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Towards pathwise stochastic fast dynamo in magneto-hydrodynamics

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Abstract. In a viscous magnetic fluid with a velocity field that is white-noise in time, the magnetic field H is a 3d-vector-valued random field on $\mathbf{R}^3$ satisfying a linear system of stochastic parabolic partial differential equations, in which the zero- and first-order terms have coefficients that are white-noise in time. The stochastic fast dynamo effect conjectures that the exponential rate of change of the norm of H for large time is a constant (almost-sure Lyapunov exponent) that remains positive even as the magnetic diffusivity decreases to zero. Existence, uniqueness, and explicit stochastic Feynman-Kac- and particle-type formulae for H are proven. Using the theory of products of random matrices, and the particle representation, a discussion of how to establish the dynamo effect is given. Ideas for proving and observing even sharper results are cast in the setting of numerical simulations.

1 Introduction

This expository paper discusses a long-standing conjecture in random magneto-hydrodynamics (MHD): the stochastic fast dynamo effect. According to this effect, the magnetic field's intensity inside a magnetic fluid can grow exponentially fast under appropriate conditions on the fluid's random velocity, even in the limit of small magnetic diffusivity.

The MHD fast dynamo problem was formulated by the Russian school of mathematical physics ([**27**], [**30**] and references therein). The problem's equations' involve

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a coupling of the magnetic field and of the velocity field. The random context, introduced by Molchanov and his collaborators, was an attempt to simplify the problem's complexity by assuming that the velocity field is known, which eliminates the nonlinearity from the equations. Indeed, instead of satisfying a Navier-Stokes-type equation, the velocity field is assumed to be a very erratic random field. This assumption accounts heuristically for a highly turbulent regime as is typical in nonlinear fluid dynamics (the case of high Reynolds number). It is then hoped that the well-known properties of exponential magnification of random flows yield the desired dynamo effect.

The fast dynamo effect was established for the statistical moments of the magnetic field by Molchanov, Ruzmaikin and Sokolov [**22**] using analytic methods. In [**23**], Molchanov and Tutubalin were the first to study pathwise random fast dynamo; they applied the theory of products of random matrices and their almost-sure exponential magnification (*Lyapunov exponents,* Oseledets' multiplicative ergodic theorem, see [**6**] Chapter VI). They only solved a simplified model with linear velocity field, however. The case of null magnetic diffusivity was treated in [**3**], but it does not constitute a fast dynamo theorem, which regards the case of *small* but nonzero diffusivity. A toy model was considered in [**21**], which we extended slightly in [**29**]. However, the difficulties encountered in the works cited above have led some to search for arguments against the random approach. Most notably in [**27**], V. Oseledets states that "Difficulties arise in this interpretation [the random approach] because chaotic flows behave differently in different directions." This paper will show that such a concern is unfounded. We will explain how to obtain random fast dynamo results in an incompressible and spatially homogeneous setting. We will show why products of random matrices are the right tool to tackle the random fast dynamo problem. Our techniques are based on a representation formula for the magnetic field using particle systems.

Let $\left\{V_j\left(t,x\right):t\geq 0;x\in\mathbf{R}^3;j=1,2,3\right\}$ be a given random field, the velocity field. Let κ be a positive constant, the magnetic diffusivity. Maxwell's equations yield that the magnetic field $\{H_j\left(t,x\right):t\geq 0;x\in\mathbf{R}^3;j=1,2,3\}$ satisfies the partial differential equation

$$\frac{\partial H_j}{\partial t}\left(t,x\right)=\kappa\Delta_x H_j\left(t,x\right)+\left(V\left(t,x\right)\cdot\nabla\right)H_j\left(t,x\right)-\frac{\partial V_j}{\partial x_k}\left(t,x\right)H_k\left(t,x\right) \qquad (1.1)$$

for all t,x,j, with initial data $H_j\left(0,x\right)=H_j^0\left(x\right)$, a given function. The repeated indices k above denote summation for $k=1,2,3$. For simplicity of exposition, we assume there are no boundary conditions on H. The following definition gives two forms of the fast dynamo effect: the pointwise effect (D1), and the energy effect (D2):

Definition 1.1 *A magnetic field H is said to exhibit the fast dynamo effect if, for small enough magnetic diffusivity $\kappa>0$, there is a $\lambda\left(\kappa\right)>0$ such that $\lim_{\kappa\to 0}\lambda\left(\kappa\right)>0$ and the large-time exponential behavior of H is bounded by $\lambda\left(\kappa\right)$ from below*

D1: *for each x:*

$$\liminf_{t\to\infty}\frac{1}{t}\log\left\|H\left(t,x\right)\right\|=\lambda\left(\kappa\right),$$

D2: *or for the magnetic energy:*

$$\liminf_{t\to\infty}\frac{1}{t}\log\left(\int\left\|H\left(t,x\right)\right\|^2dx\right)^{1/2}=\lambda\left(\kappa\right).$$

The function λ may be abusively called a *Lyapunov exponent*. If H satisfies equation (1.1) with a random velocity field V, the fast dynamo effect may be called *random*. In this article we will concentrate on a special type of random velocity field: the stochastic case, in which the behavior of V in the time parameter is of white-noise type. That is, we assume that V is the time-derivative, in the generalized sense, of a random field $\left\{v_j(t,x) : t \geq 0; x \in \mathbf{R}^3; j = 1,2,3\right\}$ whose behavior is Brownian in time. It then becomes necessary to interpret equation (1.1) in the stochastic sense, as follows

$$H_j(t,x) = H_j^0(x) + \int_0^t \kappa \Delta_x H_j(s,x)\,ds + \int_0^t (v(ds,x)\cdot\nabla) H_j(s,x) - \int_0^t \frac{\partial v_j}{\partial x_k}(ds,x) H_k(s,x).$$

The integrals in this equation may be interpreted in the Itō or the Stratonovich sense (see for example [**14**]). The stochastic fast dynamo effect simply means that the solution to equation (1) satisfies one of the two effects of definition 1.1 with probability one (almost-surely).

