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Almost sure exponential behaviour for a parabolic SPDE on a manifold

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Abstract

We derive an upper bound on the large-time exponential behavior of the solution to a stochastic partial differential equation on a compact manifold with multiplicative noise potential. The potential is a random field that is white-noise in time, and Hölder-continuous in space. The stochastic PDE is interpreted in its evolution (semigroup) sense. A Feynman–Kac formula is derived for the solution, which is an expectation of an exponential functional of Brownian paths on the manifold. The main analytic technique is to discretize the Brownian paths, replacing them by piecewise-constant paths. The error committed by this replacement is controlled using Gaussian regularity estimates; these are also invoked to calculate the exponential rate of increase for the discretized Feynman–Kac formula. The error is proved to be negligible if the diffusion coefficient in the stochastic PDE is small enough. The main result extends a bound of Carmona and Viens (Stochast. Stochast. Rep. 62 (3–4) (1998) 251) beyond flat space to the case of a manifold. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

This article deals with a linear parabolic stochastic partial differential equation on a smooth, compact, finite-dimensional manifold M with multiplicative noise V that is

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white-noise in time, and with diffusivity $\kappa > 0$:

$$\frac{\partial u}{\partial t}(t,x) = \kappa\Delta u(t,x) + V(t,x)u(t,x),$$

$$u(0,x) = 1, \quad t \geqslant 0, \quad x \in M.$$

We establish that in large time, the almost-sure exponential rate of increase of the unique solution is bounded above by a deterministic rate which tends to zero for small κ:

There is a constant c such that for small κ, almost surely, for any $x \in M$,

$$\limsup_{t\to\infty} t^{-1} \log u(t,x) \leqslant c/\log(\kappa^{-1}).$$

For $\kappa = 0$, the solution is trivially given by $u(t,x) = \exp \int_0^t V(s,x)\,\mathrm{d}s$, whose exponential rate of increase is zero since $s \mapsto \int_0^t V(s,x)\,\mathrm{d}s$ is a Brownian motion. Our estimate is thus a continuity result. An exponential behavior is to be expected in the diffusive case $\kappa > 0$ because of the equation's linear multiplicative potential. The behavior is expected to be non-trivial, as it was proved in Euclidean space by Carmona et al. (1996); Carmona and Viens (1998). Although there is no guarantee in general that $t^{-1} \log u(t,x)$ has a limit for $t \to \infty$, it has become conventional to say that the upper and lower limits are both *Lyapunov exponents*.

Our work goes beyond the estimates found in Carmona et al. (1996), Carmona and Viens (1998), who deal only with the cases of x in $\mathbb{Z}^d$ and $\mathbb{R}^d$. We show that the same bound on the exponential increase holds in the absence of spatial flatness; this indicates that a Lyapunov exponent is a local property. As such we follow the philosophy developed by Tindel and Viens (1999), in which it is shown that curved non-commutative space (Lie groups) does not effect the existence, uniqueness, and regularity properties of a stochastic PDE as characterized by the regularity properties of the driving noise. Unlike Carmona et al. (1996); Carmona and Viens (1998) and Tindel and Viens (1999), we show that the hypothesis of spatial homogeneity of the noise is not needed to estimate the large-time asymptotics. In fact, if the manifold has no group structure, homogeneity is not a relevant concept.

The tools used in this article are similar to those in Carmona and Viens (1998) insofar as a stochastic Feynman–Kac formula is used and the main estimates are consequences of Gaussian inequalities. The new difficulty lies in the fact that the Laplace Beltrami operator Δ has non-constant coefficients. We approach the problem by immersing Brownian paths in M into Euclidean space. In fact our proofs can be adapted to a stochastic PDE on any subset of Euclidean space, with any second-order differential operator L, as long as L has smooth bounded coefficients and the spatial growth of V is slow at infinity. For the sake of conciseness, clarity, and sharpness, we avoid the most general situation.

An interesting physical motivation for the present work comes from magnetohydrodynamics, as presented in Hazra and Viens (2002). Other approaches to Lyapunov exponents for stochastic PDEs can be found in Bertini and Giacomin (1999) and Berge et al. (2001).

Following the advice of an anonymous referee, we explain briefly why we make no attempt in this article at investigating a lower bound on $\liminf$ of $t^{-1}\log u(t,x)$ as $t\to\infty$. The lower bound problem is significantly harder than the upper bound problem, and indeed there is no such result even in the flat space case, except for that in Carmona and Molchanov (1994); this result is in discrete space $\mathbb{Z}^d$ with a potential W that is white-noise in space as well as in time (i.e. $\{W(\cdot,x)\colon x\in\mathbb{Z}^d\}$ is a family of IID Brownian motions); the result was recently confirmed (and admirably sharpened) by M. Cranston and T. Mountford, although a preprint is yet unavailable. This discrete-space lower bound, of the same order in κ as the upper bound, uses the independence in x in a crucial way, as well as the fact that in discrete space, the inter-jump times of the random walks in the Feynman–Kac formula for u are exponentially distributed, so that in particular they have densities that are bounded below near zero.

One of the key ingredients in the passage to continuous space, present already in Carmona and Viens (1998), is the discretization of the Brownian paths in the Feynman–Kac formula representing u. However, this discretization results in inter-jump times with densities that are not bounded below near zero, causing serious technical difficulties. Additionally, and more importantly from the physical standpoint, the hypothesis of space–time white noise is not allowed in continuous space, the Stratonovich correction term being infinite. A higher degree of dependence between the values of the potential at neighboring points in space, such as a hypothesis of almost-sure Hölder continuity in x—the weakest assumption under which it is known that the Feynman–Kac formula correctly represents the solution—will not allow the use of the strong spatial independence arguments used in discrete space. This is the main physical reason which makes the lower bound more difficult than the upper bound, even in flat space. In fact, we believe that a lower bound of the same order as the upper bound is impossible to achieve in continuous space as soon as W is almost-surely uniformly continuous in x. There should be a relation between the modulus of continuity and a Lyapunov exponent λ. In the α-Hölder-continuous case, we have strong indications that this relation may be $\lambda\asymp\kappa^{\alpha/(\alpha+1)}$, a relation which we will hope to establish in a subsequent publication.

Lastly, we note that the difficulties with the lower bound exist in full strength in the flat space case. If results of the type alluded to at the end of the previous paragraph are obtained in flat space, their extension to curved space should not represent an additional leap in difficulty; the tools used in this article should be applicable to obtaining such an extension.

This paper begins with a preliminary section containing essential results on Brownian motion in M, and existence, uniqueness, and regularity results for a stochastic PDE on M. Section 2 establishes the Feynman–Kac representation. Then the proof of our main result is separated into two sections, the first one establishing that the discretization of the Feynman–Kac formula introduces an error of lesser magnitude than our final estimate, the second one showing that the discretized solution is almost-surely bounded by a deterministic quantity whose exponential rate of increase is bounded as announced. In the remainder of the paper, c,C,K will designate some positive constants that can change from line to line.

2. Preliminaries

2.1. Brownian motion on a compact manifold

Let M be a compact and complete Riemannian manifold of dimension d, that we shall consider as a regular submanifold isometrically imbedded in $\mathbb{R}^D$ for a $D \geqslant d$. For an arbitrary point $x \in M$, set $\Pi(x): \mathbb{R}^D \to T_xM$ for the projection from $\mathbb{R}^D$ to T_xM, where T_xM stands for the tangent space to M at x. Let ρ be the Riemannian metric associated to $M \hookrightarrow \mathbb{R}^D$, d$x$ the Riemannian volume element, and denote by Δ the Laplace–Beltrami operator on M. If $f: M \to \mathbb{R}$ is a smooth function, we denote by ∇f the gradient of f. Let B be a Brownian motion on $\mathbb{R}$ defined on a complete probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ and $\{\hat{p}_t^x;\ t \geqslant 0, x \in M\}$ the solution to the following stochastic differential equation:

$$\begin{aligned} d\hat{p}_t^x &= 2\kappa \Pi(\hat{p}_t^x)\partial B_t, \quad t \geqslant 0, \\ \hat{p}_0^x &= x, \end{aligned} \tag{1}$$

where ∂B_t stands for the Stratonovich differential of B and κ is a strictly positive parameter. It is well known (see i.e. Rogers and Williams, 1987, 31.1) that (1) has a unique strong solution, which is a Markov process with infinitesimal generator $\kappa\Delta$. Furthermore, $\hat{p}_t^x$ admits a jointly continuous version in $(t,x) \in \mathbb{R}_+ \times M$, and the following composition rule holds: for any $t, s \geqslant 0$, and $x \in M$ (see the theory of stochastic flows in Karatzas and Shreve, 1989),

$$\hat{p}_t^{\hat{p}_s^x} = \hat{p}_{t+s}^x. \tag{2}$$

Notice that, since $\Pi: M \to \mathscr{L}(\mathbb{R}^D; TM)$ can be extended as a smooth function on $\mathbb{R}^D$, the Brownian motion p on M can be seen as a diffusion on $\mathbb{R}^D$ with coefficients that are smooth and bounded by a multiple of κ. In the sequel, we shall denote by $\hat{E}$ the expectation in $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$.

