2$, it is sufficient to estimate I_t optimally from above and below. We have

$$
I_t = E\left|\int_0^t e^{(t-s)A}\Phi dB^H(s)\right|_V^2 = E\left|\sum_n \int_0^t e^{(t-s)A}\Phi e_n d\beta_n^H(s)\right|_V^2
$$

Step 1 *(Upper bound).* We prove first the sufficient condition for the existence of a square integrable mild solution of equation (13). We start with the following technical Lemma (its proof is given in the Appendix).

Lemma 2 *Let*

$$
B(a, A) = \int_0^1 ds \exp(-2as)\left[\int_0^s (\exp ar - 1)\, r^{A-1} dr\right]^2
$$

where $a \geq 0$ *and* $A \in (-1/2, 0]$. *Then it holds*

$$
B(a, A) \leq K_A a^{-2A-1}
$$

with K_A *a positive constant depending only on* A.

Using (7) and the representation (8), we have

$$
\begin{aligned}
I_t &\leq 2 \sum_n \int_0^t \left| e^{(t-s)A} \Phi e_n \right|_V^2 K^2(t,s) ds \\
&+ 2 \sum_n \int_0^t \left| \int_s^t \left(e^{(t-r)A} \Phi e_n - e^{(t-s)A} \Phi e_n \right) \frac{\partial K}{\partial r}(r,s) dr \right|_V^2 ds \\
&= \sum_n (I_1(n) + I_2(n))
\end{aligned}
$$

Using the following inequality (see [10], Th. 3.2),

$$
K(t,s) \leq c(H)(t-s)^{H-\frac{1}{2}} s^{H-\frac{1}{2}}
$$

the first sum above can be majorized in the following way

$$
\begin{aligned}
\sum_n I_1(n) &\leq c(H) \sum_n \int_0^t \langle e^{2(t-s)A} \Phi e_n, \Phi e_n \rangle_V (t-s)^{2H-1} s^{2H-1} ds \\
&= c(H) \sum_n \int_0^\infty \lambda^{-2H} \left(\int_0^{2\lambda t} e^{-v} v^{2H-1} (t - \frac{v}{2\lambda})^{2H-1} dv \right) d\mu_n(\lambda) \\
&\leq c(H) \sum_n \int_0^\infty \lambda^{-2H} C\,(t,H)\, d\mu_n(\lambda) \\
&= C\,(t,H)\, Tr(\Phi^* (-A)^{-2H} \Phi)
\end{aligned}
\tag{23}
$$

where $C\,(t,H)$ depends only on t and H. Here we used the fact that

$$
\begin{aligned}
&\int_0^{2\lambda t} e^{-v} v^{2H-1} (t - \frac{v}{2\lambda})^{2H-1} dv \\
&\leq (t/2)^{2H-1} \int_0^\infty e^{-v} v^{2H-1} dv + (\lambda t)^{2H-1} \int_{\lambda t}^{2\lambda t} e^{-v} (t - \frac{v}{2\lambda})^{2H-1} dv \\
&\leq C(t,H) + (\lambda t)^{2H-1} \int_0^{\lambda t} e^{-(2\lambda t - v')} \left(v'/(2\lambda) \right)^{2H-1} dv' \\
&\leq C\,(t,H) + C\,(t,H)\, e^{-\lambda t} \, (\lambda t)^{2H} = C\,(t,H)\,.
\end{aligned}
$$

For the second sum from above, we can write

$$
\begin{aligned}
\sum_n I_2(n) &= \sum_n \int_0^t ds \int_s^t dr_1 \int_s^t dr_2 \frac{\partial K}{\partial r_1}(r_1,s) \frac{\partial K}{\partial r_2}(r_2,s) \\
&\times \langle \left(e^{(t-r_1)A} - e^{(t-s)A} \right) \Phi e_n, \left(e^{(t-r_2)A} - e^{(t-s)A} \right) \Phi e_n \rangle_V \\
&= \sum_n \int_0^t ds \int_s^t dr_1 \int_s^t dr_2 \frac{\partial K}{\partial r_1}(r_1,s) \frac{\partial K}{\partial r_2}(r_2,s) \\
&\times \langle \left(e^{(t-r_1)A} - e^{(t-s)A} \right) \left(e^{(t-r_2)A} - e^{(t-s)A} \right) \Phi e_n, \Phi e_n \rangle_V
\end{aligned}
$$

and, by the fact that $\frac{\partial K}{\partial r}(r,s) \leq 0$ for every $r, s \in T$ and $|\frac{\partial K}{\partial r}(r,s)| \leq C(H)(r-s)^{H-\frac{3}{2}}$, we get

$$\sum_n I_2(n) = \sum_n \int_0^t ds \int_s^t dr_1 \int_s^t dr_2 \frac{\partial K}{\partial r_1}(r_1, s) \frac{\partial K}{\partial r_2}(r_2, s)$$
$$\times \int_0^{+\infty} \left(e^{-\lambda(t-r_1)} - e^{-\lambda(t-s)}\right)\left(e^{-\lambda(t-r_2)} - e^{-\lambda(t-s)}\right) d\mu_n$$
$$\leq C(H) \sum_n \int_0^t du \int_0^u dv_1 \int_0^u dv_2 (u-v_1)^{H-\frac{3}{2}}(u-v_2)^{H-\frac{3}{2}}$$
$$\times \int_0^{\infty} \left(e^{-\lambda(v_1+v_2)} - e^{-\lambda(u+v_2)} - e^{-\lambda(v_1+u)} + e^{-2\lambda u}\right) d\mu_n$$

where we used the change of variables $t-s=u$, $t-r_1=v_1$, $t-r_2=v_2$ and the symmetry of A.

Let us note that the above quantities are positive and therefore we can apply Fubini theorem, obtaining

$$\sum_n I_2(n) = \sum_n \int_0^{\infty} d\mu_n \int_0^t du \int_0^u dv_1 \int_0^u dv_2 (u-v_1)^{H-\frac{3}{2}}(u-v_2)^{H-\frac{3}{2}}$$
$$\times \left(e^{-\lambda(v_1+v_2)} - e^{-\lambda(u+v_2)} - e^{-\lambda(v_1+u)} + e^{-2\lambda u}\right)$$
$$= \sum_n \int_0^{\infty} d\mu_n \int_0^t du \left(\int_0^u (u-v)^{H-\frac{3}{2}}(e^{-\lambda u} - e^{-\lambda v})dv\right)^2$$
$$= \sum_n \int_0^{\infty} d\mu_n \int_0^t e^{-2\lambda s}\left(\int_0^s (e^{\lambda r}-1) r^{H-\frac{3}{2}} dr\right)^2 ds$$
$$= \sum_n \int_0^{\infty} I_2(\lambda, t) d\mu_n(\lambda) \tag{24}$$

where on the last line we came back to the initial variables. Now, applying (24) and Lemma 2 to

$$I_2(\lambda, t) = \int_0^t e^{-2\lambda s}\left(\int_0^s (e^{\lambda r}-1) r^{H-\frac{3}{2}} dr\right)^2 ds,$$

with (23), we have the upper bound

$$I_t \leq C\,(t,H) \sum_n \int_0^{\infty} \lambda^{-2H} d\mu_n(\lambda) = C\,(t,H)\, Tr(\Phi^*(-A)^{-2H}\Phi).$$

Step 2 *(Lower bound).* To prove the necessity, note that

$$I_t = E\left[\left|\sum_n \int_0^t \left(e^{(t-s)A}\Phi e_n\right) K(t,s) d\beta_n(s)\right.\right.$$
$$\left.\left. + \int_0^t \left(\int_s^t \frac{\partial K}{\partial r}(r,s)\left(e^{(t-r)A} - e^{(t-s)A}\right)\Phi e_n dr\right) d\beta_n(s)\right|_V^2\right]$$

and this equals

$$
\begin{aligned}
I_t &= \sum_n \int_0^t \left|e^{(t-s)A}\Phi e_n\right|_V^2 K^2(t,s)ds \\
&+ 2\sum_n \int_0^t K(t,s) \int_s^t \frac{\partial K}{\partial r}(r,s)\langle e^{(t-s)A}\Phi e_n, \left(e^{(t-r)A} - e^{(t-s)A}\right) \Phi e_n\rangle_V drds \\
&+ \sum_n \int_0^t \left|\int_s^t \frac{\partial K}{\partial r}(r,s) \left(e^{(t-r)A} - e^{(t-s)A}\right) \Phi e_n dr\right|_V^2 ds
\end{aligned}
$$

We let $t = 1$ for simplicity and we use the measure $d\mu_n(\lambda) = d\langle \Phi e_n, P_\lambda \Phi e_n\rangle_V$. Taking account that $P_\lambda = 0$ outside $(-\infty, -l)$, we get

$$
\begin{aligned}
I_t &= \sum_n \int_l^\infty \left(\int_0^1 \exp(-2\lambda(1-s))K^2(1,s)ds\right) d\mu_n(\lambda) \\
&+ 2\sum_n \int_l^\infty \int_0^1 ds \exp(-2\lambda(1-s))K(1,s) \\
&\times \left(\int_s^1 (\exp((r-s)\lambda) - 1) \frac{\partial K}{\partial r}(r,s)dr\right) d\mu_n(\lambda) \\
&+ \sum_n \int_l^\infty \left(\int_0^1 \exp(-2\lambda(1-s)) \left(\int_s^1 (\exp((r-s)\lambda) - 1) \frac{\partial K}{\partial r}(r,s)dr\right)^2 ds\right) d\mu_n(\lambda) \\
&= \int_l^\infty J(\lambda)d\mu_n(\lambda)
\end{aligned}
$$

The conclusion of the theorem follows from the next lemma. □

Lemma 3 *Let*

$$
\begin{aligned}
J(\lambda) &= \int_0^1 \exp(-2\lambda(1-s))K^2(1,s)ds \\
&+ 2\int_0^1 \exp(-2\lambda(1-s))K(1,s) \left(\int_s^1 (\exp((r-s)\lambda) - 1) \frac{\partial K}{\partial r}(r,s)dr\right) ds \\
&+ \int_0^1 \exp(-2\lambda(1-s)) \left(\int_s^1 (\exp((r-s)\lambda) - 1) \frac{\partial K}{\partial r}(r,s)dr\right)^2 ds
\end{aligned}
$$

Then $J(\lambda) \geq c(H)\lambda^{-2H}$ for every $\lambda > l > 0$ with l arbitrary small.

Proof: See the Appendix. □

3.4 Extended existence for the weak equation

Assume now that V is a Hilbert space of functions on finite-dimensional Euclidean space E, and assume A is a self-adjoint operator on V. One can interpret the noise term $\Phi B^H(t)$ directly as a Gaussian field on $T \times E$ that is fBm in time and possibly a generalized function in space. For the formulation of an existence result, we keep using representation of this field via the operator $\Phi \in \mathcal{L}(V,V)$ operating on a cylindrical $B^H(t)$ in V. Equation (13) is now reads,

$$
X(dt,x) = [AX(t,\cdot)](x)\,dt + \left[\Phi B^H\right](dt,x),\ X(0) = X_0 \in V, t \geq 0, x \in E
$$

and its *weak* version is

$$\int_E \phi(x) X(t,x)\, dx = \int_E \phi(x) X_0(x)\, dx + \int_E \int_0^t X(t,x) A\phi(x)\, dx dt + \int_E \left[\Phi B^H\right](t,x)\phi(x)\, dx, \tag{25}$$

for all $t \geq 0$, $x \in E$, $\phi \in Dom(A)$. If it happens that the Gaussian field ΦB^H on $T \times E$ is generalized-function-valued in the parameter x, the last term in (25) must be interpreted as

$$\left[\Phi B^H\right](t, \phi)$$

for all test functions ϕ in $Dom(A) \cap dom\left[\Phi B^H(1)\right]$.

More generally, we can formulate a weak equation in an abstract separable Hilbert space V. We assume that A is a self-adjoint operator on V, that B^H is a cylindrical fBm in V, and that $\Phi \in \mathcal{L}(V, V)$. The generalization of (25) is

$$\langle X(t), \phi\rangle = \langle X(0), \phi\rangle + \int_0^t \langle X(s), A\phi\rangle ds + \int_0^t \langle \Phi^* \phi, dB^H(s)\rangle, \tag{26}$$

for all $t \geq 0$ and all test functions ϕ in $Dom(A)$, where $\langle\ ,\ \rangle$ denotes the scalar product in V.

The following proposition shows that the spectral gap condition for existence can be eliminated when dealing only with the weak equation.

Proposition 1 *Let $H \in (0,1)$. Let B^H be a cylindrical fBm in V, a separable Hilbert space, and let $A : Dom(A) \subset V \to V$ be a self-adjoint operator on V such that for some $\lambda_0 > 0$, $A - \lambda_0 I$ is a negative operator. Then for any fixed $\Phi \in \mathcal{L}(V, V)$, there exists a solution $(X(t, \cdot))_{t \in T}$ of (25) belonging to $L^2(\Omega; V)$ as long as $\Phi^* G_H(-A)\Phi$ is a trace class operator.*

Proof. By hypothesis we can find positive numbers μ and ε such that $A - \mu I < -\varepsilon I$, that is to say, the operator $\bar{A} = A - \mu I$ satisfies the hypotheses of both Theorem 2 and Theorem 3. Therefore, in both the cases $H < 1/2$ and $H > 1/2$, we have existence and uniqueness of a mild solution in $L^2(\Omega; V)$ to the following equation:

$$dY_t = (A - \mu I) Y_t dt + \Phi dB_t^H$$

if and only if $\Phi(\mu I - A)^{-2H} \Phi^*$ is trace class. Indeed, one should require, rather, that $\Phi G_H(\mu I - A)\Phi^*$ be trace class, but here the strict negativity of $\bar{A}$ allowed us to replace the function G_H by the function $F_H(\lambda) = \lambda^{-2H}$. Now a simple repetition of arguments of Da Prato and Zabczyk in [6] shows that for any Lipschitz function F on V, the equation

$$dZ_t = (A - \mu I) Z_t dt + F(Z_t)\, dt + \Phi dB_t^H$$

also has a unique mild solution formed by considering the semigroup of the operator $A - \mu I$. By taking $F(z) = \mu z$ we see that the following mild equation has a unique solution Z:

$$Z(t) = e^{t(A-\mu I)} x + \int_0^t e^{(t-s)(A-\mu I)} \Phi dB^H(s) + \mu \int_0^t e^{(t-s)(A-\mu I)} Z(s)\, ds. \tag{27}$$

The next step in the proof is to show that Z defined by (27) also satisfies (25). This can be checked by a classical calculation for all test functions $\phi \in Dom(A - \mu I)$. However this domain is defined as the set of all functions $\phi \in V$ such that $(A - \mu I)\phi \in V$. Thus it coincides with $Dom(A)$, and the weak equation (25) is satisfied by Z.

The last step in the proof is to show that the trace condition on $\Phi\left(\mu I-A\right)^{-2H}\Phi^{*}$ is equivalent to the condition that $\Phi G_{H}\left(-A\right)\Phi^{*}$ be trace class. Recall that for any function F we have

$$tr\left[\Phi F\left(-A\right)\Phi^{*}\right]=\sum_{n}\int_{-\infty}^{\infty}F\left(\lambda\right)d\mu_{n}\left(\lambda\right)$$

where μ_n, defined in (19), is a positive measure for any n. Therefore it is sufficient to show that the function $G_{H}\left(\lambda\right)=\left(\max\left(1,\lambda\right)\right)^{-2H}$ is commensurable with the function $\bar{G}_{H}\left(\lambda\right)=\left(\lambda+\mu\right)^{-2H}$. For $\lambda>1$ this is clear. For $\lambda<1$, we use the fact that the support of all measures $d\mu_n$ is in $[-\lambda_0;+\infty)$. Since it is no restriction to require that $\mu>\lambda_0+\varepsilon$, we have that for $\lambda\in[-\lambda_0;1]$, $\bar{G}_{H}\left(\lambda\right)$ is bounded above by ε^{-2H} and below by $\left(1+\mu\right)^{-2H}$; in this sense it is commensurable with $G_{H}\left(\lambda\right)$ since the latter is equal to 1 in that interval.□

4 Spatial regularity of the solution: the general case

In this section, we give some general results on the spatial regularity of the solution to our linear additive equation. As in Theorem 1, we assume that:

(R) the operator A is self adjoint and there exist $\varepsilon>0$ such that $A\leq-\varepsilon I$.