2 Existence, uniqueness, and representations

2.1 Existence and uniqueness: evolution approach. Existence and uniqueness of a solution to equation (1) is guaranteed by the recent works of [**16**] and [**26**], in the weak and the evolution settings respectively. The weak form replaces equation (1) by its integral against an appropriate family of test functions. The evolution form of [**26**] is based on the stochastic semigroup of operators with infinitesimal generator $\Delta + V(t,x)\cdot\nabla$. We will use this form because of its direct connection to the probabilistic representations needed for tackling the dynamo problems. We state the existence and uniqueness result:

Proposition 2.1 *Let $\left\{v_j(t,x) : t \geq 0; x \in \mathbf{R}^3; j = 1,2,3\right\}$ be a Gaussian random field defined by its covariance structure $\mathbf{E}[v(t,x)v(s,y)] = s \wedge tQ(x,y)$ on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\{b_t : t \geq 0\}$ be a 3-dimensional Brownian motion with variance 2κ on the Wiener space $(\mathcal{W}, \mathcal{G}, \mathbf{P}_b)$. If $v(1,\cdot)$ has a spatial gradient that has a bounded second moment, then*

(a): *The backwards Stratonovich stochastic differential equation*

$$\varphi_{t,s}(x) = x - \int_s^t v\left(\circ \hat{d}r, \varphi_{t,r}(x)\right) + \int_s^t b(\circ \hat{d}r) \tag{2.1}$$

on the product probability space $(\Omega \times \mathcal{W}, \mathcal{F} \times \mathcal{G}, \mathbf{P} \times \mathbf{P}_b)$ has a solution for all $0 \leq s \leq t$ and $x \in \mathbf{R}^3$ that is a stochastic flow of homeomorphisms,

(b): *The formula $T_{t,s}f(x) = \mathbf{E}_b(f(\varphi_{t,s}(x))$ defines a stochastic semigroup of operators for $0 \leq s \leq t$ in the sense of* [**26**]. *T's kernel, the marginal density $p(s,t,y,x) = \mathbf{P}_b[\varphi_{t,s}(x) \in dy]/dy$ is a stochastic kernel in the sense of conditions (i)-(iv) in* [**26**].

If moreover all 5th-order spatial derivatives of v have second moments, then

(c): *The stochastic kernel p also satisfies integrability conditions $(v)_p$, $(vi)_p$ and $(vii)_p$ in* [**26**], *as well as forward and backward Kolmogorov equations (see* [**26**]*)*

(d): *There is a unique solution in $C\left(\mathbf{R}_{+};L^{2}\left(\Omega\times\mathbf{R}^{3}\right)\right)$ to the following evolution form of equation (1), in which the anticipative integrals are of generalized-Stratonovich type:*

$$\begin{aligned} H_j(t,x) &= \int_{\mathbb{R}^3} p(0,t,y,x)\, H_j^0(y)\, dy \\ &\quad + \int_{\mathbb{R}^3}\int_0^t p(s,t,y,x)\, \frac{\partial v_j}{\partial x_k}(ds,y)\, H_k(s,y)\, dy. \end{aligned} \tag{2.2}$$

Moreover, the solution is almost-surely in $C\left(\mathbf{R}_{+};L^{2}\left(\mathbf{R}^{3}\right)\right)$, and is a classical solution to equation (1) if H^0 is smooth enough.

Proof Although the results in [**26**] are formulated for a single equation rather than a system of 3 equations, and for Skorohod integrals rather than generalized Stratonovich integrals, this proposition is a straightforward consequence of the theorems in [**26**]. For the definition and properties of generalized Stratonovich anticipative stochastic integrals, see [**25**]. □

2.2 Feynman-Kac representation. A Feynman-Kac formula for the solution to (1) or (2.2) exists. It is based on the stochastic kernel p. Before it can be given, we will define the notion of multiplicative stochastic integral.

For $m(dt)$ the differential of some matrix-valued Stratonovich integrable adapted process, let M be the solution to the generalized Stratonovich stochastic differential equation $dM_t = M_t m(dt)$ with initial data M_0 equal to the identity matrix Id. This linear stochastic flow of matrices can be understood with the following multiplicative stochastic integral notation $M_t = \prod_{[t,0]}[Id + m(ds)]$ which can be defined as a limit of appropriate Riemann products. As such, M_t appears as a (continuous) product of (infinitesimal) random matrices. Note the order of the endpoints in the interval of integration, which indicates in which order the matrices are to be multiplied. Let W be the matrix-valued Gaussian field defined by $W_{j,k}(dt,x) = -\partial v_j/\partial x_k(dt,x)$, and define the "Stratonovich correction term"

$$\gamma_{j,k}(x) = \mathbf{E}\left[-W_{j,k}(1,x)\,\nabla\cdot v(x)\right] = \left[\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_{lk}}Q_{j,l}\right](x,x).$$

Proposition 2.2 *The solution to equation (2.2) is given by the formula*

$$H(t,x) = \mathbf{E}_b\left[\prod_{[t,0]}\left[Id - W\left(\circ\hat{ds},\varphi_{t,s}(x)\right)\right]H^0\left(\varphi_{t,0}(x)\right)\right] \tag{2.3}$$

$$= \mathbf{E}_b\left[\prod_{[t,0]}\left[Id - W\left(\hat{ds},\varphi_{t,s}(x)\right) - \gamma\left(\varphi_{t,s}(x)\right)ds\right]H^0\left(\varphi_{t,0}(x)\right)\right] \tag{2.4}$$

Proof Replacing the product integral notation by its definition as the solution of a stochastic differential equation, using a conditional stochastic Fubini lemma to bring the expectation $\mathbf{E}_b$ inside the resulting formula, and using the definition of the stochastic kernel p, the proof of formula (2.3) follows. Formula (2.4) is obtained by transforming the Stratonovich integral into an Ito integral, as follows. Let t and x be fixed. Let Y be the process defined by the backwards Stratonovich equation

$$Y(ds) = -Y(s)\,W\left(\circ\hat{ds},\varphi_{t,s}(x)\right); \quad Y(t) = Id; \quad s \le t.$$