2.2. Parabolic SPDEs on M

Let $L^2(M, \mathrm{d}x)$ be the space of square integrable functions defined on M, equipped with a scalar product denoted by $\langle .,. \rangle$. Let $\{e_i; i \geqslant 1\}$ be an orthonormal basis of $L^2(M, \mathrm{d}x)$ composed of eigenvectors of $-\Delta$, and set $\{\lambda_i;\ i \geqslant 1\}$ for the associated eigenvalues. We will also suppose that, on our compact Riemannian manifold, the family $\{e_i, \lambda_i;\ i \geqslant 1\}$ satisfies the relation

$$\|\nabla e_i\|_\infty \leqslant c\|e_i\|_\infty \lambda_i^{1/2} \tag{3}$$

for a constant $c > 0$ independent of i. Note that this assumption holds true for any connected and compact Lie group.

In order to define a good function-valued solution to our parabolic SPDE on M, we shall need a cylindrical Gaussian noise with a certain space correlation, that we define as follows: let $\{W^i;\ i \geqslant 1\}$ be a family of independent Brownian motions defined on another complete probability space $(\Omega, \mathscr{F}, P)$ and $\{q_i;\ i \geqslant 1\}$ a collection of positive coefficients. Our cylindrical noise will be defined formally as

$$W(\mathrm{d}s,\mathrm{d}x)=\sum_{i=1}^{\infty} q_i^{1/2} e_i(x) W^i(\mathrm{d}s)\,\mathrm{d}x,$$

which means that for any sufficiently x-regular and t-integrable function $f:\mathbb{R}_+\times M\to\mathbb{R}$, we have

$$W(f)=\sum_{i=1}^{\infty} q_i^{1/2}\int_{\mathbb{R}_+}\langle f_s,e_i\rangle\, W^i(\mathrm{d}s),$$

where the stochastic integral is of Itô's type. Notice then that

$$E[W(f)^2]=\sum_{i=1}^{\infty} q_i\int_{\mathbb{R}_+}\langle f_s,e_i\rangle \mathrm{e}^2\,\mathrm{d}s. \tag{4}$$

We shall also consider the associated Stratonovich noise $W(\partial s,\mathrm{d}x)$, defined with Stratonovich type integrals in time instead of Itô integrals.

In this paper, we study a stochastic partial differential equation of the type

$$\begin{aligned} &u(\mathrm{d}t,x)=\kappa\Delta u(t,x)\,\mathrm{d}t+u(t,x)W(\partial t,\mathrm{d}x), \quad (t,x)\in\mathbb{R}_+\times M,\\ &u(0,x)=1 \end{aligned} \tag{5}$$

and more precisely, the evolution form of (5), that is

$$u(t,x)=1+\int_0^t\int_M H_{t-s}(x,y)u(s,y)\,W(\partial s,\mathrm{d}y), \tag{6}$$

where $H_t(x,y)$ is the heat kernel associated to $\kappa\Delta$, i.e.

$$H_t(x,y)=\hat{P}(\hat{p}_t^x\in\mathrm{d}y)/\mathrm{d}y, \quad t>0, \quad x\in M.$$

Notice that the stochastic integral in (6) is of Stratonovich type. The minimal assumption we should make in order to get a unique $L^2(M,\mathrm{d}x)$-valued solution to (6) taken in Itô's form would be $\sum_{i\geqslant 1} q_i\|e_i\|_\infty^2(1+\lambda_i)^{-1}<\infty$ (see Da Prato and Zabczyk (1992) for the abstract result in a Hilbert space, and Tindel and Viens (1999) for the case of stochastic PDEs on Lie groups). However, we will need a much more restrictive assumption on the spatial correlation of W for our purpose:

(H1) There exists a constant $\alpha>0$ such that $\sum_{i\geqslant 1} q_i\|e_i\|_\infty^2(1+\lambda_i)^{\alpha}<\infty$.

We then have the

Proposition 1. *Assume* (H1). *Then* $\mathrm{d}y\mapsto\int_0^t W(\mathrm{d}s,\mathrm{d}y)$ *is almost surely a signed measure, and has a version with a density with respect to* $\mathrm{d}y$. *Denoting this density also by* W, *namely*

$$W(\mathrm{d}s,x)=\sum_{i=1}^{\infty} q_i^{1/2} e_i(x) W^i(\mathrm{d}s), \tag{7}$$

there exists a version of $W(t,x)$ *that is almost surely* β*-Hölder continuous in the space parameter for any* $\beta<\alpha$. *This version also admits an expansion of the form* *given in this proposition. Integration of an $\mathscr{F}_t$-adapted function $R \in L^2(\mathbb{R}_+ \times \Omega \times M)$ with respect to $W(\mathrm{d}s, x)\,\mathrm{d}x$ can be expressed iteratively as*

$$\iint_{\mathbb{R}_+ \times M} R(s,x) W(\mathrm{d}s, \mathrm{d}x) = \iint_{\mathbb{R}_+ \times M} R(s,x) W(\mathrm{d}s, x)\,\mathrm{d}x$$

$$= \int_M \left[\int_{\mathbb{R}_+} R(s,x) W(\mathrm{d}s, x) \right] \mathrm{d}x.$$

Proof. Assuming we have proved the second part of the proposition, that the random field $W(\mathrm{d}s, x)$ is indeed spatially β-Hölder continuous, it is a trivial matter to show that $W(\mathrm{d}s, \mathrm{d}x)$ as defined previously, and $W(\mathrm{d}s, x)\,\mathrm{d}x$, are versions of one another. The third and last statements of the proposition are also left to the reader. To prove the second statement, we use Kolmogorov's lemma locally on W, via a local chart; i.e. for x fixed in M, we let $c : U \to V$ be a C^∞-bijection from an open set U in $\mathbb{R}^d$ to a neighborhood V of x in M. Then we only need to prove that for every $p \geqslant 1$, there is a constant $K_p < \infty$ such that

$$E[|W(1,x) - W(1,y)|^{2p}] \leqslant K_p \rho(x,y)^{2\alpha p}. \tag{8}$$

Indeed, assuming this, for $u \in U$, let $Y(u) = W(1, c(u))$. Then since by compactness and smoothness of M we have $\rho(c(u_1); c(u_2)) \leqslant K_M |u_1 - u_2|$, we obtain

$$E[|Y(u_1) - Y(u_2)|^{2p}] = E|W(1, c(u_1)) - W(1, c(u_2))|^{2p}$$

$$\leqslant K_p \rho(c(u_1); c(u_2))^{2\alpha p}$$

$$\leqslant K_p |u_1 - u_2|^{2\alpha p}.$$

By Kolmogorov's lemma (e.g. Kunita, 1990, Problem 2.2.9), we have the existence of a ε-Hölder-continuous version of Y for any $\varepsilon < (2\alpha p - d)/2p$, which can be made arbitrarily close to α for large p. This Hölder continuity transfers to W on U by composition with the deterministic differential map c^{-1}. To prove the estimate (8), since W is Gaussian, we may let $p = 1$ without loss of generality. Then write

$$E[|W(1,x) - W(1,y)|^2] = \sum_{i=1}^{\infty} q_i e_i(x)(e_i(x) - e_i(y))$$

$$+ \sum_{i=1}^{\infty} q_i e_i(y)(e_i(x) - e_i(y))$$

$$\leqslant c \sum_{i=1}^{\infty} q_i [\|e_i\|_\infty \rho(x,y) \|\nabla e_i\|_\infty]^{2\alpha} [\|e_i\|_\infty]^{1-2\alpha}$$

$$\leqslant c\rho(x,y)^{2\alpha} \sum_{i=1}^{\infty} q_i \|e_i\|_\infty^{2-2\alpha+2\alpha} \lambda_i^\alpha$$

$$\leqslant c\rho(x,y)^{2\alpha},$$

where we used the definition of W, the estimate (3), and hypothesis (H1), finishing the proof of the proposition. $\square$

Remark 2. The condition (H1) which implies Hölder continuity of W in the space variable, is sharp. Indeed the co-authors of this paper established in Tindel and Viens (1999) that for a very general class of compact Lie groups, if W is α-Hölder-continuous almost surely, then (H1) is satisfied.

Let Q be the spatial covariance of W, that is, from expression (7),

$$Q(x,y) := \sum_{i=1}^{\infty} q_i e_i(x) e_i(y). \tag{9}$$

The following lemma provides an estimate of Q's regularity:

Lemma 3. *The spatial covariance function Q of W, as defined in* (9), *satisfies for any x, y, y' in M,*

$$|Q(x,y) - Q(x,y')| \leqslant C_{Q,\alpha} \rho(y,y')^{2\alpha},$$

where $C_{Q,\alpha}$ is a constant depending only on Q and α.

Proof. The proof is nearly identical to the calculations in the proof of Proposition 1. $\square$

Remark 4. Also notice that Hypothesis (H1) on Q implies that

$$\sup_{x\in M} Q(x,x) = \sup_{x\in M} \sum_{i=1}^{\infty} q_i |e_i(x)|^2 \leqslant M \sum_{i=1}^{\infty} q_i < \infty$$

because by M's compactness, the sets of values $\{\lambda_i \colon i \in \mathbb{N}\}$ and $\{\|e_i\| \colon i \in \mathbb{N}\}$ accumulates at infinity and nowhere else.

