Stochastics and Stochastic Reports

ISSN: 1045-1129 (Print) (Online) Journal homepage: http://www.tandfonline.com/loi/gssr19

Almost-sure exponential behavior of a stochastic anderson model with continuous space parameter

René.A. Carmona & Frederi G. Viens

To cite this article: René.A. Carmona & Frederi G. Viens (1998) Almost-sure exponential behavior of a stochastic anderson model with continuous space parameter, Stochastics and Stochastic Reports, 62:3-4, 251-273, DOI: 10.1080/17442509808834135

To link to this article: https://doi.org/10.1080/17442509808834135

Published online: 22 Dec 2010.

Stochastics and Stochastics Reports, Vol. 62, pp. 251–273
Reprints available directly from the publisher
Photocopying permitted by license only

ALMOST-SURE EXPONENTIAL BEHAVIOR OF A STOCHASTIC ANDERSON MODEL WITH CONTINUOUS SPACE PARAMETER

RENÉ. A. CARMONA[a] and FREDERI G. VIENS[b,*,†]

[a]*Statistics and Operations Research Program, Princeton University, Princeton, NJ 08544;*
[b]*Department of Mathematics, University of North Texas, P.O. Box 30 5118, Denton, TX 76203-5118*

(Received 7 July 1997)

We study the almost-sure large time exponential growth of the solution to a linear stochastic parabolic equation with continuous space parameter, and a Gaussian-correlated potential that is white noise in time and homogeneous in space. We use the evolution form of that equation, for which existence and uniqueness are known. We establish a Feynman-Kac formula for the solution. By using a method of discretization of time and space, we prove that for small diffusion parameter κ, there is a deterministic constant c such that almost surely,

$$\limsup_{t\to\infty} \frac{1}{t} \log u(t, x) \leq \frac{c}{\log \kappa^{-1}}.$$

Keywords: Stochastic partial differential equation; Lyapunov exponent; spatial discretization; Gaussian estimates; stochastic Feynman-Kac formula; stochastic evolution equation

AMS Subject Classification: Primary: 60H15; Secondary: 60G17

*Research partially supported by the Air Force Office of Scientific Research NDSEG Fellowhip, and the National Science Foundation Grant No. INT 9600278.
†Corresponding author.

1. INTRODUCTION

We study the almost-sure large time exponential behavior for the parabolic equation of Anderson type with random potential:

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= \kappa \Delta u(t, x) + V(t, x) u(t, x) \\ u(0, x) &= 1, \quad t \geq 0 \end{aligned} \tag{1}$$

For discrete space parameter ($x \in \mathbf{Z}^d$), this problem has been studied in detail in Carmona and Molchanov's AMS memoir *Parabolic Anderson Problem and Intermittency* [2]. In their last chapter, with the potential V taken to be white noise in time, they investigate the so-called *almost-sure Lyapunov exponent* for the solution, i.e. the random quantity $\lim_{t \to \infty} t^{-1} \log u(t, x)$. When the limit does not exist, it is still interesting to evaluate the corresponding lim inf and lim sup. In [2], the following lower bound is obtained for small κ (smaller than some κ_0), and with some constant c_1:

$$\liminf_{t \to \infty} t^{-1} \log u(t, x) \geq \frac{C_1}{\log \kappa^{-1}}, \text{ almost surely.}$$

In [3], Gaussian estimates are used to exploit the hypothesis that the family $(V(\cdot, x)\ x \in \mathbf{Z}^d)$ of white noises that constitute the potential has a Gaussian correlation. They yield the following upper bound for $\kappa \geq \kappa_0$, and with some constant c_2:

$$\limsup_{t \to \infty} t^{-1} \log u(t, x) \leq \frac{C_2}{\log \kappa^{-1}}, \text{ almost surely.}$$

Although there is little hope of conciliating c_1 and c_2 in general, this upper bound constitutes an improvement on earlier works ([2], [8]) in which the Gaussian property had not been fully used. The upper bound's method of proof suggests that it may be adaptable to the case of continuous space parameter: then indeed, a Gaussian hypothesis on V is a natural one. In this article we show that this can indeed be done.

Specifically, a full Gaussian hypothesis on the family of white noises $(V(\cdot, x))_x$ means that V is the formal time-derivative of a random field W defined under some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ by the covariance

$$\mathbf{E}[W(t, x) W(s, y)] = s \wedge t \cdot Q(x, y)$$

where Q is the kernel of a positive operator. The usual meaning given to the pde 1 is its stochastic integral form. This equation cannot have a meaning in the strong sense unless W is spatially C^2. However the results proved herein only require that W be spatially Hölder-continuous (thus is equivalent to Hypothesis 6 below). In this case, we must (and will from now on) understand the stochastic pde in its less restrictive, evolution form, as follows:

$$u(t, x) = 1 + \int_{y \in \mathbf{R}^d} \left[\int_0^t p(t-s, x, y)\, W(ds, y)\, u(s, y) \right] dy, \qquad t \geq 0, \quad x \in \mathbf{R}^d \tag{2}$$

Here $p(s, x, y) = (2\pi\kappa s)^{-d/2} \exp(-(x-y)^2/2\kappa s)$ is the transition probability density of Brownian motion with variance 2κ. The upper bound result mentioned in the previous paragraph holds when the stochastic integral is understood in the sense of Stratonovitch.