The value $Y(0)$ is precisely the multiplicative integral in formula (2.3). Note that the process Y is not anticipative. Rather, it is backwards adapted, and therefore the stochastic differential coincides with the usual Stratonovich differential. Consequently

$$\begin{aligned}
Y_{k,l}(ds) &= -Y_{k,m}(s)\,W_{ml}\left(\hat{d}s,\varphi_{t,s}(x)\right)-\frac{1}{2}Y_{k,m}(\hat{d}s)W_{m,l}\left(\hat{d}s,\varphi_{t,s}(x)\right)\\
&= -Y_{k,m}(s)\,W_{ml}\left(\hat{d}s,\varphi_{t,s}(x)\right)\\
&\quad+\frac{1}{2}Y_{k,n}(s)\,W_{n,m}\left(\hat{d}s,\varphi_{t,s}(x)\right)W_{m,l}\left(\hat{d}s,\varphi_{t,s}(x)\right)\\
&= -Y_{k,m}(s)\,W_{ml}\left(\hat{d}s,\varphi_{t,s}(x)\right)\\
&\quad-\frac{1}{2}Y_{k,n}(s)\left[\frac{\partial}{\partial x_m}\frac{\partial}{\partial x_l}Q_{n,m}\right](\varphi_{t,s}(x),\varphi_{t,s}(x))\,ds\\
&= -Y_{k,m}(s)\,W_{ml}\left(\hat{d}s,\varphi_{t,s}(x)\right)-\frac{1}{2}Y_{k,n}(s)\,\gamma_{n,l}(\varphi_{t,s}(x))\,ds.
\end{aligned}$$

□

Remark 2.1 *The equations (1) or (2.2) can be understood in the Ito or the Stratonovich senses. We chose to use the Stratonovich sense for two reasons:*

- **(i):** *it is generally accepted that the Stratonovich integral is the appropriate limit in the setting of diffusion approximation;*
- **(ii):** *the Ito equation also has a Feynman-Kac representation, but it involves more complicated correction terms: let*

$$\begin{aligned}
G(x) &= \kappa Id-2^{-1}Q(x,x),\\
\gamma^k_{j,l}(x) &= -\left(G(x)\right)^{-1}_{k,m}\partial_{x_l}Q_{j,m}(x,y)\,|_{y=x},
\end{aligned}$$

and let b_1,b_2,b_3 be three independent Brownian motions in addition to b. The Ito equation's Feynman-Kac formula is the same as formula (2.3) with $\gamma(\cdot)\,ds$ replaced by $\gamma^k(\cdot)\,b_k(ds)$ and with the flow φ driven by $\sqrt{G(\cdot)}b(ds)$ rather than just $b(ds)$. This shows that the Ito equation can only have a Feynman-Kac representation if $\sqrt{G}$ and G^{-1} are defined, i.e. if G is a positive definite matrix (the so-called coercivity *condition), which is the case only if κ is large. The Ito Feynman-Kac formula is therefore not physically relevant for the dynamo problem for which κ must be small.*

2.3 Particle representation. Representations of stochastic PDEs with systems of exogenous particles have seen a rapid growth in recent years (see e.g. [**10**], [**12**], [**13**], [**15**]). We propose a particle representation which is akin to the Feynman-Kac formula, but has several advantages.

Proposition 2.3 *Let $(b^i)_{i\in\mathbf{N}}$ be a family of independent Brownian motions in $\mathbf{R}^3$ with variance 2κ. Let the* particles $(X^i)_{i\in\mathbf{N}}$ *be forward versions of the stochastic flow φ of equation (2.1):*

$$X^i_t=X^i_0-\int_0^t v\left(\circ dr,X^i(r)\right)+\int_0^t b^i(\circ dr)$$

The solution to (1), or to (2.2), or to the weak form of (1), satisfies the particle representation formula almost-surely

$$\int f(x)\, H(t,x)\, dx = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f\left(X_t^i\right) Z_t^i. \tag{2.5}$$

where the vector-valued weights Z^i *are defined as*

$$Z_t^i = \prod_{s\in[t,0]} \left[Id + W\left(ds, X_s^i\right) + \gamma\left(X_s^i\right) ds \right] Z_0^i \tag{2.6}$$

as long as the initial positions X_0^i *and initial vectors* Z_0^i *are such that the measure* $\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} Z_0^i \delta_{X_0^i}$ *is a vector-valued measure whose density is* H^0.

Remark 2.2 *Formula (2.5) gives the solution* H *as the density of the measure-valued stochastic process*

$$\mu_t = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i} Z_t^i$$

with respect to the Lebesgue measure. The above limit is almost-sure with respect to $\mathbf{P} \times \prod_{i=1}^{\infty} \mathbf{P}_{b^i}$*, and does not depend on the paths* $\left(b^i\right)$*, by the effect of the law of large numbers.*

Proof (of the proposition). A straightforward application of the general theory of particle representations (see Section 11.3 in [**18**]) and of the fact that $dZ_t^i = Z_t^u W\left(dt, X_t^i\right)$, yields that the measure μ_t solves the weak form of the stochastic PDE (1). However, the theory does not guarantee that this measure has a density H. If the weak equation has a unique solution, then Chapter 6 in [**26**] proves that it must coincide with the unique evolution solution of Proposition 2.1, thereby showing that μ_t has a density equal to the evolution solution H. Uniqueness of the weak solution is given by the results in [**16**]. □

3 Comparing Feynman-Kac and particle representation

3.1 Particles more efficient than Feynman-Kac. We have been involved in some of the only attempts to estimate Lyapunov exponents using stochastic Feynman-Kac formulas such as (2.3) (see [**7**], [**8**], [**9**], [**29**], [**2**]). Our Feynman-Kac approach in [**2**] is the only one not based on an approximation by some kind of independent particle system, and is too specific to be applied to a wider context. The other Feynman-Kac methods all run into common difficulties not present in the particle methods: the time-reversal, which causes anticipativity in the formula, and the need to approximate the average over all paths somehow. In the particle method, the fact that the solution is given as a measure means that there is no need for time-reversal, and the Feynman-Kac-type average is replaced by an empirical measure over a countable family of paths, whose approximation by a finite number of paths is natural.

3.2 How to discretize time and space. Discretizing the space parameter, as an effort to reduce the problem's complexity, and as a crucial step for any numerical simulation, is problematic both in particle and Feynman-Kac representations. The generator of the particles X^i or b is the random operator with white-noise drift $L_t = \Delta_x + V(t,x) \cdot \nabla_x$. It is well known that there is no Markov process in discrete space $\mathbf{Z}^d$ with a generator L containing a drift, unless the drift can be incorporated into the second order term by writing the generator in a so-called *divergence* form, $L = \partial_i (a_{ij}\partial_j)$. Even if V were a bonafide function, it is clear that there can be no divergence form for L_t since its second order term is the Laplacian, which forces the matrix a to be the identity, and the drift to be zero. This means that any discretization of X^i or b will not preserve the Markov property, and thus, the corresponding formulas will not representent a stochastic PDE in discrete space. Consequently a spatial discretization of X^i or b, such as Euler's method, will need to be justified by a convergence theorem to the continuous space stochastic PDE.