The following result is shown using classical tools such as the proof of the Proposition 1, and arguments such as those in Sections 3 and 4 in Tindel and Viens (1999). Its proof is omitted for conciseness.

Proposition 5. *Suppose that* (H1) *is satisfied.*

1. *There exists a unique $\mathscr{F}_t$ adapted solution u to* (6), *in the space $C(\mathbb{R}_+, L^2(M, \mathrm{d}x))$.*
2. *For any $\varepsilon < \frac{1}{2}$ and $\gamma < \alpha$, there exists a version of u in $C^{\varepsilon, 1+\gamma}(\mathbb{R}_+ \times M)$ almost surely.*

3. Feynman–Kac representation

We will establish in this section a Feynman–Kac representation of the solution to (6) that will be useful for the computation of the Lyapunov exponent of our equation.

Let us first change slightly the evolution form of our SPDE: we shall work from now in the space $(\Omega \times \hat{\Omega}, \mathscr{F} \otimes \hat{\mathscr{F}}, P \otimes \hat{P})$, and recall that $\hat{E}$ denotes the expectation with respect to $\hat{P}$.

Proposition 6. *Let u be the solution to* (6), *and $\hat{p}$ the Brownian motion on M defined in Section* 2.1. *Then P-almost surely we have, for any $(t,x) \in \mathbb{R}_+ \times M$,*

$$u(t,x) = 1 + \hat{E}\left[\int_0^t u(s, \hat{p}^x_{t-s}) W(\partial s, \hat{p}^x_{t-s})\right].$$

Proof. Using the integrability and continuity results of Proposition 5, it only remains to prove that the stochastic integral in the expectation on the right-hand side of the above equation is indeed jointly measurable with respect to $\hat{p}$ and ω. Indeed, then, the claim of the proposition is given by applications of stochastic Fubini lemmae, the definition of $\hat{p}$, and the last statement of Proposition 1, as follows:

$$\begin{aligned}
&\hat{E}\left[\int_0^t u(s, \hat{p}^x_{t-s}) W(\partial s, \hat{p}^x_{t-s})\right] \\
&\quad = \hat{E}\left[\sum_{i=1}^{\infty} \int_0^t u(s, \hat{p}^x_{t-s}) W_i(\partial s) \sqrt{q_i} e_i(\hat{p}^x_{t-s})\right] \\
&\quad = \sum_{i=1}^{\infty} \int_0^t \left[\int_M u(s,y) H_{t-s}(x,y) \sqrt{q_i} e_i(y)\right] W_i(\partial s) \\
&\quad = \int_0^t \left[\int_M u(s,y) H_{t-s}(x,y) W(\partial s, y)\right].
\end{aligned}$$

Similarly, to prove the required measurability, we write

$$\int_0^t u(s, \hat{p}^x_{t-s}) W(\partial s, \hat{p}^x_{t-s}) = \sum_{i=1}^{\infty} \int_0^t u(s, \hat{p}^x_{t-s}) W_i(\partial s) \sqrt{q_i} e_i(\hat{p}^x_{t-s}).$$

Any partial sum of the above series is jointly measurable in $\hat{p}$ and ω, as an $L^2(\Omega \times \hat{\Omega})$-limit of Stratonovich Riemann sums and because u and e_i are ω-almost-surely continuous. The whole sum of the series is still measurable, again because it is an $L^2(\Omega \times \hat{\Omega})$-limit of measurable terms. □

As an intermediate step towards our Feynman–Kac representation, we will need the following result.

Proposition 7. *Let us fix a path $\{\hat{p}^x_t; t \geqslant 0\}$ of the Brownian motion on M. Then, the unique P-a.s. continuous solution to the stochastic differential equation*

$$Y_{t,x}(s) = 1 + \int_0^s Y_{t,x}(r) W(\partial r, \hat{p}^x_{t-r}), \quad s \leqslant t$$

is given on $[0,t]$ *by*

$$Y_{t,x}(s) = \exp\left(\int_0^s W(\mathrm{d}r, \hat{p}^x_{t-r})\right).$$

Proof. Consider the adapted process $s \mapsto \hat{M}_s := \int_0^s W(\mathrm{d}r, \hat{p}^x_{t-r})$ where the path $\hat{p}$ is fixed. $\hat{M}$ is a square-integrable martingale. Indeed by the expansion for W and the stochastic Fubini lemma

$$\hat{M}_s = \sum_{i=1}^{\infty} \sqrt{q_i} \int_0^s W_i(\mathrm{d}r) e_i(\hat{p}^x_{t-r}).$$

In the above series, since $\hat{p}^x$ is fixed, each of the terms is a continuous Gaussian square-integrable martingale, they are all independent of each other, and the sum converges in $L^2(\Omega)$, so that $\hat{M}$ is a square-integrable (Gaussian) process; assuming the sum of all quadratic variations

$$A_i(s) := q_i \int_0^s |e_i(\hat{p}^x_{t-r})|^2 \, \mathrm{d}r$$

is a differentiable function $A(s)$, we could conclude that $\hat{M}$ is a mean-zero square-integrable martingale with quadratic variation $A(s)$. As such the unique strong solution of the Stratonovitch stochastic differential equation in the proposition would be given by $s \mapsto \exp \hat{M}_s$, which is the assertion of the proposition. To establish the existence of $\hat{M}$'s quadratic variation, we use the fact that since $\hat{p}^x$ is fixed and uniformly continuous on $[0,t]$, for any $\varepsilon > 0$, there exists $h > 0$ such that if $s \leqslant r \leqslant s+h$, then $\rho(\hat{p}^x_{t-r}; \hat{p}^x_{t-s}) \leqslant \varepsilon$. Consequently,

$$\begin{aligned}
&\left| \frac{1}{h} \sum_{i=1}^{N} [A_i(s+h) - A_i(s)] - \sum_{i=1}^{N} q_i |e_i(\hat{p}^x_{t-s})|^2 \right| \\
&\quad = \left| \frac{1}{h} \sum_{i=1}^{N} q_i \int_s^{s+h} (|e_i(\hat{p}^x_{t-r})|^2 - |e_i(\hat{p}^x_{t-s})|^2) \, \mathrm{d}r \right| \\
&\quad \leqslant \frac{1}{h} \sum_{i=1}^{N} q_i \int_s^{s+h} \mathrm{d}r (2\|e_i\|2_\infty)^{1-2\alpha} (2\|e_i\|_\infty \|\nabla e_i\|_\infty \rho(\hat{p}^x_{t-r}; \hat{p}^x_{t-s}))^{2\alpha} \\
&\quad \leqslant \frac{1}{h} \sum_{i=1}^{N} q_i \int_s^{s+h} \mathrm{d}r (2\|e_i\|^2_\infty)^{1-2\alpha} (2\|e_i\|_\infty c\sqrt{\lambda_i} \|e_i\|_\infty \varepsilon)^{2\alpha} \\
&\quad = \varepsilon \sum_{i=1}^{N} q_i 2c^{2\alpha} \|e_i\|^2_\infty \lambda_i^\alpha.
\end{aligned}$$

Since (H1) holds, we have in fact proved that $\langle \hat{M} \rangle(s)$ exists and

$$\frac{\mathrm{d}\langle \hat{M} \rangle}{\mathrm{d}s} = \sum_{i=1}^{\infty} q_i |e_i(\hat{p}^x_{t-s})|^2. \qquad \square$$

The main result of this section is the following.

Theorem 8. *Let u and p be defined as in Proposition* 6. *Then*, *P-almost surely*, *for any* $(t,x)\in\mathbb{R}_+\times M$,

$$u(t,x)=\hat{E}\left[\exp\left(\int_0^t W(\mathrm{d}s,\,\hat{p}^x_{t-s})\right)\right].$$

Proof. We will divide this proof in two steps.

Step 1: Fix $(t,x)\in\mathbb{R}_+\times M$. Let us show that, for any $s\leqslant t$, we have $Y_{t,x}(s)=Y_{s,\hat{p}^x_{t-s}}(s)$. Indeed, if $r\leqslant s\leqslant t$, then

$$Y_{t,x}(r)=1+\int_0^r Y_{t,x}(z)W(\partial z,\,\hat{p}^x_{t-z}) \tag{10}$$

and

$$Y_{s,\hat{p}^x_{t-s}}(r)=1+\int_0^r Y_{s,\hat{p}^x_{t-s}}(z)\,W(\partial z,\,\hat{p}^{\hat{p}^x_{t-s}}_{s-z}).$$

But, by composition rule (2), $\hat{p}^{\hat{p}^x_{t-s}}_{s-z}=\hat{p}^x_{t-z}$. Hence, $Y_{t,x}$ and $Y_{s,\hat{p}^x_{t-s}}$ satisfy the same equation on $[0,s]$. We get then the desired result by uniqueness of the solution to (10).