For the sake of simplicity, we assume that W is spatially homogeneous, i.e. that $Q(x, y) = Q(x-y)$. In this situation in [4], Dawson and Salehi showed that equation 2 has a unique $L^2(\Omega)$- valued solution by exhibiting its Wiener chaos expansion. This means introducing the Gaussian spectral measure of W, i.e. the unique independently scattered Gaussian random measure $M(ds, d\lambda)$ with the properties

$$W(t, x) = \int\int_{\mathbf{R}^d \times [o,t]} e^{i\lambda \cdot x} M(ds, d\lambda) \tag{3}$$

$$E\left[\int\int f(\lambda, s) M(d\lambda, ds) \overline{\int\int g(\lambda, s) M(d\lambda, ds)}\right] = \int\int f(\lambda, s) \overline{g(\lambda, s)}\, \hat{Q}(d\lambda) ds \tag{4}$$

where $\hat{Q}$ is the Fourier transform Q. See [7] and [4] for details. In the next section, we establish the following stochastic Feynman-Kac formula for this solution:

THEOREM 1 *Let $(u(t, x), t \leq 0, x \in \mathbf{R}^d)$ be the unique separable solution to the Stratonovich stochastic evolution equation* 2. *Assume condition* 6. *Let $(b_t \geq 0)$ be a standard Wiener Process started from x with diffusion* 2 κ, *under* *its canonical probability space* $(\mathcal{C}, \mathcal{F}, \mathbf{P}_x)$ *(where* $\mathcal{C} = C(\mathbf{R}^+; \mathbf{R}^d)$*). Then* **P**-*almost-surely*,

$$u(t, x) = \mathbf{E}_x\left[\exp\left(\int_{\mathbf{R}^d}\int_0^t e^{i\lambda\cdot b_{t-s}} M(d\lambda, ds)\right)\right], \quad t \geq 0, \quad x \in \mathbf{R}^d \tag{5}$$

The proof for the Ito equation identifies each term in the Wiener chaos expansion with a corresponding term in the expansion of exp in Hilbert polynomials. The Stratonovich solution is trivially obtained by multiplying the Ito one by exp $Q(0)\ t/2$, yielding the above formula. This Feynman-Kac representation confirms that for both stochastic and non-stochastic equations, spatial smoothness of the data W is not a prerequisite. Its proof indicates that the evolution equation is one step closer to a probabilistic representation than a strong equation; indeed the heat kerner $p(t-s, x, y)$, which can be written probabilistically as $\mathbf{P}_x[b_{t-s} \in dy]/dy$, appears explicity the equation.

In Section 3 and 4 we proceed to the main quantitative result of this paper. We exploit the Feynman-Kac formula by using Gaussian supremum estimates (Borell-type inequality, Fernique's theorem) and prove our main result:

THEOREM 2 *Let* $(W(t, x), t \geq 0, x \in \mathbf{R}^d)$ *be a Gaussian process on* $(\Omega, \mathcal{F}, \mathbf{P})$ *with covariance* $\mathbf{E}[W(t,x)\ W(s,y)] = s \wedge t \cdot Q(x-y)$, *where* Q *is any homogeneous covariance function such that its Fourier transform* $\hat{Q}$ *has a moment of some order* $\alpha > 0$, *i.e.*

$$\exists\alpha > 0 : \int_{\mathbf{R}^d} |\lambda|^\alpha \hat{Q}(d\lambda) = K_\alpha < \infty. \tag{6}$$

The stochastic parabolic evolution equation 2 in Stratonovitch form has a unique continuous solution taking values in $L^2(\Omega)$ *and there are positive deterministic constants* κ_0 *and* c *depending only on* Q *such that, for* $\kappa < \kappa_0$, **P**-*almost surely, the large-time exponential behavior is bounded above as follows*:

$$\limsup_{t\to\infty} \frac{1}{t} \log u(t, x) \leq \frac{c}{\log k^{-1}}$$

In Section 3 we show how to reduce the proof of this upper bound to a situation in which the spatial parameters is in $\mathbf{Z}^d$ instead of $\mathbf{R}^d$. This discretization of space allows us, in Section 4, to finish the proof of the upper bound much in the spirit of the proof of the corresponding result in discrete space in [3] (also see [2], chapter IV, paragraph 3).

Regarding the Ito equation's solution, the correction factor $\exp -Q(0)t/2$, valid in the homogeneous case, or merely if $Q(x, x)$ is constant, has a trivial effect on the upper bound in Theorem 2. When $Q(x, x)$ is not constant, the correction factor to be included inside the Feynman-Kac expection 5 is $\exp -(1/2)\int_0^t Q(b_{t-2}, b_{t-s})ds$; this modification is not trivial. We do not investigate it here.