Discretizing time faces similar difficulties in both settings. It is perhaps even more important to implement from the theoretical standpoint than space discretization, since it would lead to a discrete-time multiplicative stochastic integral, which is a bona fide product of random matrices, and which could allow the application of classical results on these products. This is evidenced by the toy model of the next section in which sharp upper and lower bounds of the fast-dynamo type are obtained thanks to large deviations estimates. These results also illustrate the difficult task of finding a lower bound.

4 A toy model

The following model ignores the first order term $V \cdot \nabla H$, and is cast in discrete time and space.

Let $\{M_n(x) : n \in \mathbf{N}; x \in \mathbf{R}^3\}$ be a family of random matrices such that $M_n(x)$ and $M_m(y)$ are independent if $n \neq m$, and such that all matrices have the same law μ. Assume μ is strongly irreducible and contractive, in the sense of Definition 5.3 below. Assume the following integrability condition: $\exists \alpha > 0$:

$$\int \mu(dm) \left(\|m\|^\alpha + \left\|m^{-1}\right\|^\alpha \right) < \infty.$$

Let b be a simple symmetric random walk in $\mathbf{Z}^3$ which jumps at times $\kappa^{-1}\mathbf{Z}_+$ (assuming κ^{-1} is an integer). Let $\mathbf{E}_b$ be the expectation with respect to b, and assume that b is independent of the matrices $M_n(x)$. Let $V \in \mathbf{R}^3$ be fixed and define the analog of the magnetic field by a discrete Feynman-Kac-type formula

$$H(n,x) = \mathbf{E}_b M_n(x) \cdots M_k(b_n - b_k + x) \cdots M_1(b_n - b_1 + x) V.$$

Theorem 4.1 *With the above hypotheses and notation, the Lyapunov exponent*

$$\gamma(0) = \lim_n n^{-1} \log \|M_n(0) \cdots M_1(0) V\|$$

exists and there exist $a, \varepsilon_0 > 0$, such that if $\varepsilon < \varepsilon_0$, the large deviations estimate

$$P\left[\left|n^{-1} \log \|M_n(0) \cdots M_1(0) V\| - \gamma(0)\right| > \varepsilon\right] \leq \exp -\frac{n\varepsilon^2}{2a^2},$$

holds; it follows that,

(a) there exist $\kappa_0, c > 0$ such that for all $0 < \kappa < \kappa_0$, almost surely,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|H(n,x)\| < \gamma(0) + c\sqrt{\kappa};$$

(b) if moreover all the coefficients of all the matrices $M_n(x)$ and all the components of V are positive, there exists $c' > 0$ such that for all $0 < \kappa < \kappa_0$, almost surely,

$$\liminf_{n\to\infty} \frac{1}{n} \log \|H(n,x)\| > \gamma(0) - c'\sqrt{\kappa}$$

Proof The first statement is the basic component of the Oseledets theorem (see Proposition 5.1 below). The second one is the large deviations result of LePage (see [**4**] Chapter 5).

To prove the upper bound (a), by the Borel-Cantelli lemma, we only need to show summability of the following probabilities in n:

$$\begin{aligned} & P\left[\|H(n,x)\| > \exp n\left(\gamma(0)+\varepsilon(\kappa)\right)\right] \\ \leq\ & P\left[E_b\left\|M_n(x)\cdots M_k(b_n-b_k+x)\cdots M_1(b_n-b_1+x)V\right\| > e^{n(\gamma(0)+\varepsilon)}\right] \\ \leq\ & \sum_{b\in\mathcal{P}_{\kappa n}} P\left[M_n(x)\cdots M_k(b_n-b_k+x)\cdots M_1(b_n-b_1+x)V > e^{n(\gamma(0)+\varepsilon)}\right]; \end{aligned}$$

here $\mathcal{P}_{\kappa n}$ is the set of all possible nearest-neighbor trajectories of length $[\kappa n]$ started from the origin in $\mathbf{Z}^3$. This set has cardinality $6^{[\kappa n]}$. Since n, x and b are now fixed, we can use the LePage large deviations result, to obtain

$$P\left[\|H(n,x)\| > \exp n\left(\gamma(0)+\varepsilon(\kappa)\right)\right] \leq 6^{\kappa n}\exp -\frac{n\varepsilon^2}{2a^2}$$

which is summable if $\varepsilon > \sqrt{\kappa 2a^2\log 6} := \varepsilon_1(\kappa)$. The large deviations result is only for small $\varepsilon \leq \varepsilon_0$, therefore we need to impose $\varepsilon_1(\kappa) < \varepsilon_0$. This holds for $\kappa < \kappa_0 = \varepsilon_0^2 2a^2\log 6$, and we obtain the result for $c = \sqrt{2a^2\log 6}$.

To prove the lower bound:

$$\begin{aligned} & P\left[\|H(n,x)\| < e^{n(\gamma(0)-\varepsilon)}\right] \\ \leq\ & P\left[\left|u(n,x)_1\right| < e^{n(\gamma(0)-\varepsilon)}\right] \\ =\ & P\left[E_b\left[\left(M_n(x)\cdots M_k(b_n-b_k+x)\cdots M_1(b_n-b_1+x)\right)_{1,j}V_j\right] < e^{n(\gamma(0)-\varepsilon)}\right] \\ \leq\ & 6^{\kappa n}P\left[\left(M_n(0)\cdots M_k(0)\cdots M_1(0)\right)_{1,j}V_j < e^{n(\gamma(0)-\varepsilon)}\right] \\ \leq\ & 6^{\kappa n}P\left[\left\|M_n(0)\cdots M_k(0)\cdots M_1(0)V\right\| < e^{n(\gamma(0)-\varepsilon/2)}\right] \\ & +6^{\kappa n}P\left[\frac{\left(M_n(0)\cdots M_k(0)\cdots M_1(0)\right)_{1,j}V_j}{\left\|M_n(0)\cdots M_k(0)\cdots M_1(0)V\right\|} < e^{-n\varepsilon/2)}\right] \end{aligned}$$