Step 2: For $(t,x)\in\mathbb{R}_+\times M$, set $v(t,x)=\hat{E}[Y_{t,x}(t)]$. Then v verifies

$$v(t,x)=1+\hat{E}\left[\int_0^t Y_{t,x}(s)W(\partial s,\,\hat{p}^x_{t-s})\right].$$

Using the result obtained at Step 1, we also have

$$\begin{aligned}v(t,x)&=1+\hat{E}\left[\int_0^t Y_{s,\hat{p}^x_{t-s}}(s)W(\partial s,\,\hat{p}^x_{t-s})\right]\\&=1+\hat{E}\left[\int_0^t \hat{E}[Y_{s,\hat{p}^x_{t-s}}(s)W(\partial s,\,\hat{p}^x_{t-s})|\hat{\mathscr{F}}_{t-s}]\right].\end{aligned}$$

Notice that the conditional expectation of $W(\partial s,\,\hat{p}^x_{t-s})$ can be understood here as an $L^2(\Omega)$-limit using the series expansion of W, and a stochastic Fubini lemma (e.g. as in the proof of Proposition 6). Since the process $\{Y_{s,\hat{p}^x_{t-s}}(r);r\leqslant s\}$ depends only on the future of $\hat{p}$ after $t-s$, the Markov property for p gives

$$\begin{aligned}v(t,x)&=1+\hat{E}\left[\int_0^t \hat{E}[Y_{s,z}(s)W(\partial s,z)]_{z=\hat{p}^x_{t-s}}\right]\\&=1+\hat{E}\left[\int_0^t \hat{E}[Y_{s,z}(s)]_{z=\hat{p}^x_{t-s}}W(\partial s,\,\hat{p}^x_{t-s})\right]\\&=1+\hat{E}\left[\int_0^t v(s,\,\hat{p}^x_{t-s})W(\partial s,\,\hat{p}^x_{t-s})\right].\end{aligned}$$

Thus, u and v satisfy the same SPDE. By uniqueness of the solution to (6) in $C(\mathbb{R}_+, L^2(M))$ (see Proposition 5), we get $u = v$ a.s., which ends the proof. □

Remark 9. This type of proof may also be used in the case of flat space ($\mathbb{R}^d$). As such, it is an improvement on the technique, based on the chaos expansion of u, used in Carmona and Viens (1998).

4. Approximation by a discrete problem

We shall follow here the line of Carmona and Viens (1998): in order to get our Lyapunov exponent on the manifold M, we shall approximate the path $\hat{p}^x$ by a discrete path $\hat{p}^{\varepsilon,x}$, show that the Lyapunov exponents of $\hat{p}^x$ and $\hat{p}^{\varepsilon,x}$ are close, and then compute the exponent for $\hat{p}^{\varepsilon,x}$.

The approximating path $\hat{p}^{\varepsilon,x}$ will be constructed as follows: recall that M is imbedded in $\mathbb{R}^D$. For a given $\varepsilon > 0$, let us divide $\mathbb{R}^D$ into cubes of length 2ε, and we call $P^{\varepsilon}_{i_1,\dots,i_D}$ the cube of length 2ε around $(\varepsilon i_1, \dots, \varepsilon i_D)$ with $(i_1, \dots, i_D) \in \mathbb{Z}^D$ (notice that those cubes are not disjoint sets). Suppose that $x \in M$, the starting point of $\hat{p}^x$, is an element of a given $P^{\varepsilon}_{y_0^{\varepsilon,x}}$, where $y_0^{\varepsilon,x} = (\varepsilon i_1^0, \dots, \varepsilon i_D^0)$ denotes the nearest neighbor of x in $\varepsilon\mathbb{Z}^D$. Set then $m_0^{\varepsilon,x}$ for the nearest point from $y_0^{\varepsilon,x}$ in M. Denote by $T_1^{\varepsilon,x}$ the first exit time of $\hat{p}^x$ from $P^{\varepsilon}_{y_0^{\varepsilon,x}}$ and set $y_1^{\varepsilon,x} = (\varepsilon i_1^1, \dots, \varepsilon i_D^1)$ for the nearest neighbor of $\hat{p}^x_{T_1^{\varepsilon,x}}$ in $\varepsilon\mathbb{Z}^D$ and $m_1^{\varepsilon,x}$ for the nearest point from $y_0^{\varepsilon,x}$ in M.

A sequence $\{T_n^{\varepsilon,x}; n \geqslant 0\}$ of stopping times, $\{y_n^{\varepsilon,x}; n \geqslant 0\} \subset \varepsilon\mathbb{Z}^D$ of nearest neighbors and $\{m_n^{\varepsilon,x}; n \geqslant 0\}$ of nearest points in M can be constructed then inductively. We will suppose that ε is small enough so that the $m_n^{\varepsilon,x}$ are still elements of $P^{\varepsilon}_{y_n^{\varepsilon,x}}$, which is always possible if M is a compact manifold. Notice also that the family $\{m_n^{\varepsilon,x}; n \geqslant 0\}$ is a subset of a fixed lattice in M. The appoximating path is then defined by

$$\hat{p}_t^{\varepsilon,x} = m_n^{\varepsilon,x}, \quad T_n^{\varepsilon,x} \leqslant t < T_{n+1}^{\varepsilon,x}.$$

Remark that $|\hat{p}_t^x - \hat{p}_t^{\varepsilon,x}|_{\mathbb{R}^D} \leqslant 2\sqrt{\mathrm{d}}\varepsilon$ for all $t \geqslant 0$ a.s., and since M is a compact manifold (with bounded curvature), we also have $\sup\{\rho(\hat{p}_t^x, \hat{p}_t^{\varepsilon,x}); t \geqslant 0\} \leqslant c\varepsilon$ for a fixed constant c.

Let us give some more notation: we will set

$$\begin{aligned}
\hat{e}^x &= \int_0^t W(\mathrm{d}s, \hat{p}^x_{t-s}) & \hat{e}^{\varepsilon,x} &= \int_0^t W(\mathrm{d}s, \hat{p}^{\varepsilon,x}_{t-s}),\\
\hat{Y}^{\varepsilon,x} &= \hat{e}^x - \hat{e}^{\varepsilon,x} & u^{\varepsilon}(t,x) &= \hat{E}[\exp(\hat{e}_t^{\varepsilon,x})],\\
\gamma_x &= \limsup_{t\to\infty} \tfrac{1}{t} \log u(t,x) & \gamma_x^{\varepsilon} &= \limsup_{t\to\infty} \frac{1}{t} \log u^{\varepsilon}(t,x).
\end{aligned}$$

Denote also by $\|f\|_{\infty,n}$ the quantity $\sup\{|f(t)|; n-1 \leqslant t \leqslant n\}$ for a continuous function f on $[n-1, n]$. In this section, we will show that γ_x and γ_x^{ε} are close when ε is small enough. The following regularity result for $\hat{p}^x$ will be an important step in that direction:

Lemma 10. *Let $\hat{c}^{x,\beta}_{p,n}$ be the β-Hölder norm of $\hat{p}^x$ on $[0,n]$ for a fixed path $\hat{p}^x$ and a given $\beta < \frac{1}{2}$, calculated over balls of maximal radius* 1:

$$\hat{c}^{x,\beta}_{p,n} = \sup_{u,v\in[0,n];0<|u-v|\leqslant 1} \frac{\rho(\hat{p}^x_u, \hat{p}^x_v)}{|u-v|^\beta}.$$

Let c be a positive constant. Then there exists a positive number α_0 such that, for any $\alpha < \alpha_0$, $n \geqslant 1$

$$\hat{E}[\exp(cn^{1/2}(\hat{c}^{x,\beta}_{p,n})^\alpha)] \leqslant \tilde{c}n\exp(n^{1/2(1-\zeta)})$$

for $\zeta > 0$ arbitrarily small and a given constant $\tilde{c} = \tilde{c}_\zeta$.