2. THE FEYNMAN-KAC FORMULA

2.1. The Wiener Chaos Expansion

In [4], the unique L^2 (Ω)-valued solution to the evolution equation 2 is exhibited as the sum in $L^2(\Omega)$ of the series $\Sigma_{n=0}^{\infty} X_n(t, x)$ defined by $X_0(t, x) = 1$ and

$$X_{n+1}(t,x) = \int_{\mathbf{R}^d}\left(\int_0^t p(t-s,x,y)X_n(s,y)W(ds,y)\right)dy. \tag{7}$$

To establish Theorem 1 we express X_n explicitly. From definition 3, first rewrite equation 7 using the spectral measure M:

$$X_{n+1}(t,x) = \int_{\mathbf{R}^d}\left(\int\int_{[0,t]\times\mathbf{R}^d} p(t-s,x,y)\,X_n(s,y)\,e^{i\lambda\cdot y}M(d\lambda, ds)\right)dy.$$

Now using a stochastic Fubini lemma (see [4], lemma 2.1) repetitively on the variables y_i, $i = n-1 \ldots 1$ we obtain, with the convention $s_0 = t$ and $y_0 = x$:

$$\begin{aligned}
X_n(t,x) &= \int\int\int_{y_1}\cdots\int\int\int_{y_n}\prod_{j=1}^{n}\left(p(s_{j-1}-s_j, y_{j-1}, y_j)\,e^{i\lambda_j\cdot y_j}\mathbf{1}_{[0,s_{j-1}]}(s_j)\right)\\
&\quad dy_n M(d\lambda_n, ds_n)\ldots dy_1 M(d\lambda_1, ds_1)\\
&= \int\int\cdots\int\int\prod_{j=1}^{n}\mathbf{1}_{[0,s_{j-1}]}(s_j)\left\{\int_{y_1}\cdots\int_{y_n}\prod_{j=1}^{n}p(s_{j-1}-s_j, e^{i\lambda_j\cdot y_j}\right.\\
&\quad \left. y_{j-1}, y_j)dy_n\ldots dy_1\right\}M(d\lambda_n, ds_n)\ldots M(d\lambda_1, ds_1).\\
&= \int\int\cdots\int\int\mathbf{E}_x\left(\prod_{j=1}^{n}\mathbf{1}_{[0,s_{j-1}]}(s_j)e^{i\lambda_j\cdot b_{t-s_j}}\right)\\
&\quad M(d\lambda_n, ds_n)\ldots M(d\lambda_1, ds_1).
\end{aligned} \tag{8}$$

In the last step, the spatial integral was represented thanks to the property of independence of increments for Brownian motion, which yields for $\tau_1 \leq \ldots \leq \tau_n$:

$$\mathbf{P}_x[b_{r_1} \in dx_1; \ldots; b_{r_n} \in dx_n]$$
$$= p(\tau_1, x, x_1)p(\tau_2 - \tau_1, x_1, x_2) \ldots p(\tau_n - \tau_{n-1}, x_{n-1}, x_n)\, dx_1 \ldots dx_n.$$

The proof given in [4], lemma 2.1, of the stochastic Fubini lemma used above is valid if one is prepared to accept the fact that if M is a Gaussian spectral measure with covariance $\hat{Q}$, and if $g(x, y)$ is a measurbale function which is in $L^2(\hat{Q})$ in the x variable, then the random variables $e(y) = \int g(x, y)$ $M(dx)$ are measurable in the parameter y. In the above use of the lemma, the parameter y is in the space C of continuous functions under the Wiener measure. Regularity conditions must then be imposed on g to obtain the required measurability. A general result in this direction is in [9], Proposition 2.1 and its corollaries. Here we will only need the following consequence of these results:

PROPOSITION 3 *Under Hypothesis 6, for all $t \geq 0$, $x \in \mathbf{R}^d$ and all positive integer n, the process*

$$b \to X_{n,t}(b, \omega) = \int \int \cdots \int \int \prod_{j=1}^{n} \mathbf{1}_{[0, s_{j-1}]}(s_j) e^{i\lambda_j \cdot b_{t-s_j}}$$
$$M(d\lambda_n, ds_n) \ldots M(d\lambda_1, ds_1),$$

has a version which is a random variable in $L^2(C \times \Omega)$.

We only recall the proof of the following analytic lemma, which is the core of the proof of the Proposition 3, and will be of crucial importance throughout this article:

LEMMA 4 *If $\hat{Q}$ is a finite measure on $\mathbf{R}^d$ (total mass $= Q(0) < \infty$) satisfying condition 6, i.e. $\exists\, \alpha > 0 : K_\alpha = \int_{\mathbf{R}^d} |\lambda|^\alpha \hat{Q}(d\lambda) < \infty$, then for any a, b in $\mathbf{R}^d$, we have*

$$\int |e^{i\lambda \cdot a} - e^{i\lambda \cdot b}|^2 \hat{Q}(d\lambda) < K_\alpha |a - b|^{2\alpha/(2+\alpha)}$$