Let $V_n := M_n(0)\cdots M_k(0)\cdots M_1(0)V$. The results of Guivarc'h and Raugi (see [**4**], Chapter 6) imply that there is a random vector Z^* such that

$$\lim_{n\to\infty}\frac{(V_n)_{j'}}{(V_n)_j} = \frac{Z^*_{j'}}{Z^*_j},$$

and make it easy to show that the second term in the last expression is summable for any ε. The first term is summable by the calculation for the upper bound. This finished the proof of the theorem. □

5 The dynamo effect

5.1 Reduction to the IID case. The particle representation shows that we are faced with the system of (infinitesimal) matrices in the weights' formula (2.6):

$$M(ds) := Id + W\left(ds, X^i_s\right) + \gamma\left(X^i_s\right)ds; \quad s \geq 0.$$

This infinitesimal increment is in the Ito sense, which guarantees that these matrices, when multiplied together, form a Markov process. There exists a theory of Lyapunov exponents for such linear multiplicative matrix-valued Markov processes (see e.g. [**6**] Chapter 4), but it is not as well-developed as the more basic theory of products of independent and identically distributed (IID) matrices. To guarantee that the matrices $M(ds)$ are IID, it is necessary that γ be constant. This is guaranteed if the velocity field v satisfies the following *weak homogeneity* condition.

Definition 5.1 *A Gaussian random field v defined on the Cartesian product of two index sets $\mathcal{T}\times\mathcal{X}$ by its covariance structure*

$$\mathbf{E}v(t,x)\,v(s,y)=R(s,t)\,Q(x,y):\quad s,t\in\mathcal{T};\quad x,y\in\mathcal{X},$$

is said to be spatially weakly homogeneous for all $x\in\mathcal{X}$ if its spatial covariance function Q is such that $Q(x,x)$ does not depend on x. In other words, the spatial variance is constant.

Note that this condition is much weaker than the more commonly used homogeneity condition $Q(x,y)=Q(x-y)$. Assuming that v is spatially weakly homogeneous, not only is γ constant, but it also follows that the matrices $W\left(ds,X_x^i\right)$ are actually IID, as the following discrete-time argument shows.

Lemma 5.2 *Let $\{M_n(x):n\in\mathbf{N}\}$ be a family of random elements indexed by time n, and depending on a another parameter x in a countable set $\mathcal{X}$. If for each fixed $x\in\mathcal{X}$, $M_\cdot(x)$ is an IID family, and if the law of this family does not depend on x, then for any random sequence $\{X_n:n\in\mathbf{N}\}$ in $\mathcal{X}$ such that X_n is measurable w.r.t. $\mathcal{F}_{n-1}:=\sigma(M_0;\cdots;M_{n-1})$, the family of elements $\{M_n(X_n):n\in\mathbf{N}\}$ is IID.*

Proof Let f,g be bounded measurable functions. Then

$$\begin{aligned}
&\mathbf{E}f(M_n(X_n))\,g(M_{n-1}(X_{n-1}))\\
=\;&\mathbf{E}\{\mathbf{E}[f(M_n(X_n))\,|\mathcal{F}_{n-1}]\,g(M_{n-1}(X_{n-1}))\}\\
=\;&\mathbf{E}\{\mathbf{E}[f(M_n(x))]\,|_{x=X_n}g(M_{n-1}(X_{n-1}))\}\\
=\;&\mathbf{E}[f(M_n(0))]\,\mathbf{E}[g(M_{n-1}(X_{n-1}))]\\
=\;&\mathbf{E}[f(M_n(0))]\,\mathbf{E}\left\{\mathbf{E}[g(M_{n-1}(x))]\,|_{x=X_{n-1}}\right\}\\
=\;&\mathbf{E}[f(M_n(0))]\,\mathbf{E}[g(M_{n-1}(0))]\,.
\end{aligned}$$

□

This argument can be adapted to the continuous-time setting for $W\left(ds,X_s^i\right)$ because this differential is in the Ito sense. It can be adapted to the continuous space setting by noting that by hypothesis, $W(ds,x)$ is continuous in the space variable x, and therefore this random field can be approximated using the same random field defined solely for $x\in\mathbf{Q}^3$, so that we may use $\mathcal{X}=\mathbf{Q}^3$ in the above lemma.

5.2 Main result.

Definition 5.3 *Let μ be a probability law on the group of invertible matrices of size d. The law μ is strongly irreducible if there is no finite union U of proper subspaces of $\mathbf{R}^d$ such that $gU=U$ for all g in the support of μ. The law μ is contractive if the semigroup generated by the support of μ contains a sequence $\{g_n:n\in\mathbf{N}\}$ such that $g_n\|g_n\|^{-1}$ converges to a rank-one matrix.*

Proposition 5.1 *Let $\{M_n:n\in\mathbf{N}\}$ be an IID family of matrices in $GL_d(\mathbf{R})$ with common law μ such that $\int 0\vee\log\|x\|\,\mu(dx)$ and $\int 0\vee\log\left\|x^{-1}\right\|\mu(dx)$ exist. There is a sequence of d nested random subspaces $\{0\}=V_{d+1}\subset V_d\subset V_{d-1}\subset\cdots\subset$*

$V_1 = \mathbf{R}^d$ *and a sequence of* d *deterministic numbers* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ *(Lyapunov exponents) such that the limit*

$$\lim_{n\to\infty} \frac{1}{n} \left\| M_n M_{n-1} \cdots M_1 M_0 V \right\|$$

is almost surely equal to λ_k *iff* $V \in V_k \backslash V_{k+1}$. *This is the* Oseledets *or* Multiplicative Ergodic *theorem.* λ_1 *is called the* top *Lyapunov exponent.*

Moreover

$$\lim_{n\to\infty} \frac{1}{n} \left\| \det \left(M_n M_{n-1} \cdots M_1 M_0 \right) \right\| = \lambda_1 + \cdots + \lambda_d.$$

If μ *is strongly irreducible and contractive, then* $\lambda_1 > \lambda_2$.

Proof See e.g. [**20**] or [**6**]. □

The following, our main result, must be stated in the form of a conjecture because a crucial step in the proof still eludes us for technical reasons.