Proof. In what follows, c is a numerical constant depending on $Q, \alpha', \beta, b, \sigma$ and whose value may change from line to line. We begin by estimating the β-Hölder norm of $\hat{p}$ on $[0,2]$ rather than $[0,n]$. Call it $\hat{d}^x$. By classical Sobolev embedding, we have for any r and p such that $r - p^{-1} > \beta$

$$(\hat{d}^x)^{\alpha'} \leqslant c\left[\int_{[0,2]^2} \frac{|\hat{p}^x_t - \hat{p}^x_s|^p}{|t-s|^{1+rp}}\,\mathrm{d}s\,\mathrm{d}t\right]^{\alpha'} := \hat{\eta}^x.$$

Recall from Section 2.1 that the decomposition of $\hat{p}^x$ (as a $\mathbb{R}^D$-valued process) can be written

$$\hat{p}^x_t - \hat{p}^x_s = \left\{\int_s^t \sigma_{i,j}(s)B_j(\mathrm{d}s) + \int_s^t b_i(s)\,\mathrm{d}s;\ i = 1,\dots,D\right\},$$

where b and σ^2 are random adapted processes that are bounded by $c\kappa$ where c is a constant depending only on M. Using Ito's formula and Burkholder's inequality (see Kunita, 1990, Chapter 3), if $m > 2$

$$E[|\hat{p}^x_t - \hat{p}^x_s|^m] \leqslant \kappa^m 2^m \mathrm{e}^{2m} m^m |t-s|^{m/2}. \tag{11}$$

Let q be such that $\alpha' q > 1$. Then by Jensen's inequality

$$(\hat{\eta}^x)^q \leqslant c^q 2^{\alpha' q - 1}\int_{[0,2]^2} \frac{|\hat{p}^x_t - \hat{p}^x_s|^{p\alpha' q}}{|t-s|^{(1+rp)\alpha' q}}\,\mathrm{d}s\,\mathrm{d}t$$

and

$$\begin{aligned} E[(\hat{\eta}^x)^q] &\leqslant (4\kappa \mathrm{e}^2 p\alpha)^{p\alpha' q} q^{p\alpha' q}\int_{[0,2]^2} |t-s|^{\alpha' q(p(1/2-r)-1)}\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant c^{p\alpha' q} q^{p\alpha' q} \end{aligned}$$

as long as $p(1/2 - r) \geqslant 1$, which can be achieved for any r as long as p is large enough. Now using the Stirling-type bound $q! > q^q 3^{-q}$, we find

$$\hat{E}[\exp\hat{\eta}^x] \leqslant \sum_{q=0}^{[(\alpha')^{-1}]} \hat{E}(\hat{\eta}^x)^q + \sum_{q=[(\alpha')^{-1}]+1}^{\infty} (3c^{p\alpha'})^q q^{-q(1-p\alpha)}.$$

The infinite series converges for any choice of the constant p, as long as α' is small enough. The estimate for the tail of the series is uniform in $x\in M$; the first $[(\alpha')^{-1}]$ terms are estimated uniformly in $x\in M$ as well using Burkholder's inequality (11). We have proved that if α' is small enough,

$$\sup_{x\in M}\hat{E}[\exp\hat{\eta}^x]:=K<\infty.$$

The same calculation would yield the same integrability if $\hat{\eta}^x$ were the the α'th power of the β-Hölder constant of $\hat{p}$ over any other interval of length 2. We let $\hat{\eta}_j^x$ be those constants over the respective intervals $[j;j+2]$ for $j\in\mathbb{N}$.

To estimate the Hölder norm over $[0,n]$ itself, since it is defined over balls of length no geater than 1, we begin by noticing that

$$(\hat{c}_{p,n}^{x,\beta})^{\alpha'}\leqslant\sup_{j\in\{0,\dots,n-1\}}\hat{\eta}_j^x.$$

Therefore, for $\gamma\in(0,1)$, $N\geqslant 0$,

$$\hat{P}_x((\hat{c}_{p,n}^{x,\beta})^{\alpha'\gamma}\geqslant N)\leqslant\sum_{j=0}^{n-1}\hat{P}_x((\hat{\eta}_j)^{\gamma}\geqslant N).$$

However, the laws of $\{\hat{p}^x: x\in M\}$ form a homogeneous Markov family, and thus, using the previous estimate and Chebyshev's inequality

$$\begin{aligned}\hat{P}_x((\hat{\eta}_j)^{\gamma}\geqslant N)&=\hat{E}[\hat{P}_{y=\hat{p}_j^x}((\hat{\eta}^y)^{\gamma}\geqslant N)]\\&\leqslant\sup_{y\in M}\hat{P}((\hat{\eta}^y)^{\gamma}\geqslant N)\\&\leqslant K\mathrm{e}^{-N^{1/\gamma}}\end{aligned}$$

and therefore

$$\hat{P}_x((\hat{c}_{p,n}^{x,\beta})^{\alpha'\gamma}\geqslant N)\leqslant Kn\mathrm{e}^{-N^{1/\gamma}},$$

and

$$\begin{aligned}\hat{E}[\exp((\hat{c}_{p,n}^{x,\beta})^{\alpha'\gamma})]&=\int_0^{\infty}\hat{P}[\exp(\hat{c}_{p,n}^{x,\beta})^{\alpha'\gamma}>z]\,\mathrm{d}z\\&\leqslant 1+\int_1^{\infty}P((\hat{c}_{p,n}^{x,\beta})^{\alpha'\gamma}>\log z)\,\mathrm{d}z\\&\leqslant 1+Kn\int_1^{\infty}\exp(-(\log z)^{1/\gamma})\,\mathrm{d}z\leqslant cn.\end{aligned}\tag{12}$$

Noticing that, with $\zeta\in(0,1)$, the function $f(z)=\exp(z-az^{\zeta})$, $z>0$, has a global minimum equal to $\exp(a^{1/(1-\zeta)}c_{\zeta})$, we obtain with $\alpha=\alpha'\gamma\zeta$

$$\hat{E}[\exp(c\sqrt{n}(\hat{c}_{p,n}^{x,\beta})^{\alpha})]\leqslant\exp(n^{1/2(1-\zeta)})\hat{E}[\exp((\hat{c}_{p,n}^{x,\beta})^{\alpha'\gamma})],$$

which gives the desired result, taking into account relation (12).

Of course, α can be chosen arbitrarily small because if Hypothesis (H1) holds for a given constant α, it holds for any smaller positive constant. □

A bound on the jumps of $\hat{p}^{\varepsilon,x}$, given in the next lemma, will also be needed later on:

Lemma 11. *For $t,\varepsilon > 0$, set $\hat{N}_t$ for the number of jumps of $\hat{p}^{\varepsilon,x}$ before t, that is*

$$\hat{N}_t = \sum_{i=1}^{\infty} \mathbf{1}_{(T_i^{\varepsilon,x} \leqslant t)}.$$

Let C be an arbitrary positive constant. Then, for some $\gamma < 1, c > 0$ and for all n large enough,

$$\hat{E}[\exp C\hat{N}_n^{1/2}] \leqslant \exp(cn^{\gamma}). \tag{13}$$

Proof. This estimate is proved using a coupling argument. Fix $x \in M$ and $i \in \{1,\dots,D\}$. Let X be defined as the martingale part of the process $\hat{p}^{x,i}$, i.e.

$$X(t) = x + \int_0^t \sigma_{i,k}(r)\,\mathrm{d}B_k(r).$$

Let $\hat{\varphi}$ be the right-continuous inverse of the increasing process $A = \langle X\rangle$. Let $\hat{B}(s) = X(\hat{\varphi}(s))$. On an enlarged probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$, the process $\hat{B}$ is a standard Brownian motion and we have the representation $X(t) = \hat{B}(A_t)$ (Kunita, 1990 Theorem 3.4.6). Notice that with $K = \|\sigma\|_\infty^2$, we have for all realizations of B, $A_t \leqslant Kt$. We can also write $\hat{p}^{x,i}(t) = \hat{B}(A_t) + g(t)$ where g is differentiable and $\sup_{t\geqslant 0}|g'(t)| \leqslant \|b\|_\infty$. In the remainder of the proof, the jth *jump time* T_j of a continuous stochastic process Y *relative to the scale* v is defined as the exit times of Y from $[Y_{T_{j-1}} - v; Y_{T_{j-1}} + v]$, with $T_0 = 0$. Let $\hat{N}_{Kn}^B$ be the number of jump times for $\hat{B}$ before time Kn relative to the scale $\varepsilon/3$. This is greater than $\hat{N}_n^X$, the number of jump times of X before time n for the same scale. Let $\hat{N}_n^g$ be the number of jump times of g before time n relative to the scale $\varepsilon/6$. We denote $\hat{N}_n^i$ the number of jump times of $\hat{p}^{x,i}$ before time n relative to the scale ε.

Step 1: Let $k_0(n) = 6n\|b\|_\infty/\varepsilon$. Let k be an integer greater than $2k_0(n)$. We will prove that, if $\hat{N}_n^i \geqslant k$ then

$$\hat{N}_{Kn}^B \geqslant \left[\frac{k}{2}\right].$$

To this purpose, notice first that the number of jump times of g before time n in the scale $\varepsilon/6$ is less than $6n\|b|_\infty/\varepsilon$. Indeed $\|g'|_\infty \leqslant \|b\|_\infty$ and the greatest possible number of jumps for g is achieved if g is linear between the jump times with constant slope, in which case the integer part of $6n\|g'\|_\infty/\varepsilon$ is exactly the number of jumps. Since the number of jumps of $g+X$ is at least 2 times larger than the number of jumps of g, then there are at least $\hat{N}_n^i/2$ inter-jump intervals for $g+X$ which are within two successive jump times of g. Between two such times, the range of g is within an interval of size $\varepsilon/3$ while the range of $g+X$ exceeds an interval of size ε. Therefore, the range of X must exceed an interval of size $2\varepsilon/3$, and therefore X must jump at least once, proving that $\hat{N}_{Kn}^B \geqslant \hat{N}_n^X \geqslant [k/2]$.