As in Remark 2 we could also allow A to have 0 as an eigenvalue, with a finite dimensional eigenspace, and then a spectral gap up to $-\varepsilon$. We omit these details.

Our regularity result is based on a proposition taken from [6], which we enunciate here for sake of completeness: let A be an unbounded operator satisfying condition (R). For $\alpha,\gamma\in(0,1)$, $p>1$ and $\psi\in L^p([0,T];V)$, set

$$R_{\alpha,\gamma}\psi(t)=\frac{\sin(\alpha\pi)}{\pi}\int_0^t(t-\sigma)^{\alpha-1}(-A)^{\gamma}e^{(t-\sigma)A}\psi(\sigma)d\sigma,$$

where A^γ has to be interpreted as in (15). It is a known fact (see [6, Proposition A.1.1]) that, if $\alpha>\gamma+\frac{1}{p}$, then

$$R_{\alpha,\gamma}\in\mathcal{L}\left(L^p([0,T];V);C^{\alpha-\gamma-\frac{1}{p}}\left([0,T];D((-A)^{\gamma})\right)\right). \tag{28}$$

Let now X be the process defined by relation (14) with $x=0$, that is the usual stochastic convolution of B^H by A. The main result of this section is the following:

Theorem 4 *Let $H\in(0,1)$, and suppose that for $\alpha\in(0,H)$, the operator*

$$\Phi^*(-A)^{-2(H-\alpha)}\Phi$$

is trace class. Then, for any $\gamma<\alpha$ and any $\varepsilon<(\alpha-\gamma)$, almost surely,

$$X\in C^{\alpha-\gamma-\varepsilon}\left([0,T];D\left((-A)^{\gamma}\right)\right).$$

In particular, for any fixed $t>0$, $X(t)\in D((-A)^\gamma)$.

Proof: Under our assumptions, it can be shown by the usual factorization method (see e.g. [6, Theorem 5.2.6]) that the process $(-A)^\gamma X$ can be written as

$$(-A)^{\gamma}X(t)=\left[R_{\alpha,\gamma}Y_{\alpha}\right](t),$$

where the process Y_α is defined by

$$Y_\alpha(s) = \int_0^s (s-\sigma)^{-\alpha} e^{(s-\sigma)A} \phi dB^H(\sigma).$$

Then, using relation (28), we are reduced to showing that $Y_\alpha \in L^p([0,T];V)$, and since Y_α is a Gaussian process, it is sufficient to prove that $Y_\alpha \in L^2([0,T];V)$.

We first treat the case of $H > \frac{1}{2}$: along the same lines as in the proof of Theorem 2, and taking up the notations introduced therein, it can be seen that

$$E\left[|Y_\alpha(t)|_V^2\right] = C(H) \sum_n \int_0^\infty \lambda^{-2(H-\alpha)} M_\alpha(\lambda,t) d\mu_n(\lambda),$$

where

$$M_\alpha(\lambda,t) = \int_0^{\lambda t} x^{-\alpha} e^{-x} \left(\int_0^x y^{-\alpha}(x-y)^{2H-2} e^{-y} dy \right) dx.$$

Since M_α is obviously bounded by a constant for all $t, \lambda > 0$, whenever $\alpha < H$, we get the desired result.

Let us now turn to the case $H < \frac{1}{2}$. Following again the proof of Theorem 3, we can decompose $E[|Y_\alpha(t)|_V^2]$ as

$$E\left[|Y_\alpha(t)|_V^2\right] = \sum_n I_1(n) + I_2(n),$$

where $I_2(n)$, that contains the main part of the contribution to the norm of $Y_\alpha(t)$, is defined by

$$I_2(n) = \int_0^t \left| \int_s^t \left((t-r)^{-\alpha} e^{(t-r)A} \phi e_n - (t-s)^{-\alpha} e^{(t-s)A} \phi e_n \right) \frac{\partial K}{\partial r}(r,s) \right|_V^2 ds.$$

Now, the same computations as in the proof of Theorem 3 yield

$$\begin{aligned} I_2(n) &\leq C(H) \int_0^\infty d\mu_n(\lambda) \int_0^t du \int_0^u dv_1 \int_0^u dv_2 (u-v_1)^{H-3/2}(u-v_2)^{H-3/2} \\ &\quad \times \left((v_1 v_2)^{-\alpha} e^{-\lambda(v_1+v_2)} - (v_1 u)^{-\alpha} e^{-\lambda(v_1+u)} - (u v_2)^{-\alpha} e^{-\lambda(u+v_2)} + u^{-2\alpha} e^{-2\lambda u} \right) \\ &= C(H) \int_0^\infty d\mu_n(\lambda) \int_0^t \left(\int_0^u (u-v)^{H-3/2} \left(u^{-\alpha} e^{-\lambda u} - v^{-\alpha} e^{-\lambda v} \right) dv \right)^2 du \\ &= C(H) \int_0^\infty \lambda^{-2(H-\alpha)} N(\lambda t) d\mu_n(\lambda), \end{aligned}$$

where $N(\tau)$ is given by

$$N(\tau) = \int_0^\tau \left(\int_0^x (x-y)^{H-3/2} \left(y^{-\alpha} e^{-y} - x^{-\alpha} e^{-x} \right) dy \right)^2 dx.$$

The following lemma ends the proof. □

Lemma 4 *If* $a < H$, *then* $\sup_{\tau \geq 0} N(\tau) < \infty$

Proof: Left to the reader. □

5 Spatial regularity of the solution: a detailed study of the circle

The purpose of this section is to present a characterization of almost-sure spatial moduli of continuity for the solution X of the linear additive equation. This characterization is new for stochastic PDEs driven by fBm, even in the Brownian case $H = 1/2$. The proofs are based on results characterizing the modulus of continuity for a purely spatial Gaussian field. As far as we know, these results are new.

In our effort to give results that are as sharp as possible, we specialize to the case of the Laplacian on the one-dimensional circle S^1. It is easy to extend all our sufficient conditions for continuity of X to higher-dimensional spaces, and/or much more general operators; the difficulty is in extending the necessary conditions. We will tackle the issue of sharp necessary conditions ("lower bounds") for more general operators and spaces in a subsequent publication. In our present situation, we will show that the necessary and sufficient conditions coincide for a nontrivial class of moduli of continuity. The reader may notice that our "lower bounds" proofs below make extensive use of a property of spatial isotropy for W. Since we always assume that W is spatially homogeneous, in the case of the circle S^1 isotropy is always satisfied. In higher-dimensional problems, we believe there is hope of extending our one-dimensional lower bound results only in the isotropic case. We consider two types of conditions for guaranteeing/characterizing the fact that X admits a given fixed function f as an almost-sure uniform modulus of continuity:

- Type I (an *intrinsic* or *pathwise* condition): the fact that the same continuity holds for the "fractional spatial antiderivative" of W, $Y := (I - \Delta)^{-H} W$;
- Type II (a condition *on the distribution*): a convergence condition on the coefficients of W's spatial covariance.

Establishing Type I and Type II conditions will benefit greatly from the sharp calculations that were performed in the previous sections to establish necessary and sufficient conditions for existence of X. In fact, our work above reduces most of our task below to regularity questions for Gaussian fields on S^1 only (spatial dependence), as opposed to fields on $[0, \infty) \times S^1$ (space-time dependence). To establish the "necessary", or "lower bound" portion of the Type I condition, we will need a technical assumption which amounts to requiring that X is not Hölder-continuous. Without this assumption, in the Hölder scale, we will show a slightly weaker result. For the Type II condition, the "necessary" condition requires an assumption which limits the regularity of X slightly, excluding the moduli that are more irregular than any function f_α defined by $f_\alpha(r) = (\log(1/r))^{-\alpha}$ for $\alpha > 0$. The "sufficient", or "upper bound" Type II condition requires a mild technical assumption which does not limit the regularity scales one may wish to consider. Summarizing,

- Type I and Type II conditions are always sufficient;
- the Type I condition is necessary if the modulus is not too regular, i.e. not Hölder; in the Hölder scale, the Type I condition is nearly necessary;
- the Type II condition is necessary if the modulus is not too irregular;
- there is a range of moduli for which both Type I and Type II conditions are necessary and sufficient; it includes the class of moduli $\{f_\alpha : \alpha > 0\}$ defined by

$$f_\alpha(r) = (\log(1/r))^{-\alpha}.$$

Extending the ranges of validity of the necessary conditions will be the subject of future work. A partial extension is presented below, in Corollary 4, where it can be seen that the Type II condition is morally necessary in all cases.

5.1 Tools

This section presents the tools that are required for our study. The results which we establish, and their proofs, are of intrinsic value in the theory of Gaussian regularity. We have specialized the study to the case of the circle. However, it is not difficult to modify the arguments to fit many one-dimensional situations, and, as alluded to above, many isotropic higher-dimensional settings as well. For the sake of continuity and readability, the proofs are presented in the Appendix, in Sections 7.2 and 7.3.

The metric $|\cdot|$ on S^1 identified to $[0,2\pi)$ coincides with the usual Euclidean distance on any subinterval of length π, normalized by the factor 2π, with an obvious extension to the whole of S^1 due to the identification of 0 and 2π. Let $\{e_n\}$ be the orthonormal basis of $L^2 = L^2\left(S^1\right)$ made of trigonometric functions, namely the set of eigenfunctions of Δ on S^1. We recall the expression of the fBm on $L^2(S^1)$ introduced in Section 2.

For $\{q_n\}_n$ a sequence of nonnegative terms, let

$$B(t,x) = B^H(t,x) = \sum_n \sqrt{q_n} e_n(x) \beta_n^H(t), \tag{29}$$

where β_n^H are IID fBm's with constant Hurst parameter $H \in (0,1)$. As before, we allow this definition to be formal in x, i.e. we allow $W(t,\cdot)$ to be generalized-function-valued. We've proved in Corollary 1 and Corollary 2 that the (unique $L^2\left(S^1\right)$-function-valued) solution X to the stochastic heat equation (1) with $F = 1$ (exists and) is given by

$$X(t,x) = \sum_n \sqrt{q_n} e_n(x) \int_0^t e^{-n^2(t-s)} \beta_n^H(ds) \tag{30}$$

if and only if

$$\sum_n q_n \frac{1}{n^{4H}} < \infty.$$

We have assumed that $X(0,\cdot) \equiv 0$. Other initial conditions would not change the arguments below. Note also that both W and X are spatially homogeneous Gaussian random fields. In particular, we get that for fixed t, $X(t,\cdot)$ is almost-surely in L^2 if and only if for each fixed x, $E\left[X(t,x)^2\right] < \infty$. Our purpose now is to seek a stronger condition on $(q_n)_n$ which characterizes existence of a solution whose almost-sure spatial modulus of continuity is specified. We start with some definitions.

Definition 1 *Let f be a continuous increasing function on $\mathbb{R}_+$ such that $lim_{0^+} f = 0$. Let $\left\{Y(x) : x \in S^1\right\}$ be a bonafide random field on S^1 ($Y(t)$ is almost-surely a bonafide function).*

- *We say that f is an almost-sure spatial uniform modulus of continuity for Y if there exists an almost-surely positive (non-zero) random variable α_0 such that*

$$\alpha < \alpha_0 \Longrightarrow \sup_{x,y \in S_1; |x-y|<\alpha} |Y(x) - Y(y)| \le f(\alpha).$$

- *The canonical metric δ of Y is defined as*

$$\delta(x,y) = \left(E\left[(Y(x) - Y(y))^2\right]\right)^{1/2}.$$

Remark 5 *If Y is a spatially homogeneous Gaussian field on S^1 (i.e. it is Gaussian and its covariance depends only on differences between points) then $\delta(x,y) = \delta(|x-y|)$ where $\mathbb{R}_+ \ni r \mapsto \delta(r)$ is some continuous function on a neighborhood of 0. Indeed, by homogeneity there exists some continuous function δ such that $\delta(x,y) = \delta(x-y)$, and by symmetry this also equals $\delta(y-x)$, i.e. $\delta(r) = \delta(|r|)$.*

Definition 2 *We call $\delta(\cdot)$ the canonical metric function of Y.*

Note for example that for scalar fBm $\left\{B^H(t) : t \in [0,1]\right\}$ we have $\delta(r) = r^H$.

Condition A We will assume throughout that δ is differentiable except at 0, and that $\lim_{0+} \delta' = +\infty$. Without loss of generality this implies that δ is concave in a neighborhood of 0.

In terms of regularity properties of Y, differentiability of δ except at 0 introduces no loss of generality. The condition that δ' at zero is infinite introduces no loss of generality outside of the very narrow class of processes Y that are a.s. β-Hölder-continuous for all $\beta < 1$ but that are not a.s. of class C^1. For this class of processes, similar results to those we prove here hold, but the methods of proof are substantially different. We do not comment on these processes further.

Remark 6 *Assumption A implies that δ is strictly increasing in a neighborhood of 0.*

Remark 7 (Random Fourier series representation) *For any spatially homogeneous Gaussian field Y on S^1 with canonical metric function δ, there exists a sequence $\{r_n\}_{n=0}^{\infty}$ of non-negative terms such that*

$$\delta(r)^2 = \sum_{n=1}^{\infty} r_n (1 - \cos(nr)). \tag{31}$$

Indeed, any such Y can be written as a random Fourier series

$$Y(x) = Y_0\sqrt{q_0} + 2\sum_{n=1}^{\infty} \sqrt{r_n}\{Y_n \cos(nx) + Z_n \sin(nx)\} \tag{32}$$

where all Y_n's and Z_n's are IID $N(0,1)$ r.v.'s. Then just calculate δ^2.

We now introduce a condition needed for the lower bound proof of the next theorem which characterizes the regularity of homogeneous Gaussian processes. This condition is satisfied for the class of functions δ defined by $\delta(r) = (\log(1/r))^{-p}$ for any $p > 0$ no matter how large, and for all functions that are more irregular than this class, but is not satisfied in the power scale defined by $\delta(r) = r^{\alpha}$ for any $\alpha \in (0,1)$.

Condition B There exists a constant $c > 0$ such that in a neighborhood of 0 we have

$$\int_0^{\alpha} \delta(r) \frac{dr}{r\sqrt{\log(r^{-1})}} > c\delta(\alpha)\sqrt{\log(\alpha^{-1})}$$

Theorem 5 *Let Y be a Gaussian random field on S^1 with canonical metric function δ. Let*

$$f_\delta(\alpha) = \int_0^{\delta(\alpha)} \sqrt{\log \frac{1}{\delta^{-1}(\varepsilon)}} d\varepsilon \tag{33}$$

$$= \int_0^{\alpha} \delta(r) \frac{dr}{2r\sqrt{\log(r^{-1})}} + \delta(\alpha)\sqrt{\log(\alpha^{-1})} \tag{34}$$

$$= \int_0^{1} \frac{\delta(\min(r,\alpha))\, dr}{2r\sqrt{\log(r^{-1})}} \tag{35}$$

$$= \int_0^{\infty} \delta\left(\min\left(e^{-x^2}, \alpha\right)\right) dx. \tag{36}$$

There is a constant K depending only on the law of Y such that the following hold.