Conjecture 5.1 *Let* $\left\{ v(t,x) : t \geq 0, x \in \mathbf{R}^d \right\}$ *be a Gaussian random field that is Brownian in* t *and weakly homogeneous in* x. *Assume that all 5th-order spatial derivatives of* v *have second moments, so that the conclusions of Proposition 2.1 and Proposition 2.3 are valid. Let* μ *be the distribution of the matrix* $M_1^{i,j} = \prod_{[1,0]} \left[Id - \partial v_i / \partial x_j (ds, 0) \right]$ *and assume that the top Lyapunov exponent* λ_1 *associated with* μ *thru the Oseledets theorem (Proposition 5.1) is positive. Then the solution to equation (1) or (2.2) has the energy fast dynamo property of definition (1.1) (D2).*

Proof (Sketch of proof).

The heart of the proof is to calculate the magnetic energy $\int \left\| H(t,x) \right\|^2 dx$ by using the particle representation. We define the Mth particle approximation to H

$$\int H^M(t,x) f(x)\, dx := \frac{1}{M} \sum_{j=1}^{M} Z^j(t) f\left(X_t^j \right)$$

and for any positive integer l, denote by $H^{M,l}(t,x)$ the average of H^M over the cube $B_l(x)$ of volume l^{-3} centered at the point x in $\left(l^{-1}\mathbf{Z} \right)^3$, so that we have

$$H^{M,l}(t,x) = \frac{l^3}{M} \sum_{j=1}^{M} Z^j(t) \mathbf{1}_{B_l(x)} \left(X_t^j \right).$$

The particle representation for H then yields, using a Riemann-sum-type limit, and the Mth particle approximation,

$$\begin{aligned}
&\int \|H(t,x)\|^2 dx \\
&= \int H(t,x)_k \, dx H(t,x)_k \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} Z_k^i(t) H_k\left(t, X_t^i\right) \\
&= \lim_{N\to\infty}\lim_{l\to\infty}\lim_{M\to\infty} \sum_{x\in(l^{-1}\mathbf{Z})^3} \frac{1}{N}\sum_{i=1}^{N} Z_k^i(t)\mathbf{1}_{B_l(x)}\left(X_t^i\right)\frac{l^3}{M}\sum_{j=1}^{M} Z^j(t)\mathbf{1}_{B_l(x)}\left(X_t^j\right) \\
&= \lim_{N\to\infty}\lim_{l\to\infty}\lim_{M\to\infty} \sum_{x\in(l^{-1}\mathbf{Z})^3} \frac{1}{N}\sum_{i=1}^{N} Z_k^i(t)^2\mathbf{1}_{B_l(x)}\left(X_t^i\right)\mathbf{1}_{B_l(x)}\left(X_i\right)\frac{l^3}{M} \\
&+ \lim_{N\to\infty}\lim_{l\to\infty}\lim_{M\to\infty} \sum_{x\in(l^{-1}\mathbf{Z})^3} \frac{1}{N}\sum_{i=1}^{N} Z_k^i(t)\mathbf{1}_{B_l(x)}\left(X_t^i\right)\frac{l^3}{M}\sum_{j=1;j\neq i}^{M} Z^j(t)\mathbf{1}_{B_l(x)}\left(X_t^j\right) \\
&= E_0(t) + E_1(t)
\end{aligned}$$

In the last step, the sum over j is separated according to whether $j=i$. It is easy to see that we may take M as a function of l in the above limits. We choose $M=l^3$. Thus the first term in the last expression is precisely equal to

$$E_0(t) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} Z_k^i(t)^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left\|Z^i(t)\right\|^2.$$

Recall that $\mathbf{E}_b$ is the expectation with respect to a Brownian motion path b of velocity 2κ, and let $X(b)$ and $Z(\cdot,b)$ be the particle and weight constructed using b. The second term can be written as

$$\begin{aligned}
&|E_1(t)| \\
&= \left|\lim_{N\to\infty}\lim_{l\to\infty}\sum_{x\in(l^{-1}\mathbf{Z})^3}\frac{1}{N}\sum_{i=1}^{N} Z_k^i(t)\, l^3\mathbf{E}_b\left[Z_k(t,b)\mathbf{1}_{B_l(x)}\left(X_t(b)\right)\right]\mathbf{1}_{B_l(x)}\left(X_t^i\right)\right| \\
&\leq \lim_{N\to\infty}\lim_{l\to\infty}\sum_{x\in(l^{-1}\mathbf{Z})^3}\frac{1}{N}\sum_{i=1}^{N} Z_k^i(t)\,\mathbf{E}_b\left[Z_k(t,b)\, l^3\mathbf{1}_{\left|X_t(b)-X_t^i\right|<l^{-1}}\right].
\end{aligned}$$

Well-known properties of IID matrices $M_1, M_2, \cdots, M_n, \cdots$ indicate that for $M^{(n)} = M_n\cdots M_2M_1$ and for any fixed vectors e_1, e_2 and a, the ratio

$$\left\langle M^{(n)}e_1; a\right\rangle / \left\langle M^{(n)}e_2; a\right\rangle$$

converges to a finite nonzero limit almost-surely (results of Guivarc'h and Raugi, see [**4**] Theorem VI.3.1). This is a strong indication that although the vectors $Z^i(t)$ and $Z(t,b)$ have norms that increase exponentially fast, their kth components are remarkably close. Consequently, the products $Z_k^i(t)Z_k(t,b)$ are presumably bounded above and below by constant multiples of $\left\|Z^i(t)\right\|^2$. On the other hand, the quantity

$$\mathbf{E}_b\mathbf{1}_{\left|X_t(b)-X_t^i\right|<l^{-1}} = \mathbf{P}_b\left[\left|X_t(b)-X_t^i\right|<l^{-1}\right]$$

has the same transience properties as the similar term for Brownian motion alone, $\mathbf{P}_b\left[|b_t| < l^{-1}\right]$, which is of order $\left(lt^{1/2}\right)^{-3}$. These elements indicate that the term $E_1(t)$ should be no greater than a constant multiple of $t^{-3/2}E_0(t)$, showing that the large-time exponential behavior of the energy $\int \|H(t,x)\|^2 dx$ is that of $E_0(t)$ alone.