Step 2: Assume k is as in the previous step. Let us show now that there is a constant C such that

$$\hat{P}[\hat{N}_n^i \geqslant k] \leqslant \frac{(C\varepsilon^{-2}\|\sigma\|_\infty^2 n)^{[k/2]}}{[k/2]!}. \tag{14}$$

Indeed, let us use the following result from Carmona and Viens (1998) (see Lemma 8, Proposition 9, their proofs, and Section 4.1 therein): if $\hat{B}$ is a standard Brownian motion under $\hat{P}$, and $\hat{N}_n$ is its number of jump times before time n in the scale ε, then there is a constant C such that $\hat{P}[\hat{N}_n > k] \leqslant (C\varepsilon^{-2}n)^k/k!$. Applying this to the previous lemma, we obtain

$$\begin{aligned}\hat{P}[\hat{N}_n^i \geqslant k] &\leqslant \hat{P}[\hat{N}_{Kn}^B \geqslant k/2]\\ &\leqslant (C\varepsilon^{-2}Kn)^{[k/2]}/[k/2]!\end{aligned}$$

Step 3: Bound (13) is now obtained as follows. Let $0 < \gamma < 12$. We have, for n large enough,

$$\begin{aligned}\hat{E}[\exp(C\hat{N}_n^{1/2})] &= \sum_{k=0}^{\infty} \hat{P}[\hat{N}_n = k] \exp Ck^{1/2}\\ &\leqslant \sum_{k=0}^{n^{2-2\gamma}} \exp Ck^{1/2} + \sum_{k=n^{2-2\gamma}}^{\infty} \hat{P}[\hat{N}_n \geqslant k] \exp Ck^{1/2}\\ &\leqslant n^2 \exp Cn^{1-\gamma}\\ &\quad + \sum_{k=n^{2-2\gamma}}^{\infty} \hat{P}[\exists i \in \{1,\ldots,D\} : \hat{N}_n^i \geqslant [k/D]] \exp Ck^{1/2}\\ &\leqslant \exp Cn^{1-\gamma} + \sum_{k=n^{2-2\gamma}}^{\infty} D\hat{P}[\hat{N}_n^i \geqslant [k/D]] \exp Ck^{1/2}.\end{aligned}$$

We now use inequality (14) on the last term. This is allowed because $2-2\gamma > 1$ so that for n large enough $n^{2-2\gamma} > k_0(n) = cn$. Let $\theta > 0$ be such that $2(1-\gamma)(1-\theta) > 1$. We get, with C a constant that may depend on $\kappa, \varepsilon, \|\sigma\|_\infty^2, D$ and change from line to line, and with $\zeta = 1/2D$:

$$\begin{aligned}&\sum_{k=n^{2-2\gamma}}^{\infty} \hat{P}[N_n^i \geqslant [k/D]] \exp Ck^{1/2}\\ &\quad\leqslant \sum_{k=n^{2-2\gamma}}^{\infty} \exp Ck^{1/2}(Cn)^{[\zeta k]}/[\zeta k]!\\ &\quad\leqslant \sum_{k=n^{2-2\gamma}}^{\infty} (3Cn)^{[\zeta k]}[\zeta k]^{-[\zeta k]}\end{aligned}$$

$$\leqslant \sum_{k=n^{2-2\gamma}}^{\infty} (Cn)^{[\zeta k]}[\zeta k]^{-(1-\theta)[\zeta k]}[\zeta k]^{-\theta[\zeta k]}$$

$$\leqslant \sum_{k=n^{2-2\gamma}}^{\infty} \frac{C^k}{k^{\theta k}}$$

$$\leqslant 1. \qquad \square$$

The following result will be essential in order to compare γ_x and γ_x^{ε}.

Proposition 12. *For a given* $\varepsilon > 0$, *there is a* n_ε *such that, for any* $n \geqslant n_\varepsilon$, *there exist some constants* c_1 *and* β *such that*

$$\hat{E}E[\exp(\|\hat{Y}^{\varepsilon,x}\|_{\infty,n})] \leqslant \exp(c_1 n\varepsilon^{\beta}).$$

Proof. We will divide the proof in several steps. From now on, c will designate a constant that can change from line to line.

Step 1: Some Gaussian estimates.

The classical tools of Gaussian analysis (see Adler (1990) and Carmona and Viens (1998) for more details) show that, for a constant $c > 0$,

$$E[\exp(\|\hat{Y}^{\varepsilon,x}\|_{\infty,n})] \leqslant c\left(1 + \sigma_n \exp\left(\frac{\sigma_n^2}{2}\right)\right)\exp(E[\|\hat{Y}^{\varepsilon,x}\|_{\infty,n}])$$

with

$$\sigma_n^2 = \sup\{E[(\hat{Y}_t^{\varepsilon,x})^2]; t \in [n-1;n]\}.$$

We shall then evaluate σ_n^2 and $E[\|\hat{Y}^{\varepsilon,x}\|_{\infty,n}]$. Notice also that

$$E[\|\hat{Y}^{\varepsilon,x}\|_{\infty,n}] \leqslant E[\|\hat{e}^x\|_{\infty,n}] + E[\|\hat{e}^{\varepsilon,x}\|_{\infty,n}],$$

and we shall estimate the two terms of the right-hand side separately.

Step 2: Evaluation of σ_n^2.

We have, using relations (4), (3) and hypothesis (H1), for $(t,x) \in \mathbb{R}_+ \times M$,

$$\begin{aligned} E[(\hat{Y}_t^{\varepsilon,x})^2] &= E\left[\left(\int_0^t W(\mathrm{d}s, \hat{p}_{t-s}^x) - \int_0^t W(\mathrm{d}s, \hat{p}_{t-s}^{\varepsilon,x})\right)^2\right] \\ &= \sum_{i=1}^{\infty} q_i \int_0^t [e_i(\hat{p}_{t-s}^x) - e_i(\hat{p}_{t-s}^{\varepsilon,x})]^2\,\mathrm{d}s \\ &\leqslant c\left(\sum_{i=1}^{\infty} q_i(1+\lambda_i)^{\alpha}\|e_i\|_\infty\right)\int_0^t [\rho(\hat{p}_{t-s}^x, \hat{p}_{t-s}^{\varepsilon,x})]^{2\alpha}\,\mathrm{d}s \\ &\leqslant c\left(\sum_{i=1}^{\infty} q_i(1+\lambda_i)^{\alpha}\|e_i\|_\infty\right)(c\varepsilon)^{2\alpha} t. \end{aligned}$$

Hence $\sigma_n^2 \leqslant cn\varepsilon^{2\alpha}$.

Step 3: Some more Gaussian estimates.

In order to compute $\|\hat{e}^x\|_{\infty,n}$, we shall use some more Gaussian inequalities, taken again from [1]: let $\hat{\delta}$ be the canonical metric on $[n, n+1]$ associated to $\hat{e}^x$, that is

$$\hat{\delta}(s,t) = E^{1/2}[(\hat{e}_t^x - \hat{e}_s^x)^2], \quad s, t \in [n, n+1].$$

Call $\hat{N}$ the entropy associated to this canonical metric, that is, $\hat{N}(\eta)$ is the minimal number of balls of radius no greater than η that are needed to cover $[n.n+1]$. Then we have the Borell-type inequality:

$$E[\|\hat{e}^x\|_{\infty,n}] \leqslant K_u \int_0^\infty [\log \hat{N}(\eta)]^{1/2} \, d\eta,$$

for a universal constant K_u. Remark that this inequality also holds for $\hat{e}^{\varepsilon,x}$. We shall now estimate $\hat{\delta}(s,t)$.

Step 4: Evaluation of $\hat{N}(\eta)$.

From definition (9), for $n-1 \leqslant s < t \leqslant n$ and $x \in M$,

$$\begin{aligned} & E\left[(\hat{e}_t^x - \hat{e}_s^x)^2\right] \\ & \quad = E\left[\left(\int_0^t W(du, \hat{p}_{t-u}^x) - \int_0^s W(du, \hat{p}_{s-u}^x)\right)^2\right] \\ & \quad = \int_s^t du Q(\hat{p}_{t-u}^x, \hat{p}_{t-u}^x) + \int_0^s du[Q(\hat{p}_{t-u}, \hat{p}_{t-u}) - Q(\hat{p}_{t-u}, \hat{p}_{s-u})] \\ & \qquad + \int_0^s du[Q(\hat{p}_{s-u}, \hat{p}_{s-u}) - Q(\hat{p}_{t-u}, \hat{p}_{s-u})]. \end{aligned}$$

Invoking Lemma 3, we obtain, with C a constant depending only on Q and α, and with $\beta \in (0, 1/2)$

$$\begin{aligned} E[(\hat{e}_t^x - \hat{e}_s^x)^2] & \leqslant C|t-s| + 2C|t-s|^{2\alpha\beta} \int_0^s \frac{\rho(\hat{p}_{t-u}^x, \hat{p}_{s-u}^x)^{2\alpha}}{|t-u-(s-u)|^{2\alpha\beta}} \\ & \leqslant C|t-s| + 2Cn|t-s|^{2\alpha\beta}[\hat{c}_{p,n}^{x,\beta}]^{2\alpha}, \end{aligned}$$

where $\hat{c}_{p,n}^{x,\beta}$ is the β-Hölder norm of $\hat{p}^x$ on $[0,n]$ for a fixed path $\hat{p}^x$, calculated over balls of maximal radius 1, defined in Lemma 10.