(a) Lower bound. *If Conditions A and B hold, if f is an almost-sure uniform modulus of continuity for Y on S^1, then for all α small enough,*

$$Kf(\alpha) \geq f_\delta(\alpha). \tag{37}$$

(b) Upper bound. *If $\lim_{0^+} f_\delta = 0$ then Kf_δ is an almost-sure modulus of continuity for Y on S^1.*

This theorem, whose upper bound is well-known, shows that the function f_δ is, up to a constant, an exact uniform modulus of continuity for Y, as long as Y is more irregular than Hölder. The next theorem shows that one can do nearly as well in the Hölder scale.

Corollary 2 *Assume $\delta(r) = r^\alpha$ for some $\alpha > 0$. Note that this is the case of spatially fractional Brownian motion. Then the Lower Bound (a) in Theorem 5 holds even though Condition B is not satisfied, if one replaces $f(\alpha)$ by $f(\alpha)\log(1/\alpha)$ in line (37).*

Theorem 5 and its Corollary are the key to our Type I characterization. Our Type II general theorem translates the magnitude of δ – and thus, by Theorems 5 and Corollary 2, the regularity of Y – into a condition on the summability of the q_n's. For this Type II characterization, the upper bound requires the following condition, which is stated relative to the upper bound g on the canonical metric in condition (c) below. One can think of this condition as a condition on $g = \delta^2$.

Condition C There exists constants $c, y_0 > 0$ such that for all $0 < x < y < y_0$

$$g(x)/x^2 - g(y)/y^2 \geq c\,(g(y) - g(x))/y^2.$$

The lower bound requires a different condition on the tail behavior of the converging series $\sum_n r_n$.

Condition C' With $B_n := \sum_{m=n}^{\infty} r_m$, with $[x]$ denoting the integer part of x, for all n large enough,

$$B_n \asymp \sum_{k=0}^{\infty} \sum_{m=[2k\pi n]+n+1}^{[2k\pi n]+5n-1} r_m.$$

Remark 8 *Condition C' can be shown to be implied the following two facts: (i) $B_n \asymp B_{[2\pi n]}$; and (ii) the sequence $(B_n - B_{n-1})_n$ is monotone.*

Conditions C and C' essentially place no restriction on the regularity of the canonical metrics that can be used in Type II characterizations. Indeed, both these conditions are satisfied for all the following basic examples:

- "Hölder" scale: $\delta(r)^2 = r^{2\alpha}$ for any $\alpha \in (0,1)$; up to logarithmic corrections, this scale corresponds to the Hölder scale of almost-sure uniform moduli of continuity $f(r) = r^{\alpha}$;

- logarithmic scale: $\delta(r)^2 = (\log(1/r))^{-1-2\varepsilon}$ for any $\varepsilon > 0$; this scale corresponds to the scale of moduli of continuity given by $f(r) = (\log(1/r))^{-\varepsilon}$;

- iterated logarithmic scale:

$$\delta(r)^2 = (\log(1/r))^{-1}\left(\log\log(1/r)\cdots\log_{(n-1)}(1/r)\right)^{-2}\left(\log_{(n)}(1/r)\right)^{-2-2\varepsilon}$$

  for any $n \in \{2,3,\cdots\}$ and any $\varepsilon > 0$; here $\log_{(n)}$ denotes the n-fold iterated logarithm; this scale corresponds to the scale of moduli of continuity given by $f(r) = \left(\log_{(n)}(1/r)\right)^{-\varepsilon}$.

Conditions C and C' even work in a scale which yields a.s. discontinuous Y, although this scale cannot be used for our purposes:

- logarithmic scale for discontinuous processes: $\delta(r)^2 = (\log(1/r))^{-\varepsilon}$ for any $\varepsilon \in (0,1]$.

Our Type II general theorem is the following.

Theorem 6 *Let Y be a homogeneous Gaussian random field on S^1 with canonical metric function δ. Let $\{r_n\}_n$ be the sequence defined by the random Fourier series representation (32) for Y. Let g be a strictly increasing continuous function on $\mathbb{R}_+$, continuously differentiable on $(0,\infty)$, with $\lim_{0^+} g = 0$. Consider the following statements:*

(c) *There exist a constant $K > 0$ such that for all $r \geq 0$, $\delta(r) \leq K\sqrt{g(r)}$*

(d) *For any strictly decreasing, positive function h on a neighborhood of 0 with $\int_0 h(x)\,dx < \infty$:*

$$\sum_n r_n h\left(g\left(\frac{1}{n}\right)\right) < \infty.$$

Under Condition C', we have (c)$\Longrightarrow$(d). The converse (d)$\Longrightarrow$(c) holds if we assume Condition C.

5.2 Type I characterization: pathwise

We now describe how to use the first theorem to compare the almost-sure regularities of W and X. We still use the notation $\asymp$ for *commensurate* quantities: for positive functions A and B of any variable ξ, $A(\xi) \asymp B(\xi)$ means their ratio is bounded away from 0 and ∞. In what follows t is a fixed positive value. The constants used below are either only dependent on H or, if they also depend on t, they are bounded away from 0 and ∞ as soon as the same holds for t.

In the proof of Theorem 1, we have established that the variance of the centered Gaussian r.v. $\int_0^t e^{-n^2(t-s)} B_n^H(ds)$ is commensurate with n^{-4H} (also see Corollary 1). Therefore, for fixed $t>0$, $X(t,\cdot)$ is a homogeneous Gaussian process whose random Fourier series expansion, given by (30), can be written as the expansion

$$X(t,\cdot) = \sum \sqrt{s_n} e_n(\cdot) W_n$$

where the W_n's are IID standard normals, where $s_0 = q_0$, and where the coefficients $s_n, n \geq 1$, which do depend on t, are nevertheless commensurate with q_n/n^{4H}:

$$s_n \asymp q_n/n^{4H}.$$

Let $Y = (I-\Delta)_x^{-H} W$. The operator $(I-\Delta)_x^{-H}$ on L^2 is defined, as in (15), by saying that

$$(I-\Delta)_x^{-H} e_n(x) = e_n(x) / \left(1+n^2\right)^H.$$

Y can be interpreted as an "antiderivative of order $2H$" for W. For fixed $t>0$, the expansion of Y is a random Fourier series of the form

$$Y(t,\cdot) = \sum \sqrt{r_n} e_n(\cdot) W_n'$$

where r_n is commensurate with s_n:

$$s_n \asymp r_n. \tag{38}$$

Now assume that Y has for fixed t, an almost-sure uniform spatial modulus of continuity f. Let δ_Y be the canonical metric function for $Y(t,\cdot)$. Assume δ_Y satisfies Assumption A and Condition B. Then by Theorem 5 part (a), for some $K>0$, for all α small enough,

$$Kf(\alpha) \geq \int_0^\infty \delta_Y\left(\min\left(e^{-x^2},\alpha\right)\right) dx.$$

Because of formula (31) and the fact that $s_n \asymp r_n$ we get that for some (possibly different) constant K, for all small r,

$$\begin{aligned}
\delta_Y(r) &= \sum_{n=1}^\infty \sqrt{r_n}\,(1-\cos(nr)) \\
&\geq K \sum_{n=1}^\infty \sqrt{s_n}\,(1-\cos(nr)) \\
&= K\delta_X(r),
\end{aligned}$$

where δ_X is the canonical metric function for $X(t,\cdot)$. Thus for some constant K,

$$Kf(\alpha) \geq \int_0^\infty \delta_X\left(\min\left(e^{-x^2},\alpha\right)\right) dx = f_{\delta_X}(\alpha),$$

Now use part (b) of Theorem 5: since $\lim_{0^+} f = 0$, the same holds for f_{δ_X}, and we get that f_{δ_X} is an almost-sure uniform modulus of continuity for $X(t,\cdot)$ up to a constant. Since $Kf \geq f_{\delta_X}$ we get that f itself is an almost-sure uniform modulus of continuity for $X(t,\cdot)$. Since $s_n \asymp r_n$, the roles of X and Y can be swapped, which proves the following theorem, modulo the statements in the Hölder case, which are clear given Corollary 2.

Theorem 7 *Let X, Y be as above, relative to W. Let the function δ_Y be defined by*

$$\delta_Y(r) = \sum_{n=1}^{\infty} q_n \frac{1}{n^{4H}} (1 - \cos(nr))$$

We assume Conditions A and B hold for δ_Y. Let f be an increasing continuous function on $\mathbb{R}_+$ with $\lim_{0^+} f = 0$. For any fixed $t > 0$, f is, up to a multiplicative constant, an almost-sure uniform modulus of continuity for $Y(t, \cdot)$ if and only if f is, up to a multiplicative constant, an almost-sure uniform modulus of continuity for $X(t, \cdot)$. Also, δ_Y is the canonical metric function of $Y(1, \cdot)$, and the function f_Y defined by

$$f_Y(\alpha) = \int_0^{\infty} \delta_Y\left(\min\left(e^{-x^2}, \alpha\right)\right) dx$$

is also an almost-sure uniform modulus of continuity for both $Y(t, \cdot)$ and $X(t, \cdot)$, and is bounded above by a constant multiple of f.

In the Hölder case $\delta_Y(r) = r^{\alpha}$ for some $\alpha \in (0, 1)$, Condition B is not satisfied. However, we can assert that if f is, up to a multiplicative constant, an almost-sure uniform modulus of continuity for $Y(t, \cdot)$, then $\bar{f}(r) = f(r) \log(1/r)$ is an almost-sure uniform modulus of continuity for $X(t, \cdot)$, and the same statement holds if one exchanges the roles of X and Y.

As an illustration, we reconsider the examples after Corollary 1. In this development, we omit the appellation "almost-sure, uniform" when talking about spatial moduli of continuity.

- Consider the second example after Corollary 1 and assume $H < 1/2$. Specifically assume that Z is a Gaussian field on $R_+ \times S^1$ that is fBm in time with parameter H and is fBm in space with parameter H', and that $B^H = (I - \Delta)^{1/2} Z$. Then we have $Y = (I - \Delta)^{1/2 - H} Z$. We see again that there is existence of X if and only if $H' > 1 - 2H$. But if we cannot guarantee that H'exceeds $1 - 2H$, Theorem 7 asserts that only a spatial "derivative" of Z of order $1-2H$ needs to exist; specifically, for example, the spatial modulus of continuity of X is commensurate with $f_\alpha(r) = (\log(1/r))^{-\alpha}$ for some fixed $\alpha > 0$ if and only if the same holds for the spatial derivative or order $1 - 2H$ of Z. In this situation, Z is spatially more regular than fBm of parameter $1 - 2H$, but is not spatially fBm for any parameter $H' > 1 - 2H$.

- Consider now the case where indeed Z is spatially fBm with parameter $H' > 1 - 2H$. One can check that a sharp spatial modulus of continuity for Y is $f(r) = r^{H'-1+2H} \log^{1/2}(1/r)$. Theorem 7 then asserts the following.

  - For the equation (13) driven by space-time fractional noise with Hurst parameters H and H' respectively, the evolution solution X admits

    $$f(r) = r^{H'-1+2H} \log^{1/2}(1/r)$$

    as a modulus of continuity. Note here that the full force of the characterization is being used because we start with a bound on the canonical metric of the potential, and can reprove Theorem 7 without needing to invoke the "lower bound" portion (a) of Theorem 5 and Corollary 2 (see Corollary 3).

– If the evolution solution of equation (13) has fractional Brownian regularity in space, in the sense that for some $H'' \in (0,1)$, it admits $f(r) = r^{H''} \log^{1/2}(1/r)$ as a spatial modulus of continuity, then the equation's potential is the spatial derivative of a Gaussian field which admits $\bar{f}(r) = r^{H''+1-2H} \log^{3/2}(1/r)$ as a spatial modulus of continuity

– In the previous "necessary condition implication", we do not know if the logarithmic corrections can be disposed of, because we do not know whether Corollary 2 is sharp. However, in the Hölder scale, these corrections can be viewed as irrelevant.

- With regards to the situation in which $H = H' = 1-2H = 1/3$, since then Y cannot be Hölder continuous, we can try to invoke Theorem 5 without needing Corollary 2. We get the following.

  – For the equation (13) driven by space-time fractional noise with common Hurst parameters $1/3$ in time and space, the evolution solution X does not exist. This can be established using the results of Section 3 only.

  – However, in the case $H = 1/3$, assume Y admits $f(r) = r^{1/3}\tilde{f}(r)$ as a spatial modulus of continuity where $\lim_0 \tilde{f} = 0$ and $\tilde{f}(r) \gg r^{\alpha}$ for all $\alpha > 0$. Then $\tilde{f}(r)$ is a spatial modulus of continuity for X, and the converse holds, still assuming $H = 1/3$.

5.3 Type II characterization: summability interpretation

A slightly weaker version of Theorem 7 can be formulated without Condition B if one is willing to change from a *pathwise* to a *distributional* hypothesis. The distributional hypothesis we make here is that the function f_Y, which can be calculate directly from the law of W, is continuous at 0. The final conclusion of the corollary seems to be a pathwise statement, but we still consider it a Type II characterization because f_Y is characterized by the distribution of W.

Corollary 3 *Let W, X, Y, δ_Y, f_Y be as in Theorem 7, and let δ_X and f_X be defined similarly relative to X. We have*

$$\lim_{r\downarrow 0} f_Y(r) = 0 \Longleftrightarrow \lim_{r\downarrow 0} f_X(r) = 0.$$

In that situation $X(1,\cdot)$ and $Y(1,\cdot)$ share both f_Y and f_X as a.s. uniform moduli of continuity. Consequently f_Y is an a.s. uniform spatial modulus of continuity of continuity for X if and only if the same holds for Y.

Proof: The proof follows from Theorem 5 part (b) and the fact, made trivial by relations (31), (35), and (38), that $f_Y \asymp f_X$. □

The next lemma shows how to invert the formula that gives the almost-sure modulus of continuity from the canonical metric function. In the notation of Theorem 5, it is interesting to note that this lemma implies that $\delta \mapsto f_\delta$ is a bijective linear map.

Lemma 5 *Let δ be an increasing continuous function on $\mathbb{R}_+$ with $\lim_{0^+} \delta = 0$. Let*

$$f(\alpha) = f_\delta(\alpha) := \int_0^\infty \delta\left(\min\left(e^{-x^2}, \alpha\right)\right) dx = \int_0^{\delta(\alpha)} \sqrt{\log 1/\delta^{-1}(\varepsilon)}\, d\varepsilon. \tag{39}$$

Then

$$\begin{aligned}
\delta(\alpha) = \delta_f(\alpha) &= \int_0^{f(\alpha)} \left(\log 1/f^{-1}(\varepsilon)\right)^{-1/2} d\varepsilon \\
&= \int_0^{\alpha} f'(r)\left(\log(1/r)\right)^{-1/2} dr \\
&= f(\alpha)\left(\log(1/\alpha)\right)^{-1/2} - \int_0^{\alpha} f(r)\left(\log(1/r)\right)^{-3/2}(2r)^{-1} dr \qquad (40)
\end{aligned}$$

Proof: trivial. □.

This lemma poses a difficulty in terms of monotonicity of δ with respect to f, because of the negative term in condition (40). We introduce the following condition to circumvent this difficulty.