Proof

$$\int |e^{i\lambda \cdot a} - e^{i\lambda \cdot b}|^2 \hat{Q}(d\lambda)$$
$$\leq \int_{\{\lambda < \eta\}} |a \cdot \lambda - b \cdot \lambda|^2 \hat{Q}(d\lambda) + \int_{\{\lambda \geq \eta\}} 2\hat{Q}(d\lambda)$$

$$\leq \eta^2|a-b|^2 Q(0) + 2\hat{Q}(\lambda \geq \eta)$$
$$\leq \eta^2|a-b|^2 Q(0) + 2\eta^{-\alpha}\int_{\mathbf{R}^d}|\lambda|^\alpha \hat{Q}(d\lambda)$$

where we used Chebyshev's inequality and the hypothesis of existence of the moment of order α. Then optimizing over $\eta > 0$ (taking η such that $\eta^2\,|a-b|^2 = \eta^{-\alpha}$), we obtain the result of the Lemma. ∎

2.2. Proof of the Feynman-Kac Formula

By virtue of Proposition 3, we may apply the stochastic Fubini lemma again to 8. We thus represent $X_n(t,\, x)$ by $\mathbf{E}_x[X_n^t(t,b)]$ where, for $0 < s < t$ and $s_0 = s$, $X_n^t(s,b)$ denotes the quantity

$$X_n^t(s,b) = \int\int\cdots\int\int\prod_{j=1}^n\left(\mathbf{1}_{[0,s_{j-1}]}(s_j)e^{i\lambda_j\cdot b_{t-s_j}}\right)M(d\lambda_n, ds_n)\ldots M(d\lambda_1, ds_1)$$
$$=\frac{1}{n!}\int\int\cdots\int\int\prod_{j=1}^n\left(\mathbf{1}_{[0,s]}(s_j)e^{i\lambda_j\cdot b_{t-s_j}}\right)M(d\lambda_n, ds_n)\ldots M(d\lambda_1, ds_1).$$

We transformed the integrand above by symmetrizing it since a multiple stochastic integral is invariant under this operation (see [7], Chapter 4); the symmetrization of $\Pi_{j=1}^n\mathbf{1}_{[0,s_{j-1}]}(s_j)$ with $s_0 = s$ is $(n!)^{-1}\Pi_{j=1}^n\mathbf{1}_{[0,s]}(s_j)$ and the rest of the integrand in $X_n^t(s,b)$ is symmetric. Now recall that by the Itô formula for multiple integrals (see [7], Theorem 4.2), if $f(\lambda,\, r)$ is a function with unit $L^2(dr\cdot\hat{Q}(d\lambda))$-norm, then with $\mathbf{H}_n$ the n-th Hermite polynomial,

$$\int\int\cdots\int\int f(\lambda_1, s_1)\ldots f(\lambda_n, s_n)M(d\lambda_n, ds_n)\ldots M(d\lambda_1, ds_1)$$
$$=\mathbf{H}_n\left(\int\int f(\lambda, r)M(d\lambda, dr)\right).$$

We can use this result for $X_n^t(s,b)$ with the function $f(\lambda, r) = \mathbf{1}_{[0,s]}(r)e^{i\lambda\cdot b_{t-r}}$ after dividing by its norm $\|f\|_{L^2(dr\cdot\hat{Q}(d\lambda))} = \left(\int_0^s\int|e^{i\lambda\cdot b_{t-r}}|^2(d\lambda)dr\right)^{1/2} = (Q(0)s)^{1/2}$, and obtain

$$X_n^t(s,b) = \frac{(Q(0)s)^{n/2}}{n!}\mathbf{H}_n\left((Q(0)s)^{-1/2}\int\int\mathbf{1}_{[0,s]}(r)e^{i\lambda_j\cdot b_{t-r}}M(d\lambda, dr)\right)$$
$$=\frac{(Q(0)s)^{n/2}}{n!}\mathbf{H}_n\left((Q(0)s)^{-1/2}e_{t,b}(s)\right) \tag{9}$$

with $e_{t,b}(s)=\int\int \mathbf{1}_{[0,s]}(r)e^{i\lambda_f \cdot b_{t-r}}M(d\lambda,dr)$. Notice that $(e_{t,b}(s)\colon s\in[0,t])$ is a continuous martingale with respect to the filtration $\mathcal{F}_s=\sigma\{W(r,x)\colon r\leq s, x\in\mathbf{R}^d\}, s\in[0,t]$. Its quadratic variation is easy to compute: $\langle e_{t,b}(\cdot)\rangle_s=Q(0)s$ (this shows that it is in fact a Brownian motion!). Also recall the property that if N is a continuous martingale, then we have

$$\exp\left(N_s-\frac{1}{2}\langle N\rangle_s\right)=\sum_{n=0}^{\infty}\frac{\langle N\rangle_s^{n/2}}{n!}\mathbf{H}_n(\langle N\rangle_s^{-1/2}N_s)$$

(see [6], exercise 3.3.31). Applied to our martingale $e_{t,b}(s)$, refering to 9, this yields

$$\exp\left(e_{t,b}(s)-\frac{1}{2}Q(0)s\right)=\sum_{n=0}^{\infty}X_n^t(s,b).$$

Taking $s=t$ and taking the expectation with respect to $\mathbf{P}_x$ of this equation, which is justified by Proposition 3 and the integrability of the right-hand side, we obtain the Itô equation's Feynman-Kac formula; the Stratonovich equation's formula 5 follows trivially $\mathbf{P}$-almost surely for t, x fixed. The left hand side may also be expressed as an expectation with respect to $\mathbf{P}_0$, a Wiener measure started from 0, not x, of the same quantity, with $b.$ replaced by $b.+x$. The resulting process t, b, $x\to\exp e_{t,b.+x}$, if chosen in its separable form, yields a separable process after the $\mathbf{P}_0$-expectation is taken. This proves that the Feynman-Kac formula 5 represents the separable version of $u(t,x)$ $\mathbf{P}$-almost-surely for all t, x. Under Hypothesis 6, by the Gronwall inequality technique, it is easy to apply the Kolmogorov lemma ([6], Problem 2.2.9) to establish the existence of a Hölder-continuous version, which of course coincides with the separable one. ■