Set now $\ell^2 = C(1 + n(\hat{c}_{p,n}^{x,\beta})^{2\alpha})$. Since $\hat{\delta}(s,t) \leqslant \ell |t-s|^{\alpha\beta}$, it is easily seen that

$$\hat{N}(\eta) \leqslant \left(\frac{\ell}{\eta}\right)^{\alpha^{-1}\beta^{-1}}.$$

Moreover, $\hat{N}(\eta) = 1$ for $\eta \geqslant \ell$. Hence, by an easy change of variable,

$$\int_0^\infty [\log \hat{N}(\eta)]^{1/2} \, d\eta \leqslant \int_0^1 [\log(\ell\eta^{-1})^{\alpha^{-1}\beta^{-1}}]^{1/2} \, d\eta$$

$$\leqslant (\alpha\beta)^{-1/2}\left(\int_0^1 [\log(u^{-1})]^{1/2}\,\mathrm{d}u\right)\ell$$

$$\leqslant c_{\alpha,\beta}(1+n^{1/2}(\hat{c}_{p,n}^{x,\beta})^{\alpha}),$$

where $c_{\alpha,\beta}$ is a deterministic constant. Thus we have

$$E[\|\hat{\mathrm{e}}^x\|_{\infty,n}] \leqslant C(1+n^{1/2}(\hat{c}_{p,n}^{x,\beta})^{\alpha}). \tag{15}$$

Step 5: Estimation of $\hat{E}[\exp(E[\|\hat{\mathrm{e}}^x\|_{\infty,n}])]$.

Thanks to Lemma 10 we see that for any $\gamma' > 1/2$, if α is small enough,

$$\hat{E}[\exp(E[\|\hat{\mathrm{e}}^x\|_{\infty,n}])] \leqslant c\exp cn^{\gamma'}.$$

Again recall that α can be chosen arbitrarily small because if Hypothesis (H1) holds for a given constant α, it holds for any smaller positive constant.

Step 6: Estimation of $\hat{E}[\exp(E[\|\hat{\mathrm{e}}^{\varepsilon,x}\|_{\infty,n}])]$.

We can again use the argument of Step 3, to see that with

$$\hat{\delta}^{\varepsilon,x}(s,t) = E^{1/2}[(\hat{\mathrm{e}}_t^{\varepsilon,x} - \hat{\mathrm{e}}_s^{\varepsilon,x})^2], \quad s,t\in[n,n+1].$$

Borell's inequality implies

$$E[\|\hat{\mathrm{e}}^{\varepsilon,x}\|_{\infty,n}] \leqslant K_u\int_0^{\infty} [\log\hat{N}^{\varepsilon}(\eta)]^{1/2}\,\mathrm{d}\eta,$$

where $\hat{N}^{\varepsilon,x}$ is the metric entropy for $\hat{\delta}^{\varepsilon,x}$. Now to estimate $\hat{\delta}^{\varepsilon,x}$:

$$\begin{aligned}\hat{\delta}^{\varepsilon,x}(s,t)^2 &= E\left[\left(\int_0^t W(\mathrm{d}u,\hat{p}_{t-u}^{\varepsilon,x}) - \int_0^s W(\mathrm{d}u,\hat{p}_{s-u}^{\varepsilon,x})\right)^2\right]\\ &= \int_s^t \mathrm{d}uQ(\hat{p}_{t-u}^{\varepsilon,x},\hat{p}_{s-u}^{\varepsilon,x}) + \int_0^s \mathrm{d}u'E(W(1,\hat{p}_{t-s+u'}^{\varepsilon,x}) - W(1,\hat{p}_{u'}^{\varepsilon,x}))^2\\ &\leqslant C|t-s| + CJ(s,t,\hat{p}^{\varepsilon,x}),\end{aligned}$$

where $J = J(s,t,\hat{p}^{\varepsilon,x})$ is the length of time in $[0,s]$ that $\hat{p}_{t-s+u'}^{\varepsilon,x}$ is not equal to $\hat{p}_{u'}^{\varepsilon,x}$ (indeed if these two sites are equal, then the above expectation is zero). This occurs for times u' that satisfy: $\exists j : T_j\in[u',u'+t-s]$ where T_j is one of the jump times of $\hat{p}$ before time t. If $|t-s|$ is smaller than all the interjump times, then this length of time is clearly equal to $|t-s|\hat{N}_t$ where $\hat{N}_t$ is the total number of jump times for $\hat{p}^{\varepsilon}$ before time t. This case of small $|t-s|$ is the worst case. Therefore we have proved:

$$\hat{\delta}^{\varepsilon,x}(s,t)^2 \leqslant C|t-s|\hat{N}_n.$$

Thus in the metric $\hat{\delta}^{\varepsilon,x}$, the diameter of $[n-1,n]$ is no greater than $(C\hat{N}_n)^{1/2}$, the entropy $\hat{N}^{\varepsilon,x}(\eta) \leqslant C\hat{N}_n\eta^{-2}$, and the entropy integral yields

$$E[\|\hat{\mathrm{e}}^{\varepsilon,x}\|_{\infty,n}] \leqslant K_u\int_0^{(C\hat{N}_n)^{1/2}} [\log(C\hat{N}_n\eta^{-2})]^{1/2}\,\mathrm{d}\eta \leqslant (C\hat{N}_n)^{1/2}.$$

Now, Lemma 11 yields

$$\hat{E}[\exp(E[\|\hat{e}^{\varepsilon,x}\|_{\infty,n}])] \leqslant \exp(cn^{\gamma})$$

for a $\gamma < 1$, which ends the proof. □

Let us recall briefly, following the lines of Carmona and Viens (1998), why the last proposition implies that γ_x can be compared with γ_x^{ε}, which is the main result of this section.

Proposition 13. *With the above notations, and β and c_1 defined in Proposition* 12, *we have*

$$|\gamma_x - \gamma_x^{\varepsilon}| \leqslant c_2 \varepsilon^{\beta},$$

for a constant $c_2 > c_1$.

Proof. By Schwarz's inequality, we have

$$\gamma_x \leqslant \gamma_x^{\varepsilon} + \limsup_{t \to 0} \frac{1}{2t} \log \hat{E}[\exp(2\hat{Y}_t^{\varepsilon,x})].$$

Moreover, it is easily shown that

$$\limsup_{t \to 0} \frac{1}{t} \log \hat{E}[\exp(\hat{Y}_t^{\varepsilon,x})] \leqslant \limsup_{n \to \infty} \frac{1}{n-1} \log \hat{E}[\exp(\|\hat{Y}^{\varepsilon,x}\|_{\infty,n})].$$

By Chebychev's inequality, we can write

$$P(\hat{E}[\exp(\|\hat{Y}^{\varepsilon,x}\|_{\infty,n})] > \lambda) \leqslant \frac{1}{\lambda} \hat{E}E[\exp(\|\hat{Y}^{\varepsilon,x}\|_{\infty,n})],$$

and choosing $\lambda = \exp(c_2 n \varepsilon^{\beta})$ with $c_2 > c_1$, by Borel–Cantelli's lemma, we have that, P-almost surely

$$\limsup_{t \to 0} \frac{1}{t} \log \hat{E}[\exp(\hat{Y}_t^{\varepsilon,x})] \leqslant c_2 \varepsilon^{\beta},$$

which shows that $\gamma_x \leqslant \gamma_x^{\varepsilon} + c_2 \varepsilon^{\beta}$. Since all the inequalities are symmetric in γ_x and γ_x^{ε}, we also have $\gamma_x^{\varepsilon} \leqslant \gamma_x + c_2 \varepsilon^{\beta}$, which ends the proof. □

5. Calculating the Lyapunov exponent

Fix $x \in M$. By Proposition 13, the error made by replacing $u(t,x)$ by $u^{\varepsilon}(t,x)$ in calculating γ_x is no greater than a constant multiple of ε^{β}. We will now show that if κ is small enough, $\gamma_x^{\varepsilon} \leqslant c/\log(\kappa^{-1})$ for some constant c independent of κ; this will be achieved by choosing $\varepsilon = \kappa^{q}$ for some small $q > 0$. Thus the result of this paper will be established for any fixed κ that is small enough. We follow and use several of the calculations in Carmona and Viens (1998), Section 4.