Condition D For some $c \in (0,1)$, for α small enough,

$$cf(\alpha)\left(\log(1/\alpha)\right)^{-1/2} \geq \int_0^{\alpha} f(r)\left(\log(1/r)\right)^{-3/2}(2r)^{-1} dr$$

This condition guarantees that the δ corresponding to f is bounded above and below by constant multiples of $f(\alpha)\left(\log(1/\alpha)\right)^{-1/2}$. This is satisfied for f of the form $\left(\log(1/\alpha)\right)^{-p}$ for some $p > 0$, and for all f of a lower order than this scale (e.g. the Hölder scale), but is not satisfied for f of the form $1/\log_n(1/\alpha)$, $n \geq 2$, where $\log_n$ denotes the n-fold iterated logarithm. We note that although this condition works in the opposite direction as Condition B, the intersection of the δ's satisfying Conditions B and D contains the class of δ's defined by $\delta(r) = \left(\log(1/r)\right)^{-(1/2+p)}$ for any $p > 0$, or equivalently the class of f's defined by $f(r) = \left(\log(1/r)\right)^{-p}$ for any $p > 0$.

Theorem 8 *Let f be an increasing continuous function on $\mathbb{R}_+$ with $\lim_{0^+} f = 0$. Let δ be given by (40). Let W be defined by (29), and X, Y be as above relative to W. The following conditions are equivalent:*

(e) *for some fixed $t > 0$, $X(t,\cdot)$ has a constant multiple of f as an almost-sure uniform modulus of continuity;*

(e') *for some fixed $t > 0$, $Y(t,\cdot)$ has a constant multiple of f as an almost-sure uniform modulus of continuity;*

(f) *for all $t > 0$, $X(t,\cdot)$ and $Y(t,\cdot)$ both have a constant multiple of f as an almost-sure modulus of continuity;*

If δ satisfies Condition C, then (e), (e') and (f) are implied by the following:

(g) *for any continuous, decreasing, differentiable function h on $(0,1]$ with $\int_0^1 h(x)\,dx < \infty$,*

$$\sum_n \frac{q_n}{n^{4H}} h\left(\delta\left(\frac{1}{n}\right)^2\right) < \infty. \qquad (41)$$

Conversely (g) is implied by any of the previous three conditions as long as f and $f_Y := f_{\delta_Y}$ satisfy Conditions B, C', and D, where δ_Y is the canonical metric function of $Y(1,\cdot)$.

Remark 9 *In [25], a conjecture in the direction of Theorem 8 was formulated. The authors believed the above result would hold with* $h(r) = r^{-1}$ *in condition (g). This Theorem shows that such a condition (g) would be too strong. In fact, one can say that the gap in regularity that is introduced by the stronger version of (g) translates into a factor of order* $\left(\log\left(\delta^{-2}(r)\right)\right)^{1/2}$*; this factor is not visible in the Hölder scale, which explains why in [24], in which only the Hölder scale is considered, it had been possible to formulate necessary and sufficient conditions whose naive generalization would lead the authors to the slightly erroneous conjecture of [25].*

Remark 10 *In [24] and [25], the authors had formulated results similar to Theorems 7 and 8 in the belief that a Type II characterization was a necessary intermediate step in the proof of a Type I characterization. The proofs we propose here show that the two types of characterizations can be established independently of each other.*

Proof. We first prove the "Converse" part. Note that under Condition B, the equivalence of (e), (e') and (f) follows from Theorem 7. So we only need to prove (e') implies (g). First note by Theorem 7, with δ_Y the canonical metric function of $Y(1,\cdot)$, (e') implies

$$f_Y(\alpha) := \int_0^\infty \delta_Y\left(\min\left(e^{-x^2},\alpha\right)\right)dx \le Kf(\alpha)$$

for some constant $K > 0$. Therefore, using equation (40) and Condition D, for some constants K_1, K_2, K_3,

$$\begin{aligned}\delta_Y(r) &\le K_1 f_Y(\alpha)\left(\log(1/\alpha)\right)^{-1/2}\\ &\le K_2 f(\alpha)\left(\log(1/\alpha)\right)^{-1/2}\\ &\le K_3\delta(\alpha)\end{aligned} \tag{42}$$

Recall that $r_n = q_n n^{-4H}$ are the coefficients of the expansion of δ_Y in the form of (31). Assuming Condition C', inequality (42) allows us to use the implication "(c) implies (d)" from Theorem 6 with $g = \delta^2$ to conclude that for any decreasing, differentiable function h on $[0,1]$ with $\int_0^1 h(x)\,dx < \infty$,

$$\sum_n \frac{q_n}{n^{4H}} h\left(\delta\left(\frac{1}{n}\right)^2\right) < \infty.$$

We have thus proved (e') implies (g).

For the first statement of the theorem, assuming (g), and since $r_n = q_n n^{-4H}$ are still the coefficients of the expansion of δ_Y, assuming Condition C, the implication "(d) implies (c)" in Theorem 6 proves that there exists $K > 0$ such that the inequality $\delta_Y(\alpha) < K\delta(\alpha)$ holds for small α. Applying the transformation $\delta \mapsto f_\delta$ to this inequality yields for small α,

$$f_Y(\alpha) \le Kf_\delta(\alpha) = Kf(\alpha).$$

where the last equality is by the definition of δ. Theorem 7 could now be used directly to conclude on the moduli of continuity of X and Y. However, our claim is that condition B is not needed. To see this, note that by hypothesis $\lim_{0+} f = 0$, so that the previous inequality justifies invoking Corollary 3, which does not require any conditions, and implies here that both X and Y share both f and f_Y as a.s. uniform spatial moduli of continuity. □.

The presence of Condition D in Theorem 8 masks the fact that the summability condition (g) is a necessary condition for continuity even when D is not satisfied. We can state this by rephrasing Theorem 8 in a slightly weaker form, assuming only conditions C and C':

Corollary 4 *Let W be as in (29), and assume that $Y = (I - \Delta)^{-H} W$ has canonical metric function δ satisfying conditions C and C'. Define $f = f_\delta$ as in (39). Then conditions (e') and (g) are equivalent. Moreover, they are equivalent to each of the following:*

(i) $Y(1, \cdot)$ *is almost-surely bounded;*

(ii) $Y(1, \cdot)$ *is almost-surely continuous;*

(iii) $\lim_{0+} f_\delta = 0$

All these conditions are also equivalent to (e) and to (f) if we assume Condition B.

Proof. That (e') is equivalent to (i), (ii) and (iii) is a well-known fact from the general theory of homogeneous Gaussian processes (see [1]). The only part that has not already been established is (e')$\Longrightarrow$(g). This follows by the proof of the same implication in the proof of Theorem 8 because here since $\delta = \delta_Y$, inequality (42) holds automatically, so there is no need to invoke condition D, and for the same reason there is no need to use part (a) of Theorem 5, which makes condition B superfluous. The last statement is obvious by Theorem 7. $\square$

6 Existence and uniqueness of the solution and Feynman-Kac formula in the linear multiplicative case

We study in this section the existence and uniqueness of the solution and we derive a Feynman-Kac formula for the following parabolic stochastic partial differential equation

$$X(dt, x) = \Delta X(t, x)dt + B^H(dt, x)X(t, x) \tag{43}$$

$$X(0, x) = 1 \ , \ t \in T = [0, 1]$$

We will assume that B^H is an infinite-dimensional fractional Brownian motion as in the previous sections, with $H > 1/2$, with space variable x in R and the stochastic integral in (43) is a Skorohod integral introduced in Section 2. Even in the case $H = 1/2$, contrary to the linear additive equation, (43) does not known to have a mild solution if the behavior of B^H is worse than white-noise in the space parameter. Further, the Feynman-Kac formula can only be established if B^H is assumed to be a bonafide function in the space parameter. Since our goal is the proof of a Feynman-Kac formula, we will make the assumption that $B^H(t, \cdot)$ is a function throughout. This will simplify the proof of existence as well.

Equation (43) can be written in its evolution form

$$X(t, x) = 1 + \int_R \int_0^t p_{t-s}(x, y) X(s, y) B^H(ds, y) dy \tag{44}$$

In order to prove the existence and the uniqueness of the solution and to derive a Feynman-Kac formula for the solution of (44), we will use the properties of the multiple stochastic integrals with respect to the fractional Brownian motion. We begin by recalling some elements on multiple integrals with respect to fBm in the one-dimensional case ($t \in T = [0; 1]$).

6.1 Multiple fractional integrals

6.1.1 Finite dimensional theory

We refer to [19] ,[7] and [8] for the notions presented below. Consider $(B_t^H)_{t\in T}$ the one-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$ and consider the $\mathcal{H}$-indexed process $(B(\phi))_{\phi\in\mathcal{H}}$ with $E(B(\phi)B(h)) = \langle \phi, h\rangle_{\mathcal{H}}$ defined in Section 2. Such a process is an isonormal process and it is possible to construct multiple integrals with respect to an isonormal process (see [18]). More precisely, in the case of fBm, define $L_H^2(T^n)$ the space of functions $f : T^n \to C$ such that $\|f\|^2_{L_H^2(T^n)} := \langle f; f\rangle_H < \infty$ where the scalar product $\langle \cdot;\cdot\rangle_H$ is defined by

$$\langle f, g\rangle_H := \int_{T^{2n}} \chi(x_1,y_1)\cdots\chi(x_n,y_n) f(x_1,\cdots,x_n)\bar{g}(y_1,\cdots,y_n)\prod_{i=1}^n dx_i dy_i,$$

where

$$\chi(x,y) = H(2H-1)|x-y|^{2H-2}.$$

Define the operator $(K^{*,n})$ between $L_H^2(T^n)$ and $L^2(T^n)$ as follows:

$$(K^{*,n}f)(t_1,\cdots,t_n) = d_H^n (t_1\cdots t_n)^{\frac{1}{2}-H}$$
$$\times \left(I_-^{H-\frac{1}{2},n}\right)\left((x_1\cdots x_n)^{H-\frac{1}{2}} f(x_1,\cdots,x_n)\right)(t_1,\cdots,t_n)$$

where

$$\left(I_-^{\alpha,n} f\right)(x_1,\cdots,x_n) = \frac{1}{(\Gamma(\alpha))^n}\int_{x_1}^1\cdots\int_{x_n}^1 \frac{f(t_1,\cdots,t_n)}{(t_1-x_1)^{1-\alpha}\cdots(t_n-x_n)^{1-\alpha}} dt_1\cdots dt_n$$

denotes the Liouville fractional integral for $0<\alpha<1$ and $n \in N$ and the constant d_H is given by

$$d_H = \left[\frac{2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})}{\Gamma(2-2H)}\right]^{\frac{1}{2}}.$$

The operator $K^{*,n}$ is a transfer operator: it transports the integration with respect to Brownian motion to the fractional Brownian motion and therefore we have the following relation between the multiple integral with respect to fBm and the multiple integrals with respect to the Wiener process

$$I_n^H(f) = I_n((K^{*,n}f)), \text{ for } f \in L_H^2(T^n) \tag{45}$$

The operator $K^{*,n}$ is an isometry (see [2] for $n=1$ and [19] for the general case):

$$\|K^{*,n}f\|_{L^2(T^n)} = \|f\|_{L_H^2(T^n)} \tag{46}$$

We will also recall the isometry property of the fractional multiple integrals

$$E\left(I_n^H(f) I_m^H(g)\right) = 0 \text{ for } m \neq n \tag{47}$$

and

$$E\left(I_n^H(f) I_n^H(g)\right) = n!\langle \tilde{f}, \tilde{g}\rangle_H \tag{48}$$

where $\tilde{f}$ denotes the symmetrization of f. We will need the following inequality (see [19]) for $H > \frac{1}{2}$

$$\|K^{*,n}f\|_{L^2(T^n)} \leq \|f\|_{L^2(T^n)} \tag{49}$$

Note that in fact $L_H^2(T)$ coincides with the Reproducing kernel Hilbert space of fBm.

6.1.2 Infinite dimensional theory

We can apply the above ideas to construct multiple integrals with respect to the infinite-dimensional fBm $B^H(t,x)$. Now, the process $B^H(t,x)$ will be assumed to be a Gaussian random field on some probability space $(\Omega, \mathcal{F}, P)$ with covariance

$$E\left(B^H(t,x)B^H(s,y)\right) = R(t,s)Q(x,y)$$

where Q is the kernel of a positive operator. Note that we do not allow the dependence of B on x to be in the sense of generalized functions. Instead, we are assuming that $Q\,(x,y)$ is defined for each pair (x,y). Moreover, we will suppose that B^H is spatially homogeneous in the sense that $Q(x,y) = Q(x-y)$. The use of R^1 as our space is a convenience. The results presented would work equally well for any R^d or any finite-dimensional Lie group. Let M^H be the Gaussian random spectral measure associated to B^H. That is, M^H is the unique Gaussian random measure such that

$$B^H(t,x) = \int_0^t \int_R e^{i\lambda x} M^H(ds, d\dot{\lambda}).$$

Equivalently the law of M^H is characterized by

$$E\int_T\int_R f(s,\lambda)M^H(ds,d\lambda)\overline{\int_T\int_R g(s,\lambda)M^H(ds,d\lambda)} = \int_R \langle f(s,\lambda), \overline{g(s,\lambda)}\rangle_{L^2_H(T)}\hat{Q}(d\lambda)$$

if $\hat{Q}$ denotes the Fourier transform of Q. The spectral measure M^H can be also given by its relationship with the spectral measure M associated with an infinite-dimensional Brownian motion W with covariance $|t-s|\,Q\,(x-y)$:

$$\begin{aligned} M^H(t,\lambda) &= \int_0^t K(t,s)M(ds,\lambda), \\ W(t,x) &= \int_0^t \int_R e^{i\lambda x} M(ds, d\dot{\lambda}). \end{aligned}$$

One can define multiple integrals with respect to the Gaussian process B^H (or, with respect to its associated spectral measure, see the book of P. Major [15]), by putting, for every $f \in L^2_H(T^n) \times L^2\left(R^n, \hat{Q}^{\otimes n}\right)$, and using the shorthand notation $f(s_i,\lambda_i)$ for $f((s_i,\lambda_i) : i = 1, \cdots, n)$,

$$J_n^H(f) = \int_T\int_R \ldots \int_T\int_R f(s_i,\lambda_i)M^H(ds_1,d\lambda_1)\ldots M^H(ds_n,d\lambda_n). \tag{50}$$

By definition and using (46) we have

$$\begin{aligned} E\left|J_n^H(f)\right|^2 &= \int_R \cdots \int_R |f(\cdot,\lambda_i)|^2_{L^2_H(T^n)}\hat{Q}(d\lambda_1)\ldots\hat{Q}(d\lambda_n) \\ &= \int_T\int_R \cdots \int_T\int_R \left|K^{*,n}(f(\cdot,\lambda_i))\,(s_i)\right|^2 \hat{Q}(d\lambda_1)\ldots\hat{Q}(d\lambda_n)ds_1\cdots ds_n \end{aligned}$$

Here and henceforth, for any function f that depends on both the time variables $(s_i)_i$ and the Fourier variable $(\lambda_i)_i$, and perhaps other variables z, $\left(K^{*,n}f\right)(s_i,\lambda_\iota,z)$ denote the action of $K^{*,n}$ on the time variables only, that is $\left(K^{*,n}f\right)(s_i,\lambda_\iota) = K^{*,n}(f(\cdot,\lambda_i,z))\,(s_i)$. Since the dependence on λ is the same for W and B^H, the arguments of the finite-dimensional case can be used to prove that, as in (45),

$$J_n^H(f) = \int\int \cdots \int\int \left(K^{*,n} f\right)(s_i, \lambda_\iota)\, M(ds_1, d\lambda_1) \cdots M(ds_n, d\lambda_n). \tag{51}$$

Note that the isometry formulas (47) and (48) are also valid for the integral J_n^H, relative to the inner product in $L_H^2(T^n) \times L^2\left(R^n, \hat{Q}^{\otimes n}\right)$.