Let us remark that for t and x fixed, the processes $s\to b_{t-s}$ and $s\to x+b_t-b_s$ have the same distribution under $\mathbf{P}_x$, so that formula 5 is the same as

$$u(t,x)=\mathbf{E}_0\left[\exp\left(\int_{\mathbf{R}^d}\int_0^t e^{i\lambda\cdot(x+b_t-b_s)}M(d\lambda,ds)\right)\right],\quad t\geq 0,\quad x\in\mathbf{R}^d. \quad (10)$$

We will prefer to work with formula 10 in the sequel.

3. REDUCTION TO THE DISCRETE CASE

To prove the main Theorem 2, we decide to replace the Brownian path $(w_s\colon s\in[0,t])$ in the Feynman-Kac formula 10 by a path $\tilde{b}$ that stays close to w

proceeds only by jumps and lives in $\mathbf{Z}^d$. Our task begins by controlling the error made by this substitution.

The proof of the upper bound in Theorem 2 being indifferent of the value of x, we let $x = 0$ for notational simplicity.

3.1. The Approximating Path

Let ε be a positive real and let b^j be the j-th component of the d-dimensional path b; let T_1^j be the first time b^j exits the interval $(-\varepsilon, \varepsilon)$; let T_{m+1}^j be the first time after T_m^j that b^j exits $(b_{T_m^j} - \varepsilon, b_{T_m^j} + \varepsilon)$. From the homogeneous Markov property, for fixed j, the times $(T_{m+1}^j - T_m^j)_{m=0}^{\infty}$ are *independent and identically distributed* (*I.I.D.*) and the successive positions $x_m^j = b_{T_m^j}^j$, which are independent of the jump times, form a one-dimensional symmetric random walk on $\varepsilon\mathbf{Z}$ in discrete time ($x_{m+1}^j = x_m^j$ are I.I.D., taking values $+\varepsilon$ and $-\varepsilon$ with probability 1/2). Now let $(T_n)_{n=0}^{\infty}$ be the increasing sequence of all the $(T_m^j)_{j,m}$ and let $(x_n)_{n=0}^{\infty}$ be the nearest neighbour path in $\varepsilon\mathbf{Z}^d$ with $x_0 = 0$ whose j-th component takes the same step as x_m^j at time T_m^j. We let $\tilde{b}$ be *the path that jumps at time T_n to site x_n and is constant between jumps.* We must note that $(T_n)_n$ and $(x_n)_n$ are not independent sequences and do not have I.I.D. increments. At any time s, each coordinates of $\tilde{b}_s$ is within ε of the corresponding one of b_s, so that the distance separating the two paths is never more than $\varepsilon\, d^{1/2}$. We record this property for future reference:

PROPOSITION 5 *With $\tilde{b}$ defined above, we have for all $s \geq 0$, $|b_s - \tilde{b}_s| \leq \varepsilon d^{1/2}$.*

3.2. Controlling the Error: Strategy

Let $e_{t,b} = \int\int \mathbf{1}_{[0,t]}(s) e^{i(b_t - b_s)\cdot\lambda} M(d\lambda, ds)$, $\tilde{e}_{t,b} = e_{t,\tilde{b}}$, and $\tilde{u}(t, x) = \mathbf{E}_0[\exp \tilde{e}_{t,b}]$. If we decide to seek an upper bound for the quantity $\tilde{\gamma} = \limsup t^{-1} \log \tilde{u}(t, 0)$ instead of $\gamma = \limsup t^{-1} \log u(t, 0)$, we should evaluate the "error" commited by this switch. Using Schwartz's inequality and the Feynman-Kac formula 10 we may write $\mathbf{P}$-a.s.:

$$u(t, 0) \leq \mathbf{E}_0[\exp 2\tilde{e}_{t,b}]^{1/2} \mathbf{E}_0[\exp 2e_{t,b} - 2\tilde{e}_{t,b}]^{1/2}$$

so that

$$\gamma \leq \tilde{\gamma} + \limsup_{t\to\infty} \frac{1}{2t} \log \mathbf{E}_0[\exp 2e_{t,b} - 2\tilde{e}_{t,b}].$$

We refer to the second term in this formula as the *error*. The goal of this paper is in fact to show that we can chose the edge-size ε of the approximation lattice $\varepsilon\mathbf{Z}^d$ as a function of κ in such a way that in the regime of small κ, this error is negligeable in front of the contribution of $\tilde{\gamma}$. In this section we compute the error. We have abused the notation $\tilde{\gamma}$ because we are disregarding the factors of 2, which are irrelevant in the absence of information on the value of the final constant c in Theorem 2.