Using the concavity of the logarithm and Fatou's lemma, the estimation of E_0 reduces to showing that part (D2) of Definition 1.1 holds for any given Z^i rather than H. It also shows that H satisfies part (D2) of Definition 1.1 with $\lambda(\kappa) \geq \lambda(0)$. Since the infinitesimal matrices used to form any given Z^i have the same law as $M(ds) := Id + W(ds,0)$, $\lambda(0)$ is the top Lyapunov exponent λ_1 corresponding to the law of $M_1^{i,j} = \prod_{[1,0]}[Id + W_{i,j}(ds,0)]$, the positivity assumption on λ_1 finishes the proof. □

Definition 5.4 *A velocity field v defined on* $\mathbf{R}^3$ *is called* incompressible *or* divergence free *if* $\nabla \cdot v = 0$.

Corollary 5.4.1 *In the case of an incompressible velocity field v, the positivity of the top Lyapunov exponent in Conjecture (5.1) is satisfied. Hence the energy fast dynamo property is established in this case.*

Proof The incompressibility of v implies that the correction term γ vanishes and that each infinitesimal matrix $W(ds,0)$ has trace zero, since $traceW = -\nabla \cdot v = 0$. Again, we only need to consider the infinitesimal matrices $M(ds) := Id + W(ds,0)$; because $\gamma = 0$, the Ito and Stratonovich interpretations of this differential coincide, and therefore the product $M_t = \prod_{[t,0]} M(ds)$ is a member of the Lie group whose algebra is the set of trace-zero matrices, i.e. the matrices with determinant one. By the Oseledets theorem 5.1, this proves that the three Lyapunov exponents of the law μ of M_1 are related by $\lambda_1 + \lambda_2 + \lambda_3 = 0$. On the other hand, λ_1 and λ_2 are distinct, by a well-known property of solutions to linear matrix SDEs, which says in particular that μ is strongly irreducible and contractive (section 5.9 in [**4**]). Therefore, λ_1 must be positive. □

6 Further Conjectures

Although Conjecture 5.1 and its corollary already constitutes a fast dynamo theorem, we will also seek the pointwise fast dynamo theorem, as well as, in the spirit of the toy model of section 4, the behavior of $\lambda(\kappa)$ near zero, and an upper bound for the exponential behavior.

6.1 Sharp upper bounds.

Conjecture 6.1 *There is a constant $c' > 0$ such that parts (D1) and (D2) in 1.1 hold with the* $\liminf$ *replaced by a* $\limsup$ *and with $\lambda(\kappa)$ repalced by a $\lambda'(\kappa)$ that satisfies $\lambda'(\kappa) < \lambda(0) + c'/\log\left(\kappa^{-1}\right)$.*

Proof (Ideas for proof).

Energy and pointwise sharp upper bounds can be proven using the large deviations ideas of section 4, combined with the Lyapunov exponent estimation in continuous time found in [**8**], and with spatial discretization techniques employed in [**9**]. Although this may prove to be technically challenging, we believe the upper bound is inherently easier than the lower bound. □

6.2 Pointwise fast dynamo.

Conjecture 6.2 *The pointwise fast dynamo property (Part (D1) of Definition 1.1) holds.*

Proof (Ideas for proof).

The difficulty with the pointwise case is that a formulation of the energy as in the sketch of the proof of Conjecture 5.1, with a norm *inside* the average $\frac{1}{N}\sum_{i=1}^{N}$ of the leading term $E_0(t)$, is not available in the pointwise case. The lower bound in the toy model of section 4, under the additional restriction that all components of all matrices and vectors involved are positive, and using the lower bound time-discretization technique of [**7**] Chapter IV, could yield $\lambda(\kappa) > \lambda(0) - c/\log\kappa^{-1}$. Since one may take κ small, and since $\lambda(0)$ is positive, this would indicate a possible pointwise fast dynamo theorem (D1) if the argument could be extended beyond the hypothesis of global positivity. Imposing a strong drift term on the velocity field v, which is a reasonable physical hypothesis, could allow such an extension. □

6.3 Sharp lower bound for the energy.

Conjecture 6.3 *There is a constant $c > 0$ such that part (D2) of Definition 1.1 holds with $\lambda(\kappa) > \lambda(0) + c/\log(\kappa^{-1})$.*

Proof (Ideas for proof).

To prove such a result, discretization techniques such as those used in [**7**] Chapter IV, [**8**] and [**9**], would reduce the problem to the following question, which remains the most fundamentally difficult question in this study: can one maximize the top Lyapunov exponent of a product of random matrices in which, at each time step, there is a choice between several matrices with the same law and a certain amount of independence? There is hope to discover special cases in which this well-known open problem ([**19**]) can be solved, such as a global positivity condition. □

6.3.1 *Low discrepancy particle sequence and simulations.* We propose a simulation-assisted method. The particle representation of Proposition 2.3 claims that for *any given* "typical" realization of the auxiliary Brownian family (b^i), the approximate empirical sum $N^{-1}\sum_{i=1}^{N} Z_t^i \delta_{X_t^i}$ converges to the solution of (1). Therefore, to represent this solution, one may choose any such typical realization. Choosing one is a non-trivial task. An efficient way to do so is to identify a so-called *low-discrepancy* sequence of Brownian paths.

Low discrepancy is a well-known concept in random number simulation; such sequences have the advantage of being much more efficient than pseudo-random number generators (see [**5**]). Moreover, their precision is given explicitly by the Koksma-Hlawka inequality (Theorem 2.C.1.4 in [**5**]), which would make them convenient to use in a mathematical proof. There is a natural notion of low discrepancy for which it is known that (see [**28**]) there does not exist a low discrepancy sequence for the infinite dimensional random variable that is uniform on $[0,1]$ in each of its coordinates. Although our situation is also infinite dimensional, one would prove the sharp lower bound conjecture by a Borel-Cantelli argument, using only probability calculations for fixed time. Consequently, a notion of low discrepancy that is uniform in time would then be sufficient. Because the empirical mean in the particle representation formulas only averages out the randomness in the family of paths $(b^i)_i$, the low discrepancy would be conditional on the value of the random medium v. For example, for the leading term $E_0(t)$ we would have the

Conjecture 6.4 *There exists a realization* $\left(\hat{b}^i\right)_i$ *of the Brownian family so that uniform low discrepancy holds for* E_0 *almost-surely:*

$$\sup_{t\geq 0}\left|E_0\left(t\right)-\frac{1}{N}\sum_{i=1}^{N}\left|Z^i\left(t\right)\right|^2\right| := Discrep\left(N,\hat{b}\right)\rightarrow 0 \text{ as } N\rightarrow\infty.$$

One could argue that uniformity in time will be difficult to establish because of an increased amount of information for large t. This can be overcome by normalizing the above equation by the deterministic exponential rate that is expected. In this sense, we would prove a weighted low-discrepancy result.