6.2 Existence and uniqueness of the solution

Let $L^2(B^H)$ be the set of square integrable random variables F admitting a fractional orthogonal decomposition $F = \sum_{n\geq 0} J_n^H(f_n)$, with $f_n \in L_H^2(T^n)$ and $\sum_n \|f_n\|_H^2 < \infty$. Since we are using non-compact space with a constant initial condition, it is convenient to look for the solution to (44) in a weighted L^2 space. Therefore by $L^2(R)$ we understand a space $L^2(R, w(x)\,dx)$ where w is a positive integrable function on R. The main result of this section is the following.

Theorem 9 *Let $H > \frac{1}{2}$. Let B^H be an infinite-dimensional fBm with values in $L^2(\mathbf{R})$, and assume that $B^H(1,\cdot)$ is a bonafide spatially homogeneous Gaussian process, in the sense that $E\left[B^H(1,x)\, B^H(1,y)\right] = Q(x-y) < \infty$ for all $x, y \in \mathbf{R}^d$. Then equation (44) admits a unique solution $X = (X(t,x), t \in T, x \in R)$ in $L^2(T \times R \times \Omega)$ such that for every $t \in T, x \in R$, $X(t,x)$ belongs to $L^2(B^H)$.*

Proof of existence We introduce the usual Picard iterations $X_0(t,x) = 1$ and

$$\begin{aligned}
X_{n+1}(t,x) &= \int_R \int_0^t p_{t-s}(x,y) X_n(s,y) B^H(ds,y)dy \\
&= \int_R \left(\int_R \int_0^t p_{t-s}(x,y) X_n(s,y) e^{i\lambda y} M^H(ds,d\lambda) \right) dy \\
&= \int_R \int_0^t \left(p_{t-s}(x,y) X_n(s,y) e^{i\lambda y} dy \right) M^H(ds,d\lambda)
\end{aligned}$$

We can compute iteratively the process X_n as in [4]. We will have

$$X_n(t,x) = \int_R \int_0^t \cdots \int_R \int_0^t M^H(ds_1, d\lambda_1) \cdots M^H(ds_n, d\lambda_n) \prod_{j=1}^n 1_{[0,s_{j-1}]}(s_j)$$

$$\times \int_R \cdots \int_R \prod_{j=1}^n p_{s_{j-1}-s_j}(y_{j-1}, y_j) e^{i\lambda_j y_j} dy_n \cdots dy_1$$

where $y_0 = x$ and $s_0 = t$, and by the Markov property of Brownian motion, this expression equals

$$X_n(t,x) = \int_R \int_0^t \cdots \int_R \int_0^t E_x \left(\prod_{j=1}^n 1_{[0,s_{j-1}]}(s_j) e^{i\lambda_j b_{t-s_j}} \right) M^H(ds_1, d\lambda_1) \cdots M^H(ds_n, d\lambda_n)$$

where, under the new probability measure P_x, b is a standard Brownian motion started at x. We can also write

$$X_n(t,x) = \int_R \int_0^t \cdots \int_R \int_0^t f_n(t,x,s_i,\lambda_i) M^H(ds_1, d\lambda_1) \cdots M^H(ds_n, d\lambda_n)$$

where

$$f_n(t,x,s_i,\lambda_i) = E_x\left(\prod_{j=1}^{n} 1_{[0,s_{j-1}]}(s_j)e^{i\lambda_j b_{t-s_j}}\right)$$

Now, using relation (45) between the multiple fractional integrals and the multiple integrals with respect to the Brownian motion, we obtain

$$X_n(t,x) = \int_R\int_0^t \cdots \int_R\int_0^t (K^{*,n}f_n)(t,x,s_i,\lambda_i)M(ds_1,d\lambda_1)\cdots M(ds_n,d\lambda_n)$$

with $M(d\lambda, ds)$ the orthogonally scattered Gaussian spectral measure associated to W. Using inequality (49), the norm of $X_n(t,x)$ in $L^2(R\times T\times\Omega)$ will be given by

$$\begin{aligned}
&\|X_n\|^2_{L^2(R\times T\times\Omega)}\\
&= \int_R dx\int_0^1 dt\int_R\cdots\int_R \|f_n(t,x,\lambda_i,s_i)\|^2_{L^2_H(T^n,ds_i)}\hat{Q}(d\lambda_1)\cdots\hat{Q}(d\lambda_n)\\
&\le \int_R dx\int_0^1 dt\int_R\cdots\int_R \|f_n(t,x,\lambda_i,s_i)\|^2_{L^2(T^n,ds_i)}\hat{Q}(d\lambda_1)\cdots\hat{Q}(d\lambda_n)\\
&= \|X'_n\|^2_{L^2(R\times T\times\Omega)}
\end{aligned}$$

where

$$X'_n(t,x) = \int_R\int_0^t\cdots\int_R\int_0^t E_x\left(\prod_{j=1}^{n} 1_{[0,s_{j-1}]}(s_j)e^{i\lambda_j b_{t-s_j}}\right)M(ds_1,d\lambda_1)\cdots M(ds_n,d\lambda_n).$$

But, if we define

$$X'(t,x) = \sum_{n\ge 0} X'_n(t,x)$$

then $X'(t,x)$ satisfies the evolution form of the equation (see [9], [4])

$$X'(dt,x) = \Delta X'(t,x)dt + W(dt,x)X'(t,x)$$

$$X(0,x) = 1\ ,\ t\in T=[0,1]$$

By the classical theory of evolution equations with respect to the infinite dimensional Wiener process , we know that $\sum_{n\ge 0} X'_n(t,x)$ converges in $L^2(R\times T\times\Omega)$ and therefore the process defined by

$$X(t,x) = \sum_{n\ge 0} X_n(t,x)$$

exists and belongs to $L^2(R\times T\times\Omega)$. To show that X is a solution of the equation (44), one only needs to note that by definition we obtain

$$\sum_{n\ge 0} X_{n+1}(t,x) = \int_R\int_0^t p_{t-s}(x,y)\left(\sum_{n\ge 0} X_n(s,y)\right)B^H(ds,y)dy$$

and therefore equation (44) is satisfied, since is not difficult to observe that the right side of the above expression is convergent.

Proof of uniqueness. We show that the equation has a unique solution in $L^2(B^H)$. Consider $Y(t,x)$ another solution of (44):

$$Y(t,x) = 1 + \int_R\int_0^t p_{t-s}(x,y)Y(s,y)B^H(ds,y)dy \tag{52}$$

and let $Y_n(t,x)$ its projection on the n^{th} chaos of the fractional Brownian motion. Since $Y(t,x)$ belongs to $L^2(B^H)$ we can write its fractional chaos decomposition:

$$Y(t,x) = \sum_{n\geq 0} Y_n(t,x).$$

Note first that

$$Y_0(t,x) = E(Y(t,x)) = 1 = X_0(t,x)$$

and, from (52) we have

$$\sum_{n\geq 1} Y_n(t,x) = \int_R \int_0^t p_{t-s}(x,y) \sum_{n\geq 0} Y(s,y) B^H(ds,y)dy$$

Denote by f_n, resp. g_n the kernel of X_n resp. Y_n, i.e.

$$X_n(t,x) = I_n^H(f_n(t,x,\cdot)) \ , \ Y_n(t,x) = I_n^H(g_n(t,x,\cdot))$$

and put $Z_n = X_n - Y_n$, $Z(t,x) = \sum_{n\geq 0} Z_n(t,x)$. Thus the process Z verifies $Z_0(t,x) = 0$ for every t, x and

$$\sum_{n\geq 1} Z_n(t,x) = \int_R \int_0^t p_{t-s}(x,y) \sum_{n\geq 0} Z_n(s,y) B^H(ds,y)dy$$

Let us identify the first chaos in both sides of the above relation. It holds that

$$I_1\left(f_1(t,x,\cdot) - g_1(t,x,\cdot)\right) = \int_R \int_0^t p_{t-s}(x,y) Z_0(s,y) B^H(ds,y)dy = 0$$

and since the multiple integral is an isometry, we find

$$f_1(t,x,\cdot) = g_1(t,x,\cdot) \text{ in } L^2_H(T)$$

In this way, we can kill step by step the chaos of any order $n \geq 1$ and we obtain that $X = Y$ in $L^2(B^H)$. $\square$

6.3 Fractional Feynman-Kac formula

In this section we establish the following stochastic Feynman-Kac formula for the solution of equation (44).

Theorem 10 *Let $(X(t,x), t \in T, x \in R)$ be the unique solution of (44) with $X(t,x) \in L^2(B^H)$ for every t, x. Suppose moreover that there exists $\alpha > 0$ such that*

$$\int_R |\lambda|^\alpha \hat{Q}(d\lambda) < \infty \tag{53}$$

and let $(b_t)_{t\in T}$ a standard Wiener process starting from x on the standard Wiener space $(\mathcal{C} = \mathcal{C}(T;R), \mathcal{F}, P_x)$. Then, it holds that

$$X(t,x) = E_x\left[\exp\left[\int_0^t \int_R e^{i\lambda b_{t-r}} M^H(d\lambda, dr) - \frac{1}{2}t^{2H}Q(0)\right]\right], \ t \in T, x \in R \tag{54}$$

$$= E_x\left[\exp\left[\int_0^t B^H\left(dr, b_{t-r}\right) - \frac{1}{2}t^{2H}Q(0)\right]\right], \ t \in T, x \in R \tag{55}$$

Remark 11 *As defined in this theorem, E_x denotes the expectation with respect to b, which bears no connection with the expectation E, which is with respect to W.*

Remark 12 *Condition (53) is essentially equivalent to requiring that B^H is not only a bonafide function in the variable x but that it is almost-surely β-Hölder-continuous for any $\beta < \alpha/2$ in the variable x. The reader will easily be convinced of this fact by comparing with the results in Section 5.1. A more difficult issue, which occurs also for $H = 1/2$, is whether condition (53) can be weakened beyond the Hölder scale. The ideas in Section 5.1 can be used to prove that one may replace the function λ^α in (53) by any of the first three examples for δ^2 in Section 5.1, since they yield fields B^H that are a.s. in x. We do not know if one can push the argument to include x-discontinuous fields, such as in the fourth example in Section 5.1.*

Proof. The equality of the right-hand sides of (54) and (55) can be understood as a definition of the stochastic integral in (55). From the proof of existence in Theorem 9, the solution of (44) has the fractional chaos expansion, for every $t \in T$ and $x \in R$,

$$X(t,x) = \sum_{n=0}^{\infty} X_n(t,x)$$

where the multiple integral of order n is given by

$$\begin{aligned} &X_n(t,x) \\ &= \int_R \int_0^t \cdots \int_R \int_0^t E_x \left[\prod_{j=1}^n 1_{[0,s_{j-1}]}(s_j) e^{i\lambda_j b_{t-s_j}} \right] M^H(ds_1, d\lambda_1) \cdots M^H(ds_n, d\lambda_n) \\ &= \int_R \int_0^t \cdots \int_R \int_0^t K^{*,n} \left(E_x \left[\prod_{j=1}^n 1_{[0,s_{j-1}]}(s_j) e^{i\lambda_j b_{t-s_j}} \right] \right) M(ds_1, d\lambda_1) \cdots M(ds_n, d\lambda_n) \\ &= \int_R \int_0^t \cdots \int_R \int_0^t E_x \left[K^{*,n} \left(\prod_{j=1}^n 1_{[0,s_{j-1}]}(s_j) e^{i\lambda_j b_{t-s_j}} \right) \right] M(ds_1, d\lambda_1) \cdots M(ds_n, d\lambda_n) \end{aligned}$$

since $K^{*,n}$ is a deterministic linear operator acting only on the time variables s_i so we can interchange it with the operation E_x. Now, we can use (53) and the same argument as in [4] to apply Fubini's theorem, that is, the mapping

$$b \to \int_R \int_0^t \cdots \int_R \int_0^t K^{*,n} \left(\prod_{j=1}^n 1_{[0,s_{j-1}]}(s_j) e^{i\lambda_j b_{t-s_j}} \right) M(ds_1, d\lambda_1) \cdots M(ds_n, d\lambda_n)$$

has a version in $L^2(\mathcal{C} \times \Omega)$. By Fubini's theorem, it holds that

$$X_n(t,x) = E_x \left[X_n^t(t,b) \right]$$

where, for $0 < s < t$,

$$X_n^t(s,b) = \int_R \int_0^t \cdots \int_R \int_0^t K^{*,n} \left(\prod_{j=1}^n 1_{[0,s_{j-1}]}(s_j) e^{i\lambda_j b_{t-s_j}} \right) M(ds_1, d\lambda_1) \cdots M(ds_n, d\lambda_n)$$

with $s_0 = s$. But, since the multiple integral with respect to the fractional Brownian motion is invariant under the symmetrization of the kernel, we can write

$$
\begin{aligned}
&X_n^t(s,b) \\
&= \int_R \int_0^t \cdots \int_R \int_0^t \prod_{j=1}^n 1_{[0,s_{j-1}]}(s_j) e^{i\lambda_j b_{t-s_j}} M^H(ds_1, d\lambda_1) \cdots M^H(ds_n, d\lambda_n) \\
&= \frac{1}{n!} \int_R \int_0^t \cdots \int_R \int_0^t \prod_{j=1}^n 1_{[0,s]}(s_j) e^{i\lambda_j b_{t-s_j}} M^H(ds_1, d\lambda_1) \cdots M^H(ds_n, d\lambda_n) \\
&= \frac{1}{n!} \int_R \int_0^t \cdots \int_R \int_0^t K^{*,n} \left(\prod_{j=1}^n 1_{[0,s]}(s_j) e^{i\lambda_j b_{t-s_j}} \right) M(ds_1, d\lambda_1) \cdots M(ds_n, d\lambda_n) \\
&= \frac{1}{n!} \int_R \int_0^t \cdots \int_R \int_0^t \prod_{j=1}^n K^{*,1} \left(1_{[0,s]}(s_j) e^{i\lambda_j b_{t-s_j}} \right) M(ds_1, d\lambda_1) \cdots M(ds_n, d\lambda_n) \\
&= \frac{1}{n!} \int_R \int_0^t \cdots \int_R \int_0^t \prod_{j=1}^n 1_{[0,s]}(s_j) e^{i\lambda_j b_{t-s_j}} M^H(ds_1, d\lambda_1) \cdots M^H(ds_n, d\lambda_n)
\end{aligned}
$$

For the last equality, we used the definition of the operator $K^{*,n}$ to observe that

$$
K^{*,n}(f_1 \otimes \cdots \otimes f_n) = (K^{*,1} f_1) \otimes \cdots \otimes (K^{*,1} f_n) \text{ , if } f_i \in L^2_H(T)
$$

We will apply now Prop. 3.5. a) of [19] and we obtain

$$
\begin{aligned}
\sum_{n \geq 0} X_n^t(s,b) &= \exp\left(\int_0^t \int_R 1_{[0,s]}(r) e^{i\lambda b_{t-r}} M^H(dr, d\lambda) - \frac{1}{2} \| 1_{[0,s]}(r) e^{i\lambda b_{t-r}} \|^2_{L^2_H(T) \otimes L^2(d\hat{Q})} \right) \\
&= \exp\left(\int_0^t \int_R 1_{[0,s]}(r) e^{i\lambda b_{t-r}} M^H(dr, d\lambda) - \frac{1}{2} Q(0) s^{2H} \right) \qquad (56)
\end{aligned}
$$

Putting $s = t$ and taking the expectation E_x in the above relation, we obtain the Feynman-Kac formula (54) for the solution of the Itô evolution equation (44). □

7 Appendix

7.1 Proofs of Lemmas 1, 2 and 3.

Proof of Lemma 1.