First we operate a kind of discretization of time, using the continuity in t of e and $\tilde{e}$ in the last step for measurability:

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\log \mathbf{E}_0[\exp e_{t,b}-\tilde{e}_{t,b}] \\ &= \limsup_{n\to\infty}\sup_{t\in[n-1,n]}\frac{1}{t}\log \mathbf{E}_0[\exp e_{t,b}-\tilde{e}_{t,b}] \\ &\leq \limsup_{n\to\infty}\frac{1}{n}\log \mathbf{E}_0\left[\exp\sup_{t\in[n-1,n]}(e_{t,b}-\tilde{e}_{t,b})\right]. \end{aligned} \tag{11}$$

By Chebyshev's inequality, and using Fubini's theorem, which again is justified by the measurability of e and $\tilde{e}$, we may write

$$\mathbf{P}\left[\mathbf{E}_0\left[\exp\sup_{t\in[n-1,n]}(e_{t,b}-\tilde{e}_{t,b})\right]>\lambda\right]\leq\frac{1}{\lambda}\mathbf{E}_0\mathbf{E}\left[\exp\sup_{t\in[n-1,n]}(e_{t,b}-\tilde{e}_{t,b})\right]. \tag{12}$$

From Proposition 6 below, choosing $\lambda=\exp c'n\varepsilon^\beta$ with $c'>c$ the left-hand side of 12 becomes summable in n, so that, by the Borel-Cantelli lemma, the upper bound on the error in inequality 11 contributes less that $c'\varepsilon^\beta$, which means that

$$\limsup_{t\to\infty}\frac{1}{t}\log \mathbf{E}_0[\exp e_{t,b}-\tilde{e}_{t,b}]<c'\varepsilon^\beta,\quad \mathbf{P}-a.s. \tag{13}$$

3.3. Controlling the Error: Main Estimate

PROPOSITION 6 *There exist positive constants c and β such that*

$$\mathrm{E}_0\mathrm{E}\left[\exp\sup_{t\in[n-1,n]}(e_{t,b}-\tilde{e}_{t,b})\right]\leq\exp cn\varepsilon^\beta. \tag{14}$$

This entire section is devoted to the proof of this proposition. We begin by observing that for a fixed path b, $e_{t,b}-\tilde{e}_{t,b}$ is a Gaussian process. We will estimate the left-hand side of equation 14 before $\mathbf{E}_0$ is taken by using the following elements of Gaussian theory.

Let $X=(X_t, t\in T)$ be a centered Gaussian process indexed by some metrizable space T. Let $\|X\|=\sup_{t\in T}X_t$. Computing $E\|X\|$ is generally an impossible task. however, sharp upper bounds on $\mathbf{E}\|X\|$ and $\mathbf{P}[\|X\| > \lambda]$ are made accessible via the notion of *canonical metric*, namely the quantity

$$\delta(s,t) = (\mathbf{E}(X_t - X_s)^2)^{1/2} \quad s,t \in T, \tag{15}$$

together with its associated *entropy* $N(\eta)$, the *smallest number of balls of diameter* η *in the metric* δ *needed to cover the index set* T. With K_{univ} a universal constant and $\sigma^2 = \sup_{t\in T}\mathbf{E}(X_t^2)$ the upper bounds are as follows ([1], Corollary 4.15 and Theorem 2.1 (Borell-type inequality) respectively):

$$\mathbf{E}\|X\| \leq K_{\text{univ}} \int_0^\infty \log^{1/2} N(\eta) d\eta \tag{16}$$

$$\mathbf{P}[\|X\| > \lambda] \leq 2\exp - \frac{1}{2\sigma^2}(\lambda - \mathbf{E}\|X\|)^2. \tag{17}$$

Now notice that for any Gaussian process X, using 17,

$$\begin{aligned}
\mathbf{E}[\exp\|X\|] &= \int_0^\infty \mathbf{P}[\exp\|X\| > v] dv \\
&= \int_0^{\exp \mathbf{E}\|X\|} \mathbf{P}[\|X\| > \log v] dv + \int_{\exp \mathbf{E}\|X\|}^\infty \mathbf{P}[\|X\| > \log v] dv \\
&\leq \exp \mathbf{E}\|X\| + 2\int_{\exp \mathbf{E}\|X\|}^\infty \exp - \frac{1}{2\sigma^2}(\log v - \mathbf{E}\|X\|)^2 dv \\
&= \exp \mathbf{E}\|X\| \left[1 + (8\pi\sigma^2)^{1/2}\exp\left(\frac{\sigma^2}{2}\right)\right].
\end{aligned} \tag{18}$$

In our case, by the covariance relation 4, and using proposition 5 and Lemma 4

$$\begin{aligned}
\sigma^2 &= \sup_{t\in[n-1,n]} \int\int_0^t |e^{i\lambda\cdot(b_t - b_s)} - e^{-i\lambda\cdot(\tilde{b}_t - \tilde{b}_s)}|^2 \hat{Q}(d\lambda) ds \\
&\leq \sup_{t\in[n-1,n]} \int_0^t ds K_\alpha (\varepsilon d^{1/2})^{2\alpha/(2+\alpha)} \\
&= n K_\alpha \varepsilon^{2\alpha/(2+\alpha)},
\end{aligned} \tag{19}$$