The idea then is to identify a scheme to construct a sequence of paths with the low discrepancy property and such that the maximization effect described above occurs for the empirical mean $\frac{1}{N}\sum_{i=1}^{N}\left|Z^i\left(t\right)\right|^2$ for all N large enough. It would occur on a small proportion of the sequence of paths, small enough to ensure low discrepancy, and large enough to dictate the behavior of the whole mean without too many cancellations.

References

[1] Alòs, E.; Nualart, D.; Viens, F. G. *Stochastic heat equation with white noise drift.* Ann. Inst. H. Poincaré. Accepted (1999).

[2] Atar, R.; Viens, F.; Zeitouni, O. *Robustness of Zakai's equation via Feynman-Kac representations.* Stochastic analysis, control, optimization and applications. Eds: W. M. McEneaney, G. Yin, Q. Zhang. Birkhauser, 1999, 339-352

[3] Baxendale, P.H.; Rozovskii, B.L. *Kinematic Dynamo and Intermittency in a Turbulent Flow.* Geophys. Astrophys. Fluid Dynamics **73** (1993), 33-60.

[4] Bougerol, Ph.; Lacroix, J. *Products of random matrices and applications to Schrödinger operators.* Birkhäuser, 1985.

[5] Bouleau, N.; Lépingle, D. *Numerical Methods for Stochastic Processes.* John Wiley & Sons, 1994.

[6] Carmona, R.; Lacroix, J. *Spectral theory of random Schrödinger operators.* Birkhäuser, 1990.

[7] Carmona, R.; Molchanov, S. A. *Parabolic Anderson Model and Intermittency.* Memoirs Amer. Math. Soc. **518**, 1994.

[8] Carmona, R.; Molchanov, S. A.; Viens, F.G. *Sharp upper bound on the almost-sure exponential behavior of a stochastic parabolic partial differential equation.* Random Oper. Stochastic Equations **4** (1996), no. 1, 43–49

[9] Carmona, R.A.; Viens, F.G. *Almost-sure exponential behavior of a stochastic Anderson model with continuous space parameter.* Stochastics Stochastics Rep. **62** (1998), no. 3-4, 251–273

[10] Crisan, D.; Gaines, J.; Lyons, T. *Convergence of a branching particle method to the solution of the Zakai equation.* SIAM J. Appl. Math. **58** (1998), no. 5, 1568–1590.

[11] Da Prato, G.; Zabczyk, J. *Stochastic equations in infinite dimensions.* Cambridge Univ. Press, 1992.

[12] Del Moral, P. *Nonlinear filtering: interacting particle resolution.* Markov Processes rel. fields **2** (1996), 555–579

[13] Del Moral, P.; Guionnet, A. *On the stability of measure-valued processes. Applications to non linear linear filtering and interacting particle systems.* Preprint (1998).

[14] Karatzas, I.; Shreve, S.E. *Brownian motion and stochastic calculus.* 2nd ed. Springer, 1991.

[15] Kotelenez, P. *A class of quasilinear stochastic partial differential equations of Mc-Kean-Vlasov type with mass conservation.* Probab. Theory Rel. Fields **102** (1995), 159–188.

[16] Krylov, N.V. *An analytic approach to SPDEs.* Stochastic partial differential equations: six perspectives. Math. Surveys Monogr. **64**, Amer. Math. Soc. 1999, 185–242.

[17] Kunita, H. *Stochastic flows and stochastic differential equations.* Cambridge Univ. Press, 1990.

[18] Kurtz, T.; Protter, Ph. *Weak convergence of stochastic integrals and differential equations II: Infinite dimensional case.* Lecture Notes, 1996 CIME school in probab.

[19] Lacroix, J. Private communication.

[20] Ledrappier. *Quelques propriétés des exposants caractéristiques.* Ecole d'été de Saint-Flour 12, 1982. P.L. Hennequin, ed. Lect. Notes Math. **1097.** Springer, 1984.

[21] Molchanov, S.A.; Ruzmaikin, A.A. *Lyapunov exponents and distributions of magnetic fields in dynamo models.* The Dynkin Festschrift. Progr. Probab. **34.** Birkhäuser, 1994, 287–306.

[22] Molchanov, S. A.; Ruzmaikin, A. A.; Sokolov, D. D. *Short-correlated random flow as a fast dynamo.* Soviet Phys. Dokl. **32** (1987), no. 7, 569–570.

[23] Molchanov, S.A.; Tutubalin, V.N. *A Linear Model of Hydromagnetic Dynamo and Products of Random Matrices.* Theory Prob. Appl. **29** (1984), 231–246.

[24] Neuberger, J.W.; Viens, F.G. *Simulation of the nonlinear filter via particle methods.* In preparation.

[25] Nualart, D. *Malliavin Calculus and related topics.* Springer, 1995.

[26] Nualart, D.; Viens, F.G. *Evolution equation of a stochastic semigroup with white-noise drift.* Annals of Probability. Accepted (1998).

[27] Oseledets, V. *Fast dynamo problem.* XIth International Conference of Mathematical Physics (Paris, 1994). Cambridge, 1995, 288-296.

[28] Sobol', J.M. *Infinite dimensional uniformly distributed sequences in numerical mathematics.* Inst. Prikl. Mat. Akad. Nauk. SSR, 1974.

[29] Viens, F. G. *Almost-sure Exponential Behavior of a Linear System of Stochastic Partial Differential Equations.* Preprint, 1997.

[30] Zel'dovich, Ya.B.; Molchanov, S.A.; Ruzmakin, A.A.; Sokoloff, D.D. *Intermittency, diffusion and generation in a nonstationary random medium.* Soviet Sci. Rev. Sect. C: Math. Phys. Rev., Harwood A.P, **7** (1988), 3–110.