If $\lambda \geq 1$, note that, by (20),

$$
\begin{aligned}
A(\lambda) &\leq C(H) \left(\int_0^\lambda v^{2H-2} e^{-v} dv \right) \\
&\leq C(H) \left(\int_0^\infty v^{2H-2} e^{-v} dv \right) = C(H)
\end{aligned}
$$

and also

$$
\begin{aligned}
A(\lambda) &\geq \int_0^1 v^{2H-2} e^{-v} \left[1 - e^{-2(\lambda - v)} \right] dv \\
&\geq \int_0^1 v^{2H-2} e^{-v} \left[1 - e^{-2(-v)} \right] dv
\end{aligned}
$$

and this a positive constant denoted generically by $c(H)$. The assertion (i) is proved.

Suppose now that $\lambda \leq 1$. We use the following facts: for $0 \leq x \leq 1$,

$$\begin{aligned}
&2x \geq 1 - e^{-2x} \geq 2x/3 \\
&1 \geq e^{-x} \geq 1/3.
\end{aligned}$$

We use the notation $A \asymp [c, C]B$ to mean $c < A/B < C$. We obtain

$$\begin{aligned}
A(\lambda) &\asymp [1/3, 1] \int_0^\lambda v^{2H-2} e^{-v} 2(\lambda - v)\, dv \\
&\asymp [1/3, 1] \cdot [\lambda \int_0^{\lambda/2} v^{2H-2} e^{-v} dv; 2\lambda \int_0^\lambda v^{2H-2} e^{-v} dv] \\
&\asymp \lambda \cdot [1/3, 1] \cdot [1/3 \int_0^{\lambda/2} v^{2H-2} dv; \int_0^\lambda v^{2H-2} dv] \\
&= \lambda \cdot [1/3, 1] \cdot \frac{\lambda^{2H-1}}{2H-1} [(1/3)(1/2)^{2H-1}; 1] \\
&= \lambda^{2H} \cdot [c(H); C(H)].
\end{aligned}$$

□

Proof of Lemma 2.

We need to show first another lemma.

Lemma 6 *For all $a \geq 0$ and for all $A \in (-1, 0]$, there exists a positive constant K_A such that*

$$\sup_{t \geq 0} \int_0^t r^{A-1} \left(1 - e^{-ar}\right) dr \leq K_A a^{-A}.$$

Proof: First assume $t \geq a^{-1}$. Then we separate two pieces:

$$\begin{aligned}
\int_0^t r^{A-1} \left(1 - e^{-ar}\right) dr &= \int_0^{1/a} r^{A-1} \left(1 - e^{-ar}\right) dr + \int_{1/a}^t r^{A-1} \left(1 - e^{-ar}\right) dr \\
&\leq \int_0^{1/a} r^{A-1} ar dr + \int_{1/a}^t r^{A-1} dr \\
&= a \int_0^{1/a} r^A dr + \int_{1/a}^t r^{A-1} dr \\
&= \frac{a}{A+1} a^{-A-1} + \frac{1}{-A} \left[a^{-A} - t^A\right] \\
&\leq a^{-A} \left(\frac{1}{A+1} + \frac{1}{-A}\right).
\end{aligned}$$

If $t \leq (2a)^{-1}$, the same calculation can be used, omitting the integrals from $(2a)^{-1}$ to t, which ends the proof. □

We decompose $B(a,A)$ into three pieces $B(a,A) \leq C(P_1+P_2+P_3)$ where

$$P_1 = \int_0^1 ds \exp(-2as) \left[\int_0^{(1/a)\wedge(s/2)} (\exp ar - 1)\, r^{A-1} dr \right]^2,$$

$$P_2 = \int_0^1 ds \exp(-2as) \left[\int_{(1/a)\wedge(s/2)}^{s/2} (\exp ar - 1)\, r^{A-1} dr \right]^2,$$

$$P_3 = \int_0^1 ds \exp(-2as) \left[\int_{s/2}^{s} (\exp ar - 1)\, r^{A-1} dr \right]^2,$$

where $A = (-1/2, 0)$. For the first term we note that since $ar \leq 1$, $e^{ar} - 1 \leq 2ar$ and we obtain

$$\begin{aligned}
&\int_0^{1/a} (\exp ar - 1)\, r^{A-1} dr \\
&\leq \int_0^{1/a} 2ar^A dr = \frac{2a}{A+1}\left(\frac{1}{a}\right)^{A+1} = \frac{2}{A+1} a^{-A},
\end{aligned}$$

so that

$$\begin{aligned}
P_1 &\leq \int_0^t ds \exp(-2as) \frac{4}{(A+1)^2} a^{-2A} \\
&= \left(1 - e^{-2at}\right) \frac{2}{(A+1)^2} a^{-(2A+1)} \\
&\leq K_A a^{-(2A+1)}
\end{aligned}$$

for $K_A = 2(A+1)^{-2}$. For the second term, if $s/2 < 1/a$ we have $P_2 = 0$, and otherwise we note that Lemma 2 yields

$$\begin{aligned}
&\int_{1/a}^{s/2} \left(e^{ar} - 1\right) r^{A-1} dr \\
&\leq e^{as/2} \int_{1/a}^{s/2} \left(1 - e^{-ar}\right) r^{A-1} dr \\
&\leq e^{as/2} K_A a^{-A},
\end{aligned}$$

for some constant K_A, so that

$$\begin{aligned}
P_2 &\leq \int_0^t ds e^{-2as} \left(e^{as/2} K_A a^{-A}\right)^2 \\
&= K_A^2 a^{-2A} \int_0^t ds e^{-as} = K_A^2 a^{-(2A+1)} \left(1 - e^{-at}\right) \\
&\leq K_A a^{-(2A+1)}
\end{aligned}$$

for some constant K_A. The last term to estimate is

$$P_3 = \int_0^t ds \exp(-2as) \left[\int_{s/2}^{s} (\exp ar - 1)\, r^{A-1} dr \right]^2$$

we note first that $r^{A-1} \in [s^{A-1}; (s/2)^{A-1}]$, so for some constant K_A the following estimate is sharp:

$$
\begin{aligned}
P_3 &\leq K_A \int_0^t ds \exp(-2as)\, s^{2A-2} \left[\int_{s/2}^{s} (\exp ar - 1)\, dr \right]^2 \\
&= K_A \int_0^t ds \exp(-2as)\, s^{2A-2} a^{-2} \left[e^{as} - e^{as/2} - as/2 \right]^2 \\
&= K_A a^{-2} \int_0^t s^{2A-2} \left[1 - e^{-as/2} - (as/2)\, e^{-as} \right]^2 ds.
\end{aligned}
$$

It is again convenient to decompose this integral into two pieces: $P_3 \leq Q_1 + Q_2$ where

$$
\begin{aligned}
Q_1 &:= K_A a^{-2} \int_0^{1/a} s^{2A-2} \left[1 - e^{-as/2} - (as/2)\, e^{-as} \right]^2 ds, \\
Q_2 &:= K_A a^{-2} \int_{(1/a)\wedge t}^{t} s^{2A-2} \left[1 - e^{-as/2} - (as/2)\, e^{-as} \right]^2 ds.
\end{aligned}
$$

We assume $t > 1/a$, since otherwise $Q_2 = 0$. For Q_2 then, since $s > 1/a$, we get $1 - e^{-as/2} \in [1 - e^{-1/2}; 1]$, while $(as/2)\, e^{-as} \leq 2^{-1} e^{-1} < 1 - e^{-1/2}$, so that it is a sharp estimate to write that for some new K_A

$$
\begin{aligned}
Q_2 &\leq K_A a^{-2} \int_{(1/a)\wedge t}^{t} s^{2A-2} ds \\
&= K_A a^{-2} \frac{1}{1-2A} \left(a^{-(2A-1)} - t^{2A-1} \right) \\
&\leq K_A a^{-(2A+1)}.
\end{aligned}
$$

To estimate the last term Q_1, we perform the change of variable $\tau = as/2$, to obtain

$$
\begin{aligned}
Q_1 &= K_A a^{-2} \int_0^{1/2} \frac{2}{a} d\tau \left(\frac{2\tau}{a} \right)^{2A-2} \left(1 - e^{-\tau} - \tau e^{-2\tau} \right)^2 \\
&= K_A \left(\frac{1}{a} \right)^{2A+1} 2^{2A-1} \int_0^{1/2} \tau^{2A-2} \left(1 - e^{-\tau} - \tau e^{-2\tau} \right)^2 d\tau.
\end{aligned}
$$

The last integral is in fact finite for any $A \in (-1/2, 0)$. Indeed we can check by using a power-series expansion that $\left(1 - e^{-\tau} - \tau e^{-2\tau} \right) / \tau^2$ is bounded. Consequently for some new $K_A > 0$,

$$
Q_1 \leq a^{-(2A+1)} K_A \int_0^{1/2} \tau^{2A+2} d\tau = K_A a^{-(2A+1)}
$$

□

Proof of Lemma 3.

By the expression of K and $\partial_1 K$, we note that, when $H < \frac{1}{2}$

$$
K(t,s) \geq 0 \text{ and } \frac{\partial K}{\partial t}(t,s) \leq 0 \text{ , for all } t, s \text{ with } t > s.
$$

Throughout this proof c_H will be the constant appearing in the formula (4). To prove this lemma we need to do a direct estimation of the kernel K. Integrating by parts in the right side of (4), we will find the following expression for K

$$K(1,s) = c_H(1-s)^{H-\frac{1}{2}}s^{\frac{1}{2}-H} + c_H\left(\frac{1}{2}-H\right)s^{\frac{1}{2}-H}\int_0^{1-s} u^{H-\frac{1}{2}}(s+u)^{H-\frac{3}{2}}du$$

Denote by $g(s) = \int_0^{1-s} u^{H-\frac{1}{2}}(s+u)^{H-\frac{3}{2}}du$. Note also that, by the change of variables $r-s=u$ and relation (6) it holds that

$$\begin{aligned}
&\int_s^1 (\exp{(r-s)\lambda}-1)\frac{\partial K}{\partial r}(r,s)dr \\
&= c_H\left(\frac{1}{2}-H\right)s^{\frac{1}{2}-H}\int_0^{1-s}(e^{\lambda u}-1)u^{H-\frac{3}{2}}(u+s)^{H-\frac{1}{2}}du = c_H\left(\frac{1}{2}-H\right)s^{\frac{1}{2}-H}f(\lambda,s)
\end{aligned}$$

where we denoted

$$f(\lambda,s) = \int_0^{1-s}(e^{\lambda u}-1)u^{H-\frac{3}{2}}(u+s)^{H-\frac{1}{2}}du.$$

Therefore, we have, with the notation $e(\lambda,s) = \exp{(-2\lambda(1-s))}$,

$$\begin{aligned}
J(\lambda) &= c_H^2\int_0^1 e(\lambda,s)(1-s)^{2H-1}s^{1-2H}ds + 2c_H^2\left(\frac{1}{2}-H\right)\int_0^1 e(\lambda,s)(1-s)^{H-\frac{1}{2}}s^{1-2H}g(s)ds \\
&+ c_H^2\left(\frac{1}{2}-H\right)^2\int_0^1 e(\lambda,s)s^{1-2H}g^2(s)ds - 2c_H^2\left(\frac{1}{2}-H\right)\int_0^1 e(\lambda,s)(1-s)^{H-\frac{1}{2}}s^{1-2H}f(\lambda,s)ds \\
&- 2c_H^2\left(\frac{1}{2}-H\right)^2\int_0^1 e(\lambda,s)s^{1-2H}g(s)f(\lambda,s)ds + c_H^2\left(\frac{1}{2}-H\right)^2\int_0^1 e(\lambda,s)s^{1-2H}f^2(\lambda,s)ds \\
&= c_H^2\int_0^1 e(\lambda,s)(1-s)^{2H-1}s^{1-2H}ds \\
&+ c_H^2\left(\frac{1}{2}-H\right)^2\int_0^1 e(\lambda,s)s^{1-2H}\left(f(\lambda,s)-g(s)\right)^2 ds \\
&- 2c_H^2\left(\frac{1}{2}-H\right)\int_0^1 e(\lambda,s)s^{1-2H}(1-s)^{H-\frac{1}{2}}\left(f(\lambda,s)-g(s)\right)ds \\
&= c_H^2\int_0^1 e(\lambda,s)(1-s)^{2H-1}s^{1-2H}ds + c_H^2\left(\frac{1}{2}-H\right)\int_0^1 e(\lambda,s)s^{1-2H}\left(f(\lambda,s)-g(s)\right) \\
&\times\left[\left(\left(\frac{1}{2}-H\right)\left(f(\lambda,s)-g(s)\right) - 2(1-s)^{H-\frac{1}{2}}\right)\right]ds = A+B
\end{aligned}$$

Now, concerning the term A we can write

$$A = c_H^2\lambda^{-1}\int_0^\lambda e^{-2u}u^{2H-1}(\lambda-u)^{1-2H}du.$$

Assume first that $\lambda \le 1$. In this case it holds that

$$\begin{aligned}
A &\ge c_H^2\lambda^{-1}e^{-2}\int_0^l u^{2H-1}(\lambda-u)^{1-2H}du \ge c_H^2\lambda^{-1}e^{-2}l^{2H-1}\int_0^l(l-u)^{1-2H}du \\
&= c_H^2(2-2H)^{-1}e^{-2}l\lambda^{-1} \ge c(H,l)\lambda^{-2H}
\end{aligned}$$

where for every $H \in (0,1/2)$ and every $l>0$ the constant $c(H,l)$ is also positive. Note that $c(H,l) \to 0$ when $l \to 0$, which indicates that our bound is of decaying quality for decreasing spectral gap... If $\lambda > 1$, we find a lower bound by only integrating for $u \in [0; 1/2]$:

$$\begin{aligned} A &\geq c_H^2 \lambda^{-1} \int_0^{1/2} e^{-2u} u^{2H-1} (\lambda - u)^{1-2H} \, du \\ &\geq c_H^2 \lambda^{-1} e^{-1} (\lambda/2)^{1-2H} \int_0^{1/2} u^{2H-1} du \\ &= C(H) \lambda^{-2H}. \end{aligned}$$

It now suffices to prove that the term $B = B(\lambda)$ is positive. To this end we will prove that the function

$$h(\lambda, s) = (f(\lambda, s) - g(s)) \left((\frac{1}{2} - H) (f(\lambda, s) - g(s)) - 2(1-s)^{H-\frac{1}{2}} \right)$$

is positive. Note that, for every $s \in [0, 1]$, it holds

$$\frac{\partial h}{\partial \lambda}(\lambda, s) = \alpha(\lambda, s) h(\lambda, s) + \beta(\lambda, s)$$

where $\alpha(\lambda, s) = 2 (f(\lambda, s) - g(s))^{-1} (\partial f / \partial \lambda) (\lambda, s)$ and $\beta(\lambda, s) = 2(1-s)^{H-\frac{1}{2}} (\partial f / \partial \lambda) (\lambda, s)$. Therefore we can write, for every $\lambda \in [0, +\infty)$ and $s \in [0, 1]$

$$h(\lambda, s) = h(0, s) e^{\int_0^\lambda \alpha(t,s) dt} + \int_0^\lambda \beta(t, s) e^{\int_t^\lambda \alpha(r,s) dr} dt$$

It is easy to see that $h(0, s)$ and $\beta(t, s)$ are positive for every t, s and this finishes the proof. $\square$

7.2 Proof of Theorem 5

7.2.1 Proof of the lower bound of Theorem 5.

Let $X = \{X(t) : t \in I\}$ be a bounded Gaussian field on an index set I. Let δ be its canonical metric, and $B(x, \varepsilon)$ be the ball of radius ε centered at x in this metric. In this general situation, we introduce some notation. For a fixed measure m on I, let

$$\begin{aligned} \gamma_m(\eta) &= \sup_{x \in I} \int_0^\eta \sqrt{\log(1/m(B(x, \varepsilon)))} d\varepsilon, \\ \theta_m(\eta) &= \sup_{x \in I} \int_0^\eta \sqrt{\log(1/\sup\{m(\{u\}) : u \in B(x, \varepsilon)\})} d\varepsilon, \\ \phi_\delta(\eta) &= E[\sup\{X(x) - X(y) : x, y \in I; \delta(x, y) < \eta\}] \\ \beta_\delta(\eta) &= \sup_{x \in I} E[\sup\{|X(x) - X(y)| : y \in I; \delta(x, y) < \eta\}]. \end{aligned}$$

We note that $\theta_m \geq \gamma_m$. Since X is centered, we have $\beta_\delta \leq 2\phi_\delta$. Also we introduce the metric entropy of δ: $N(\varepsilon)$ is the smallest number of balls of radius ε in the metric δ that are needed to cover I. Let D be the diameter of I in the metric δ. Recall the following result from Fernique's general theory of suprema for Gaussian processes.