According to 16, controlling $e^{\mathbf{E}\|X\|}$where $\|X\| = \sup_{t\in[n,n+1]}(e_{t,b} - \tilde{e}_{t,b})$ reduces to evaluating the canonical metric of the process $t \rightarrow e_{t,b} - \tilde{e}_{t,b}$. It turns out that an estimate sufficient to complete the proof of proposition 6 is obtained by separating $\mathbf{E}\|X\|$ into $\mathbf{E}\|e., b\|$ and $\mathbf{E}\|\tilde{e}., b\|$. This fact means that the error, which is of the form $\mathbf{E}\exp\|X\|$, can be measured via equation 18by the contribution involving σ^2 only, which is typical of a Gaussian situation. The contribution involving $\mathbf{E}\|X\|$, which is of the form $\exp\mathbf{E}\|X\|$, is altogether of a lower order, consistent with the fact that the application of Jensen's inequality yielding $\exp\mathbf{E}\|X\|\le\mathbf{E}\exp\|X\|$ should be a crude estimate.

For any fixed function b on $[0, t]$, we can compute the canonical metric for $t\to e_{t,b}$ explicitly:

$$\begin{aligned}\delta_b(s,t)^2 &= \mathbf{E}\left[\int\int\left(\mathbf{1}_{[0,t]}(u)e^{i\lambda\cdot(b_t-b_u)} - \mathbf{1}_{[0,s]}(u)e^{i\lambda\cdot(b_s-b_u)}\right)M(d\lambda,du)\right]^2\\ &= (t+s)Q(0) - 2\int\int_{[0,s\wedge t]} e^{i\lambda\cdot(b_t-b_u-(b_s-b_u))}\hat{Q}(d\lambda)du \qquad (20)\\ &= (t+s)Q(0) - 2.s\wedge tQ(b_t-b_s)\\ &= |t-s|Q(0) + 2.s\wedge t(Q(0)-Q(b_t-b_s)).\end{aligned}$$

Different tools must then be used depending on whether the path b is Brownian motion or its ε-approximation $\tilde{b}$.

3.3.1. Control of $E_0[\exp E\|\tilde{e}_{\cdot,b}\|]$

By lemma 4, $Q(0)-Q(x) = \int(1-e^{ia\cdot x})\hat{Q}(d\lambda) \le K_\alpha^1|x|^{\alpha/(2+\alpha)}$. Define the 1/4 -Hölder constant of uniform modulus of continuity for b on the interval $[n-1, n]$ as

$$C_{b,n} = \sup_{u',u\in[n-1,n]}\frac{|b_u-b_{u'}|}{|u'-u|^{1/4}}.$$

Since moreover $|t-s| < |t-s|^{\alpha/4(2+\alpha)}$,

$$\delta_b(s,t)^2 \le Q(0)|t-s|^{\alpha/4(2+\alpha)}(1+2nK'_\alpha C_{b,n}^{\alpha/(2+\alpha)}). \qquad (21)$$

The entropy of the metric on the right-hand side is bigger than that of δ's. Thus with the notation $c = Q(0)^{1/2}(1+2nK'_\alpha C_{b,n}^{\alpha(2+\alpha)})^{1/2}$ and $\gamma = 8(2+\alpha)/\alpha$, the quantity $1+(c/\eta)^\gamma$ serves as an upper bound for $N_{\delta_b}(\eta)$.

Now in property 16, we may replace the infinite upper bound of integration by $\eta_{\max}$, the diameter of $[n-1, n]$ in the metric δ_b, because for

$\eta > \eta_{\max}$, $N(\eta) = 1$. Clearly $\eta_{\max} < (2n+1)^{1/2}\, Q\,(0)^{1/2} = c'$ since δ_b cannot exceed $(2n+1)\, Q\,(0)$. We obtain:

$$\begin{aligned}
\mathbf{E}\,\|e, b\| &\leq K_{\text{univ}} \int_0^{c'} \log^{1/2}\left(1 + \frac{c^\gamma}{\eta^\gamma}\right) d\eta \\
&\leq K_{\text{univ}} (\gamma/2)^{1/2} \int_0^{c \wedge c'} \log^{1/2}(2c^2/\eta^2) d\eta + K_{\text{univ}} \int_{c \wedge c'}^{c'} \log^{1/2} 2 d\eta \\
&\leq K_{\text{univ}} (\gamma/2)^{1/2} \int_0^1 \log^{1/2}(1/\eta'^2) d\eta' 2^{1/2} c + c' K'_{\text{univ}} \\
&= c K_{\text{univ}} (\gamma/2)^{1/2} (\pi/2)^{1/2} + c' K'_{\text{univ}}.
\end{aligned}$$

We used the fact that $\int_0^1 \log^{1/2}(1/\eta'^2) d\eta' = \int_0^\infty \exp(-x^2/2) dx = (\pi/2)^{1/2}$. Hence there is some constant K''_α depending only on Q (and α) such that

$$\mathbf{E}_0[\exp \mathbf{E}\|e_{\cdot,b}\|] \leq \mathbf{E}_0[\exp(2n)^{1/2} K''_\alpha \{(C_{b,n})^{\alpha/2(2+\alpha)} + 1\}].$$