For $t \geqslant 0$ and $k \in \mathbb{N}$, we call $\mathscr{S}(t,k)$ the simplex set

$$\mathscr{S}(t,k) = \{s = (s_1; \cdots ; s_k) \in (\mathbb{R}_+)^D \colon 0 \leqslant s_1 \leqslant \cdots \leqslant s_k \leqslant t\}.$$

If the total number of jumps times of $\hat{p}^{x,\varepsilon}$ before time t is equal to k, then the sites visited by $\hat{p}^{\varepsilon,x}$ form a nearest neighbor path in $x + \varepsilon\mathbb{Z}^D$ of length k that starts at x. We call $\mathscr{P}_k$ the set of all such possible paths in reverse order. This is a set of cardinality no greater than $(2D)^k$. First note that

$$u^\varepsilon(t,x) = \sum_{k\in\mathbb{N}} \hat{P}_x[\hat{N}_t = k]\hat{E}_x^k \left[\exp \sum_{l=0}^{k} W((T_l^{\varepsilon,x}, T_{l+1}^{\varepsilon,x}); m_l^{\varepsilon,x})\right],$$

where $W((s,t),x)$ denotes $W(t,x) - W(s,x)$, where we now call $T_l^{\varepsilon,x}$ and $m_l^{\varepsilon,x}$ the jump times and the sites visited by $s \mapsto \hat{p}_{t-s}^{\varepsilon,x}$, with $T_0^{\varepsilon,x} = 0$ and $T_{k+1}^{\varepsilon,x} = t$, and where $\hat{E}_x^k$ is the expectation conditional on the number of jumps k. If we define $\tilde{W}_m(s) = \sum_{l=0}^{k} W((s_l, s_{l+1}); m_l)$, for $m \in_k$, then $\tilde{W}_m$ is a Gaussian process on $\mathscr{S}(t,k)$. We can now write for $n \in \mathbb{N} - \{0\}$

$$\sup_{t\in[n-1,n]} u^\varepsilon(t,x) \leqslant \sum_{k\in\mathbb{N}} \hat{P}_x[\hat{N}_n \geqslant k] \sum_{m\in\mathscr{P}_k} \exp \sup_{s\in\mathscr{S}(n,k)} \sum_{l=0}^{k} W((s_l, s_{l+1}); m_l).$$

The Gaussian method of Carmona and Viens (1998) (Section 4) is now invoked to estimate the suprema of the Gaussian processes. This yields the existence of a deterministic function $\lambda(n)$ such that P-almost surely, for n large enough, $\lambda(n)$ exceeds $\sup_{t\in[n-1,n]} u^\varepsilon(t,x)$, with

$$\lambda(n) := \sum_{k\varepsilon\mathbb{N}} \hat{P}_x[\hat{N}_n \geqslant k](2D)^k \exp(Z_n),$$

where

$$Z_n = \frac{(2Q^* nk)^{1/2}}{c' + (\log(4D) + \log(1 + k^2) + \log n^2/k)^{1/2}},$$

c' is a universal constant and $Q^* := \sup_{x\in M} Q(x,x)$ is finite (consequence of Hypothesis (H1)). In particular,

$$\limsup_{t\to\infty} \frac{1}{t} \log u^\varepsilon(t,0) \leqslant \limsup_{n\to\infty} \frac{1}{n} \log \lambda(n), \quad \mathbb{P}\text{-almost-surely.} \tag{16}$$

To find the right-hand Lyapunov exponent in this inequality, we separate the sum defining $\lambda(n)$ into three pieces: those terms for which $0 \leqslant k \leqslant \log n^2$, those for which $\log n^2 \leqslant k \leqslant \delta n \log^{-2}(\kappa^{-1})$ where $\delta > 0$ will be chosen below, and those for larger k. Estimating the jump probabilities in the first piece brutally by 1 yields a bound of the form $\exp c\sqrt{n \log n}$, whose Lyapunov exponent is 0. The same brutal estimation bounds the second piece by

$$\delta n \log^{-2}(\kappa^{-1}) \exp\left[n\left(\frac{\delta \log(2D)}{\log^2(\kappa^{-1})} + \frac{Q^*\sqrt{\delta}(c' + \sqrt{2\log(4D) + 2})}{\log(\kappa^{-1})}\right)\right].$$

For small κ, this quantity's Lyapunov exponent is less than $c/\log(\kappa^{-1})$ for some constant $c>0$. For the last piece, we estimate the jump probabilities using relation (14): for $k\in\mathbb{N}$ such that $k\geqslant k_0(n):=6n\|b\|_\infty/\varepsilon$

$$\hat{P}_x[\hat{N}_n\geqslant k]\leqslant \hat{P}_x[\exists i\in\{1,\dots,D\}:\hat{N}_n^i\geqslant k/D]$$
$$\leqslant \frac{D(C\varepsilon^{-2}\|\sigma\|_\infty^2 n)^{[k/2D]}}{[\frac{k}{2D}]!}.$$

In order to be allowed to use this estimate on the last piece of $\lambda(n)$, we need only check that $\delta n\log^{-2}(\kappa^{-1})>6n\|b\|_\infty/\varepsilon$ with $\varepsilon=\kappa^q$ for some small $q>0$. To prove this, we notice that σ is the diffusion coefficient of the Markov process on M with generator $\kappa\Delta$, which means that $\sigma=\sqrt{\kappa}\sigma^{(1)}$ where $\sigma^{(1)}$ is the σ corresponding to $\kappa=1$; and therefore $\|\sigma\|_\infty^2=\kappa c$ where $c=\|\sigma^{(1)}\|_\infty^2$ is a constant depending only on M. Moreover, b is the Stratonovich correction formed on σ in the diffusion equation defining $\hat{p}$. Therefore, with $b^{(1)}$denoting the b corresponding to $\kappa=1$, we have $b=\kappa b^{(1)}$. Thus we only need to check that $\delta\log^{-2}(\kappa^{-1})>6\kappa\|b^{(1)}\|_\infty\kappa^{-q}$ which is true for small κ if $q<1$. We thus get the following quantity U as an upper bound for the tail of $\lambda(n)$, with C a constant depending on M:

$$U:=\sum_{k\geqslant\delta n\log^{-2}(\kappa^{-1})}\frac{(C\kappa\varepsilon^{-2})^{k/(2D)}}{[k/2D]!}\exp(CQ^*nk)^{1/2}.$$

Letting $L=\sqrt{CQ^*}$, $\varepsilon=\kappa^q$ and $\eta=(1-2q)/(2D)$ we calculate

$$\begin{aligned}
U&\leqslant\sum_{k\geqslant\delta n\log^{-2}(\kappa^{-1})}\frac{(C\kappa^\eta)^k}{[k/2D]!}\exp(\log(\kappa^{-1})L\delta^{-1/2}k)\\
&=\sum_{k\geqslant\delta n\log^{-2}(\kappa^{-1})}\frac{(C\kappa^{\eta-L\delta^{-1/2}})^k}{[k/2D]!}\\
&\leqslant(C\kappa^{\eta-L\delta^{-1/2}})^{\delta n\log^{-2}(\kappa^{-1})}\sum_{k'\in\mathbb{N}}(C\kappa^{\eta-L\delta^{-1/2}})^{k'}\\
&=\exp\left(n\delta\left(\frac{\log C}{\log^2(\kappa^{-1})}-\frac{\eta-L\delta^{-1/2}}{\log(\kappa^{-1})}\right)\right).
\end{aligned}$$

Choosing δ large and κ small yields a negative Lyapunov exponent for this last quantity. This ends the proof that there is a constant c such that

$$\limsup_{n\to\infty}\frac{1}{n}\log\lambda(n)\leqslant\frac{c}{\log(\kappa^{-1})}$$

and the main result of this paper is established.

Remark 14. The use of the large constant δ enables an easy final estimation above; in particular, the tail term's Lyapunov exponent is shown to be negative, and the contribution of $[k/2D]!$ is not even needed. A more careful calculation would require using the presence of the factorial, would yield a Lyapunov exponent for the tail of the same order as that of the second piece, but the only gain would be to allow a smaller δ. Since the the value of the constant in the final result lacks sharpness for several other reasons, we chose not to seek the smallest possible δ.

References

Adler, R., 1990. An introduction to continuity, extrema, and related topics for general Gaussian processes. Institute of Mathematical Statistics, Hayward, CA.

Berge, B., Chueshov, I.D., Vuillermot, P.-A., 2001. On the behavior of the solutions to certain Parabolic SPDE's driven by Wiener processes. Stochastic Process. Appl. 92, 237–263.

Bertini, L., Giacomin, G., 1999. On the long-time behavior of the stochastic heat equation. Probab. Theory Related Fields 114 (3), 279–289.

Carmona, R.A., Molchanov, S.A., 1994. Parabolic Anderson model and intermittency. Memoirs of the American Mathematical Society #518.

Carmona, R.A., Molchanov, S.A., Viens, F.G., 1996. Sharp upper bound on exponential behavior of a stochastic partial differential equation. Random Operators Stochastic Equations 4 (1), 43–49.

Carmona, R., Viens, F., 1998. Almost-sure exponential behavior of a stochastic Anderson model with continuous space parameter. Stochastics Stochastics Rep. 62 (3–4), 251–273.

Da Prato, G., Zabczyk, J., 1992. Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge.

Hazra, S.B., Viens, F., 2002. Towards pathwise stochastic fast dynamo in magneto-hydrodynamics. Fields Institute Communications. AMS, to appear.

Karatzas, I., Shreve, S.E., 1989. Brownian Motion and Stochastic Calculus, Graduate Texts in Mathematics. Springer, Berlin.

Kunita, H., 1990. Stochastic flows and Stochastic Differential Equations. Cambridge University Press, Cambridge.

Rogers, L., Williams, D., 1987. Diffusions, Markov Processes, and Martingales. Itô calculus, Vol. 2. Wiley, New York.

Tindel, S., Viens, F., 1999. On space–time regularity for the stochastic heat equation on Lie groups. J. Funct. Anal. 169 (2), 559–603.