Proposition 2 *[Theorem 17, part (a) in [23].] There exists a universal constant K (not dependent on X) such that with the notation as above, for any probability measure m on I,*

$$\phi_\delta(\eta) \leq K \gamma_m(\eta).$$

Theorem 17 in [23] also establishes the following lower bound which is original to Talagrand's paper: for some probability measure m on I,

$$\theta_m(\eta) \leq K\beta_\delta(\eta) + \eta\left(\log\left(2N(\eta)\right) + 2\log\left(2D/\eta\right)/\log 2\right)^{1/2}.$$

Yet this estimate is not sufficient for our purposes. In our specific situation however, we are able to bring a slight improvement to Talagrand's original lower bound proof, by assuming condition B.

Step 1. *A Talagrand-type lower bound.* We begin with a lemma inspired by Talagrand's lower bound [proof of Theorem 17 part (b) in [23]].

Lemma 7 *With the notation as above, with K denoting a universal constant (not dependent on X), let $\eta > 0$ be fixed. Let $\{B_j : j = 1, \cdots, N(\eta)\}$ be a covering of I with balls of radius no greater than η. There exists a probability measure m_j on B_j such that for all $x \in B_j$*

$$\int_0^{diam(B_j)} \left(\log\left[\left(\sup\left\{m_j(\{u\}) : \delta(x,u) \leq \varepsilon\right\}\right)^{-1}\right]\right)^{1/2} d\varepsilon \leq K\beta_\delta(\eta), \tag{57}$$

and moreover, defining the probability measure $m = N(\eta)^{-1}\sum_{i=1}^{N(\eta)} m_j$,

$$\theta_m(\eta) \leq \sup_{j\in\{1,\cdots,N(\eta)\}} \sup_{x\in B_j} \int_0^\eta \left(\log N(\eta) + \log\left[\left(\sup\left\{m_j(\{u\}) : \delta(x,u) \leq \varepsilon\right\}\right)^{-1}\right]\right)^{1/2} d\varepsilon \tag{58}$$

Proof. Let u_j be the center of the ball B_j. On each ball $B_j : j = 1, \cdots, N(\eta)$ we apply Theorem 14 in [23] to obtain the existence of a probability measure m_j on B_j such that

$$\int_0^{diam(B_j)} \left(\log\left[\left(\sup\left\{m_j(\{u\}) : \delta(x,u) \leq \varepsilon\right\}\right)^{-1}\right]\right)^{1/2} d\varepsilon \leq K\mathbf{E}\left[\sup_{x\in B_j} X(x)\right]$$

$$= K\mathbf{E}\left[\sup_{x\in B_j} X(x) - X(u_j)\right] \leq K\mathbf{E}\left[\left|\sup_{x\in B_j} X(x) - X(u_j)\right|\right] \leq K\beta_\delta(\eta),$$

proving the lemma's first assertion. For the second, fix $x \in I$ and let j be such that $x \in B_j$; then for any $\varepsilon \leq \eta$,

$$\sup\left\{m(\{u\}) : \delta(x,u) \leq \varepsilon\right\} \geq \frac{1}{N(\eta)} \sup\left\{m_j(\{u\}) : \delta(x,u) \leq \varepsilon\right\},$$

so that

$$\begin{aligned}
&\int_0^\eta \left(\log\left(1/\sup\left\{m(\{u\}) : u \in B(x,\varepsilon)\right\}\right)\right)^{1/2} d\varepsilon \\
&\leq \int_0^\eta \left(\log N(\eta) + \log\left(1/\sup\left\{m_j(\{u\}) : u \in B(x,\varepsilon)\right\}\right)\right)^{1/2} d\varepsilon,
\end{aligned}$$

and the result follows. □.

Step 2. *First majorizing measure integral estimation.* Assume I is a compact Lie group, let $|\cdot|$ and dx denote the Haar measure on I, and its differential, and assume X is homogeneous on I. Fix x_0 in I. Let $B(x_0,\varepsilon)$ be the ball in the metric δ centered at x_0 with radius ε. Let

$$\tau(\eta) = \int_0^\eta \sqrt{\log\left(1/\left|B(x_0,\varepsilon)\right|\right)}d\varepsilon.$$

Since X is homogeneous, τ does not depend on x_0, and we have the following equalities for any fixed probability measure m on I:

$$\begin{aligned}
\int_I m(B(x,\varepsilon))\,dx &= \int_I m(x - x_0 + B(x_0,\varepsilon))\,dx \\
&= \int_I m\left(x' + B(x_0,\varepsilon)\right)dx' \\
&= \int_I \left|y + B(x_0,\varepsilon)\right| m(dy) \\
&= \left|B(x_0,\varepsilon)\right|.
\end{aligned}$$

Since the function $u(z) = \sqrt{\log(1/z)}$ is convex for $0 < z < 1/2$, for small η, we can use

$$u\left(\int_I m(B(x,\varepsilon))\,dx\right) \le \int_I u(m(B(x,\varepsilon)))\,dx.$$

Therefore

$$u(|B(x_0,\varepsilon)|) \le \int_I u(m(B(x,\varepsilon)))\,dx,$$

which implies

$$\begin{aligned}
\tau(\eta) &\le \int_I dx \int_0^\eta u(m(B(x,\varepsilon)))\,d\varepsilon \\
&\le \sup_{x\in I}\int_0^\eta u(m(B(x,\varepsilon)))\,d\varepsilon \\
&= \gamma_m(\eta).
\end{aligned}$$

Step 3. *Using the Talagrand-type lower bound.* The last inequality, together with inequality (58), proves, with the measures m and $m_j : j = 1,\cdots,N(\eta)$, identified in Lemma 7,

$$\begin{aligned}
\tau(\eta) &\le \gamma_m(\eta) \le \theta_m(\eta) \\
&\le \sup_{j\in\{1,\cdots,N(\eta)\}}\sup_{x\in B_j}\int_0^\eta \left(\log N(\eta) + \log\left[\left(\sup\{m_j(\{u\}) : \delta(x,u)\le\varepsilon\}\right)^{-1}\right]\right)^{1/2} d\varepsilon.
\end{aligned} \tag{59}$$

Since we have no control over the term involving m_j in the above expression, in comparison to $N(\eta)$, we have no choice, as did Talagrand himself, but to use the estimate $\sqrt{A+B} \le \sqrt{A} + \sqrt{B}$. Then, with inequality (57), we obtain

$$\begin{aligned}
\tau(\eta) &\le \eta\left(\log N(\eta)\right)^{1/2} + K\beta_\delta(\eta) \\
&\le \eta\left(\log N(\eta)\right)^{1/2} + K\phi_\delta(\eta).
\end{aligned} \tag{60}$$

Step 4. *Calculation of the majorizing measure integral.* The next step in the proof is to calculate τ. Here we specialize to the case of δ on the circle $I = S^1$. We denote by $\check{\delta}$ the inverse function of δ. We have

$$|B(x_0,\varepsilon)| = |\{x : \delta(|x - x_0|) < \varepsilon\}| = \left|\left\{x : |x - x_0| < \check{\delta}(\varepsilon)\right\}\right| = 2\check{\delta}(\varepsilon).$$

Therefore τ becomes

$$\tau(\eta) := \int_0^\eta \sqrt{\log(1/|B(x_0,\varepsilon)|)}d\varepsilon$$
$$= \int_0^\eta \sqrt{\log\left(1/\left(2\check{\delta}(\varepsilon)\right)\right)}d\varepsilon. \tag{61}$$

We also note that because we are working on a one-dimensional index set, we have

$$N(\eta) = \frac{1}{2\check{\delta}(\eta)} \tag{62}$$

for any value of η that yields an integer in this formula.

Step 5. *Using the hypothesis of a.s. modulus of continuity with a zero-one law of Fernique.* To complete the proof of the lower bound, we need to estimate ϕ_δ. To this end, we use a zero-one-type result due to Fernique. Let $C^f(I)$ be the space of continuous functions on I that have f as an almost sure uniform modulus of continuity, up to a multiplicative constant. For any $\alpha \leq 1$, for any function g defined on I, set

$$A_\alpha(g) = \sup_{|x-y|\leq\alpha} |g(x) - g(y)|, \qquad N_f(g) = \sup_{\alpha\leq 1} \frac{A_\alpha(g)}{f(\alpha)}.$$

Then, following Fernique's definitions [22, Definition 1.2.1], N_θ is a gauge on $C^f(G)$. Indeed, it suffices to see that N_f is lower-semi-continuous, that is, for every $M > 0$, the set $K_M = \{\Phi; N_f(\Phi) \leq M\}$ is closed. Let Φ_n be a sequence in K_M converging uniformly to Φ. Then, for every n and for every $\alpha \leq 1$ and for every x, y such that $|x-y| \leq \alpha$, we have

$$|\Phi_n(x) - \Phi_n(y)| \leq Mf(\alpha).$$

We obtain that $\Phi \in K_M$ when we let $n \to \infty$.

By assumption, we have the existence of an almost surely positive random variable α_0 such that, if $\alpha < \alpha_0$ then

$$A_\alpha(Y) \leq f(\alpha).$$

Since Y is almost surely continuous, it is also almost surely bounded. This, together with the last inequality, implies $N_f(Y)$ is almost surely finite. A theorem of Fernique [22, Lemma 1.2.3] implies $E[N_f(Y)] := c < \infty$ where $c = c(f, Y)$ is a constant depending only on f and the law of W. Therefore

$$\phi_\delta(\eta) \leq E \sup_{\delta(|x-y|)\leq\eta} |Y(x) - Y(y)|$$
$$= E \sup_{|x-y|\leq\check{\delta}(\eta)} |Y(x) - Y(y)|$$
$$\leq cf\left(\check{\delta}(\eta)\right). \tag{63}$$

Step 6. *Conclusion.* Combining (60), (61), and (62), we obtain

$$\int_0^\eta \sqrt{\log\left(1/\left(2\check{\delta}(\varepsilon)\right)\right)}d\varepsilon \leq K\phi_\delta(\eta) + \eta\left(\log\left(\frac{1}{2\check{\delta}(\eta)}\right)\right)^{1/2}$$

Now let α be defined by $\alpha = \check{\delta}$. Then for small α, with (63) and formula (34),

$$\int_0^\alpha \delta(r)\frac{dr}{2r\sqrt{\log(r^{-1})}} + \delta(\alpha)\sqrt{\log(\alpha^{-1}/2)} \leq \delta(\alpha)\sqrt{\log(\alpha^{-1}/2)} + cKf(\alpha),$$

or in other words

$$\int_0^\alpha \delta(r)\frac{dr}{2r\sqrt{\log(r^{-1})}} \leq cKf(\alpha). \tag{64}$$

Condition B and the formula for f_δ in (34) finish the proof of (a). □

7.2.2 Proof of the upper bound of Theorem 5.

Part (b) is a consequence of a well-known property of homogeneous Gaussian processes and the general theory of Gaussian regularity. Indeed, one only needs to apply Theorem 4.4 and Corollary 4.7 in [1]. The details are left to the reader. □

7.2.3 Proof of Corollary 2.

In the proof of Theorem 5, in the Hölder case $\delta(r) = r^{\alpha}$, all lower bound calculations are valid up to inequality (64). The conclusion of the corollary follows by calculating the left-hand side of (64) and comparing it to $f(\alpha)\log(1/\alpha)$. □

7.3 Proof of Theorem 6

7.3.1 Proof of (c)⟹(d)

Since δ is a canonical metric, the series $\sum_n r_n$ converges. Let $B_n = \sum_{m=n}^{\infty} r_m$. The hypothesis on δ in condition (c) means that for all n,

$$\begin{aligned}
Kg\left(\frac{1}{n}\right) &\geq \sum_{j=1}^{\infty} r_j \left(1 - \cos(j/n)\right) \\
&\geq \frac{1}{3}\sum_{k=0}^{\infty} \sum_{m=[2k\pi n]+n+1}^{[2k\pi n]+5n-1} r_m
\end{aligned}$$

since for any integers n, k, and for any $x \in [[2k\pi n]/n + 1 + 1/n; [2k\pi n]/n + 5 - 1/n]$, $1 - \cos x > 1/3$. Condition C' implies that for some constant K', with $K'' = 3KK'$, we have

$$B_n \leq K'' g(1/n).$$

For convenience we chose a function $\tilde{g}$ defined and increasing on $[0;1]$, with $\lim_{0+}\tilde{g} = 0$, such that $B_n = \tilde{g}(1/n)$. The above inequality means that $\tilde{g}(1/n) \leq K'' g(1/n)$. With h as in condition (d), what we want to prove is that the following series converges:

$$I = \sum_{n=1}^{\infty} h\left(g(1/n)\right)\left[\tilde{g}(1/n) - \tilde{g}\left(1/(n+1)\right)\right].$$

Since h is decreasing, we obtain

$$\begin{aligned}
I &\leq K'' \sum_{n=1}^{\infty} h\left(\frac{1}{K''}\tilde{g}(1/n)\right)\left[\frac{1}{K''}\tilde{g}(1/n) - \frac{1}{K''}\tilde{g}\left(1/(n+1)\right)\right] \\
&\leq K'' \int_0^{\tilde{g}(1)/K''} h(x)\,dx < \infty.
\end{aligned}$$

modulo the fact that if h is not defined up to $\tilde{g}(1)$, one should repeat the proof starting from a higher value of n in I. This concludes the proof of (c)⟹(d).

7.3.2 Proof of (d)⟹(c)

The following lemma on summation by parts will be useful.

Lemma 8 *Let* $(A_n)_n$ *and* $(B_n)_n$ *be sequences of real numbers. Let* $a_1 = A_1$, $b_1 = B_1$, *and for all* $j \geq 2$, $a_j = A_j - A_{j-1}$ *and* $b_j = B_j - B_{j-1}$. *Then*

$$A_n B_n = \sum_{j=1}^{n} A_j b_j + \sum_{j=2}^{n} B_{j-1} a_j.$$

Proof: iterate the relation:

$$A_n B_n = A_n \left(B_n - B_{n-1}\right) + B_{n-1} \left(A_n - A_{n-1}\right) + A_{n-1} B_{n-1}.$$

from n to 1. □

Step 1. *Space discretization.* We first show that it is sufficient to show the conclusion of (c) for the x's of the form $x = 1/n$ where n is an integer. Indeed assume that there exist $K > 0$ and $n_{\min}$ an integer such that for all $n \geq n_{\min}$:

$$\delta \left(1/n\right)^2 \leq K g \left(1/n\right).$$

For an arbitrary $x \in (0; 1/n_{\min}]$, let n be such that $x \in (1/(n+1); 1/n]$. We have

$$\begin{aligned} \delta^2 (x) &\leq \delta^2 (1/n) \leq K g (1/n) \qquad (65) \\ &= K g (x) \left(1 + \frac{g (1/n) - g (x)}{g (x)}\right). \end{aligned}$$

By Condition A, without loss of generality, we can assume that $g(0) = 0$ and that $\tilde{g} = g^{1/2}$ is concave near 0. This implies that if $b > x > 0$, $[\tilde{g}(b) - \tilde{g}(x)] / [b - x] \leq g(x)/x$. Using this fact and the fact that $1/(n+1) < x$ implies $1/n < 2x$ as long as $x < 1/2$, we obtain:

$$\begin{aligned} \frac{\tilde{g}(1/n) - \tilde{g}(x)}{\tilde{g}(x)} &= \frac{\tilde{g}(1/n) - \tilde{g}(x)}{1/n - x} \cdot \frac{1/n - x}{g(x)} \\ &\leq \frac{g(x)}{x} \cdot \frac{1/n - x}{g(x)} \\ &= \frac{1/n - x}{x} < 1. \end{aligned}$$

Then we can estimate

$$\begin{aligned} \frac{g(1/n) - g(x)}{g(x)} &= \frac{\tilde{g}(1/n) - \tilde{g}(x)}{\tilde{g}(x)} \cdot \frac{\tilde{g}(1/n) + \tilde{g}(x)}{\tilde{g}(x)} \\ &= \frac{\tilde{g}(1/n) - \tilde{g}(x)}{\tilde{g}(x)} \cdot \left[\frac{\tilde{g}(1/n) - \tilde{g}(x)}{\tilde{g}(x)} + 2\right] \\ &< 3. \end{aligned}$$

Returning to (65) we get

$$\delta^2 (x) \leq 3 K g (x).$$

Step 2. *Separating the head and the tail of* δ. Let n_0 be a fixed integer larger than $n_{\min}$. We have

$$\begin{aligned} \delta^2 (1/n_0) &= \sum_{n=1}^{n_0 - 1} r_n \left(1 - \cos (n/n_0)\right) + \sum_{n=n_0}^{\infty} r_n \left(1 - \cos (n/n_0)\right) \\ &\leq \sum_{n=1}^{n_0 - 1} r_n (n/n_0)^2 + \sum_{n=n_0}^{\infty} r_n. \end{aligned}$$

We only need to show that there exists $K > 0$ such that for all $n_0 > n_{\min}$, the following two inequalities hold:

$$\sum_{n=1}^{n_0} r_n \left(n/n_0\right)^2 \leq K g\left(1/n_0\right), \tag{66}$$

$$\sum_{n=n_0}^{\infty} r_n \leq K g\left(1/n_0\right). \tag{67}$$

We will assume (d) holds and will assume successively that each of these two inequalities does not hold; we will obtain a contradiction in each case.

Step 3. *Controlling the tail.*

Step 3.1. *Assuming the tail is unbounded.* The negation of inequality (67) is equivalent to the existence of a sequence of integers $(N_m)_m$ that increases to $+\infty$, and a sequence of positive reals $(K_m)_m$ that increases to $+\infty$, satisfying for all $m \in \mathbf{N}$,

$$\sum_{n=N_m}^{\infty} r_n \geq g\left(1/N_m\right) K_m. \tag{68}$$

It will be convenient below to use the fact that without loss of generality, we can choose K_m to increase to infinity as slowly as we want, without effecting the sequence $(N_m)_m$. Let h be a function as in (d). Recall that $n \mapsto h\left(g\left(1/n\right)\right)$ is strictly increasing. We introduce the following notation:

$$\begin{aligned}
B^{(m)} &:= \sum_{n=N_m}^{\infty} r_n, \\
-b^{(m+1)} &:= B^{(m)} - B^{(m+1)} = r_{N_m} + \cdots + r_{N_{m+1}-1}, \\
A^{(m)} &:= h\left(g\left(1/N_m\right)\right) \\
a^{(m+1)} &:= A^{(m+1)} - A^{(m)} > 0.
\end{aligned}$$

By hypothesis (d) the tail $\sum_{n=N_m}^{\infty} r_n h\left(g\left(1/n\right)\right)$ converges to 0 as $m \to \infty$. We will calculate this tail using the summation-by-parts lemma 8 with the A's and B's as above. This will enable us to use the hypothesis (68) on this tail, and another application of Lemma 8 will yield a contradiction thanks to an appropriately chosen h. Let m_0 be fixed. We have

$$\begin{aligned}
\sum_{n=N_{m_0}}^{\infty} r_n h\left(g\left(1/n\right)\right) &= \sum_{m=m_0}^{\infty} \sum_{n=N_m}^{N_{m+1}-1} r_n h\left(g\left(1/n\right)\right) \\
&\geq \sum_{m=m_0}^{\infty} A^{(m)} \left(-b^{(m+1)}\right) \\
&= \sum_{m=m_0-1}^{\infty} a^{(m+1)} B^{(m+1)} - \lim_{m\to\infty} A^{(m)} B^{(m)},
\end{aligned}$$

where the last equality is by Lemma 8. We can prove that the last limit does in fact exist and is equal to 0. Indeed

$$\begin{aligned}
A^{(m)} B^{(m)} &= h\left(g\left(1/N_m\right)\right) \sum_{n=N_m}^{\infty} r_n, \\
&\leq \sum_{n=N_m}^{\infty} r_n h\left(g\left(1/n\right)\right)
\end{aligned}$$

and the last term converges to 0 as $m \to \infty$ by hypothesis (d). Now we use (68), which yields

$$\sum_{n=N_{m_0}}^{\infty} r_n h\left(g\left(1/n\right)\right) \geq \sum_{m=m_0}^{\infty} a^{(m)} g\left(1/N_m\right) K_m$$
$$\geq K_{m_0} \sum_{m=m_0}^{\infty} a^{(m)} g\left(1/N_m\right).$$

We calculate this last series by Lemma 8 again, using the notation $C^{(m)} := g\left(1/N_m\right)$ and $-c^{(m)} := g\left(1/N_{m-1}\right) - g\left(1/N_m\right)$:

$$\sum_{m=m_0}^{\infty} a^{(m)} g\left(1/N_m\right) = \sum_{m=m_0+1}^{\infty} A^{(m-1)} \left(-c^{(m)}\right) + \lim_{m\to\infty} A^{(m)} C^{(m)}$$
$$= \sum_{m=m_0+1}^{\infty} h\left(g\left(1/N_{m-1}\right)\right)\left[g\left(1/N_{m-1}\right) - g\left(1/N_m\right)\right]$$

where the fact that the last limit is zero is a trivial consequence of the integrability of h at 0. To summarize we have proved:

$$\sum_{n=N_{m_0}}^{\infty} r_n h\left(g\left(1/n\right)\right) \geq K_{m_0} \sum_{m=m_0+1}^{\infty} h\left(g\left(1/N_{m-1}\right)\right)\left[g\left(1/N_{m-1}\right) - g\left(1/N_m\right)\right]. \tag{69}$$

It is now sufficient to show that h can be chosen integrable at 0 and strictly decreasing, and such that for all m_0 large enough,

$$\sum_{m=m_0}^{\infty} h\left(g\left(1/N_{m-1}\right)\right)\left[g\left(1/N_{m-1}\right) - g\left(1/N_m\right)\right] \geq \frac{1}{\sqrt{K_{m_0}}}. \tag{70}$$

Step 3.2. *Choosing h.* We let $g_m = g\left(1/N_{m-1}\right)$ and introduce a arbitrary sequence $\left(k_m\right)_m$ such that $\lim k_m = 0$ and $k_m \geq \left(K_m\right)^{-1/2}$. First we show that we can reduce the problem of finding h as above to the problem of finding a strictly increasing sequence of positive numbers $\left(h_m\right)_m$ such that

$$\sum_{m=m_0}^{\infty} h_m\left[g_m - g_{m+1}\right] \geq k_{m_0}, \tag{71}$$

and such that the series on the left converges. Indeed define a function h as follows: for each fixed m, define h to be linear on the interval $(g_m; g_{m-1}]$, with endpoints set to

$$h\left(g_{m-1}\right) = h_{m-1},$$
$$\lim_{x\downarrow g_m} h\left(x\right) = \min\left(h_m; 2h_{m-1}\right).$$

Since $\left(h_m\right)_m$ is strictly increasing, this h is strictly decreasing. Moreover it is clear that

$$\sum_{m=m_0}^{\infty} h_m\left[g_m - g_{m+1}\right]$$
$$= \sum_{m=m_0}^{\infty} h\left(g\left(1/N_{m-1}\right)\right)\left[g\left(1/N_{m-1}\right) - g\left(1/N_m\right)\right]$$
$$\leq \int_0^{g_{m_0}} h\left(x\right) dx \leq 2\sum_{m=m_0}^{\infty} h_m\left[g_m - g_{m+1}\right].$$

This implies that (70) holds and also that h is integrable at 0, which is what we want. Thus we only need to find $(h_m)_m$ as in (71). Because of the flexibility we have on the choice of $(K_m)_m$ (being able to decrease all the values of K_m as long as the resulting sequence still converges to $+\infty$), there is no actual loss of generality in fixing the values of h_m and searching for new values of K_m such that (71) holds and $k_m \geq (K_m)^{-1/2}$. More precisely we *choose* $h_m = f(g_m)$ where f is any positive strictly decreasing integrable function (e.g. $f(x) = x^{-1/2}$). Then we can simply define $(k_m)_m$ by imposing that (71) hold as an equality. Note that we have

$$\int_0^{g_1} f(x)\,dx \geq \sum_{m=1}^{\infty} f(g_m)\left[g_m - g_{m+1}\right].$$

Therefore, k_m is the tail of this convergent series, and so it tends to zero. Therefore there is no loss of generality in reassigning the values of K_m to satisfy for all $m \geq 1$

$$K_m \leq \frac{1}{\left(\sup_{l\geq m} k_l\right)^2}.$$

Step 4. *Controlling the head.* This step follows a similar structure to Step 3.

Step 4.1. *Negating the head bound.* The negation of inequality (66) is equivalent to the existence of a sequence of integers $(N_m)_m$ that increases to $+\infty$, and a sequence of positive reals $(K_m)_m$ that increases to $+\infty$, satisfying for all $m \in \mathbf{N}$,

$$\sum_{n=1}^{N_m} n^2 r_n \geq (N_m)^2\, g(1/N_m)\, K_m. \tag{72}$$

Let h be a function as in (d). We introduce the following notation:

$$\begin{aligned}
B^{(m)} &:= \sum_{n=1}^{N_m} n^2 r_n,\\
b^{(m)} &:= B^{(m)} - B^{(m-1)} = \sum_{n=N_{m-1}+1}^{N_m} n^2 r_n,\\
A^{(m)} &:= \frac{1}{(N_m)^2} h\left(g\left(\frac{1}{N_m}\right)\right),\\
a^{(m)} &:= A^{(m)} - A^{(m-1)}.
\end{aligned}$$

We will show in Step 4.3 that h can be chosen in such a way that with $A_n := n^{-2} h(g(1/n))$, the sequence $(A_n)_n$ is decreasing. This yields

$$\begin{aligned}
\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) &= \sum_{n=N_{m_0}}^{\infty} n^2 r_n \frac{1}{n^2} h(g(1/n))\\
&\geq \sum_{n=N_{m_0}}^{\infty} \frac{1}{(N_m)^2} h\left(g\left(\frac{1}{N_m}\right)\right) \sum_{n=N_{m-1}+1}^{N_m} n^2 r_n\\
&= \sum_{m=m_0}^{\infty} A^{(m)} b^{(m)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=m_0+1}^{\infty} \left(-a^{(m)}\right) B^{(m-1)} + \lim_{m\to\infty} A^{(m)} B^{(m)} \\
&\geq \sum_{m=m_0+1}^{\infty} \left(-a^{(m)}\right) K_{m-1} g\left(\frac{1}{N_{m-1}}\right) (N_{m-1})^2 \\
&\geq K_{m_0+1} \sum_{m=m_0+1}^{\infty} \left(-a^{(m)}\right) g\left(\frac{1}{N_{m-1}}\right) (N_{m-1})^2 \\
&= K_{m_0+1} \sum_{m=m_0}^{\infty} A^{(m)} c^{(m)} - \lim_{m\to\infty} A^{(m)} C^{(m)},
\end{aligned}
$$

where $C^{(m)} = g(1/N_m)(N_m)^2$. We note that $A^{(m)}C^{(m)} = g(1/N_m)\, h(g(1/N_m))$. However, since we assumed that h is decreasing and $\int_0 h < \infty$, we get immediately that $\lim_{x\to 0} xh(x) = 0$, so that the last limit above is zero.

Step 4.2. *Applying the method of Step 3.* Thanks to the previous step we have:

$$
\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) \geq K_{m_0} \sum_{m=m_0}^{\infty} h(g(1/N_m)) \left[g(1/N_m) - \frac{(N_{m-1})^2}{(N_m)^2} g(1/N_{m-1}) \right]
$$

In fact this implies that inequality (69) holds. Indeed by Condition C, we get

$$
\sum_{n=N_{m_0}}^{\infty} r_n h(g(1/n)) \geq c K_{m_0} \sum_{m=m_0}^{\infty} h(g(1/N_m)) \left[g(1/N_{m-1}) - g(1/N_m) \right]
$$

which is stronger than (69) since $h(g(1/N_m)) > h(g(1/N_{m-1}))$. We now apply the strategy of Step 3 by saying that it is sufficient to find a function h integrable at 0 and strictly decreasing, and such that for all m_0 large enough, (70) holds.

Step 4.3. *Choosing h.* We choose $h(y) = 1 - y$, defined in a neighborhood of 0. Clearly this h is integrable at 0 and strictly decreasing. Now for each integer m_0 we define k_{m_0} by

$$
\sum_{m=m_0}^{\infty} h(g(1/N_m)) \left[g(1/N_{m-1}) - g(1/N_m) \right] = k_{m_0}.
$$

Since h is bounded by 1, the left-hand side of this equality is the tail of a converging series. Therefore k_m decreases to 0, and (70) holds by invoking the reassignment of the values of $(K_m)_m$ described at the end of Step 3.2.

The only thing left to prove is that this h is consistent with the condition, announced at the beginning of Step 4.1, that $A_n = n^{-2} h(g(1/n))$ is a decreasing sequence for n large enough. That is, we want to show that for all n large enough,

$$
\left(\frac{1}{n+1}\right)^2 (1 - g(1/(n+1))) \leq \left(\frac{1}{n}\right)^2 (1 - g(1/n))
$$

which is equivalent to:

$$
\frac{1 - g(1/(n+1))}{1 - g(1/n)} < 1 + 2/n + 1/n^2.
$$

To see this we can assume without loss of generality that near 0, either g is concave or $g(x) \leq x$. In the first case, we have

$$
\frac{1 - g(1/(n+1))}{1 - g(1/n)} = 1 + \left[1 + g(1/n) + o(g(1/n))\right] g'(\xi_n) \frac{1}{n(n+1)}
$$

where $\xi \in (1/(n+1); 1/n)$. Since g is concave we have $g'(\xi_n) \leq g'(1/(n+1)))$, and also $g'(x) \leq g(x)/x < 1/x$ for small x. Thus we get:

$$\frac{1-g(1/(n+1))}{1-g(1/n)} < 1 + 2/n.$$

For the other case, $g(x) \leq x$, we can also assume without loss of generality that $g(x) \geq x^2$ because of Condition A. Then we get for large n:

$$\frac{1-g(1/(n+1))}{1-g(1/n)} \leq \frac{1-(n+1)^2}{1-n^{-1}} = 1 + \frac{1}{n} + o\left(\frac{1}{n}\right) < 1 + \frac{2}{n}.$$

$\square$.

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