Now for any $\gamma' > 0$ and $a > 0$, simple calculus yields that there exists a constant $K_{\gamma',a}$ such that for every x, $K \geq 0$,

$$\exp(Kx^{\gamma'}) \leq \exp(K^{2/(2-\gamma')} K_{\gamma',a}) \exp(ax^2).$$

Using $\gamma' = \alpha/2\,(2+\alpha) < 1/2$, we obtain $2/(2-\gamma') < 4/3$. Thus

$$\mathbf{E}_0[\exp \mathbf{E}\,\|e_{\cdot,b}\|] \leq C_a \exp(n^{2/3} K'''_\alpha) \tag{22}$$

where a and C_a are defined by Fernique's theorem (see for example lemma 1.3.25 in [5]) applied to the modulus of continuity of Brownian motion:

$$\exists a > 0 : C_a = \mathbf{E}_0 \exp a \sup_{s,t \in [0,1]} \left(\frac{|b_t - b_s|}{|t-s|^{1/4}} \right)^2 < \infty. \qquad \blacksquare$$

3.3.2. Control of $E_0[\exp E\|\tilde{e}_{\cdot,b}\|]$

Whenever two times s and t are between two consecutive jumps of the approximating path $\tilde{b}$, $Q(\tilde{b}_t - \tilde{b}_s) = Q(0)$ and the explicit formula 20 for $\delta_{\tilde{b}}$ becomes

$$\delta_{\tilde{b}}(s,t)^2 = Q(0)|t-s| + 0$$

so that calling $M-1$ the number of jumps of $\tilde{b}$ in the interval $[n-1, n]$, and covering each of the M intervals determined by the jumps separately, we obtain

$$N(\eta) \leq M(1 \vee (Q(0)/\eta^2)). \tag{23}$$

We apply this entropy estimate to property 16 with the help of the following elementary

LEMMA 7 *For any $A \in [0, 1]$,*

$$\int_0^A \log^{1/2}(1/\eta^2) d\eta \leq A((\pi/2)^{1/2} + \log^{1/2}(1/A^2)),$$

and the fact that on $[n-1, n]$, $\delta_{\tilde{b}}$ does not exceed $c' = (Q(0)\ (2n+1))^{1/2}$, to obtain:

$$\begin{aligned}
&\mathbf{E}\|\tilde{e}_{\cdot,b}\| \\
&\leq K_{\text{univ}}\left(\int_0^{Q(0)^{1/2}} \log^{1/2}(MQ(0)/\eta^2)d\eta + \int_{Q(0)^{1/2}}^{c'} \log^{1/2} M d\eta\right) \\
&= K_{\text{univ}}\left((MQ(0))^{1/2} \int_0^{M^{-1/2}} \log^{1/2}(1/\eta'^2)d\eta' + \log^{1/2} M(c' - Q(0)^{1/2})\right) \\
&\leq K_{\text{univ}}\left((MQ(0))^{1/2} M^{-1/2}((\pi/2)^{1/2} + \log^{1/2} M) + (2nQ(0)\log M)^{1/2}\right) \\
&\leq K_Q(2n \log M)^{1/2}
\end{aligned} \tag{24}$$

where K_Q is a constant depending only on Q. Now notice that $M-1 = N$, the number of jumps between times n and $n-1$, has the same distribution as N_1, the number of jumps between times 0 and 1 except we must replace the first jump by an independent "remaining life" jump; at any rate, if f is an increasing function, $\mathbf{E}f(N) \leq \mathbf{E}f(N_1+1)$. By definition $N_1 = N_1^1 + \cdots + N_1^d$ where N_1^j refers to the j-th component of $\tilde{b}$. Thanks to inequality 24, with the notation $c = K_Q(2n)^{1/2}$, and by the independence of the N^j's,

$$\begin{aligned}
\mathbf{E}_0 \exp \mathbf{E}\|\tilde{e}_{\cdot,b}\| &= \mathbf{E}_0 \exp c \log^{1/2}(2 + N_1^1 + \cdots + N_1^d) \\
&\leq e^{c \log^{1/2} 2}(\mathbf{E}_0 \exp c \log^{1/2} N_1^1)^d
\end{aligned}$$

The following lemma allows to finish the proof.

LEMMA 8 *There is a constant $K = K_{\kappa,\varepsilon}$ depending only on κ and ε such that*

$$\mathbf{P}_0[N_1^1 = k] \leq K^k/k!$$

Indeed, with the help of the Stirling-type estimate $k! > k^k 3^{-k}$, this lemma implies

$$\begin{aligned}
&\mathbf{E}_0 \exp c \log^{1/2} N_1^1 \\
&\leq \sum_{k=0}^{\infty} \frac{K^k}{k!} \exp c \log^{1/2} k \\
&\leq \sum_{k=0}^{[c]} \frac{K^k}{k!} \exp c \log^{1/2} c + \sum_{k=[c]+1}^{\infty} (3K)^k \exp - k(\log k - \log^{1/2} k) \\
&\leq \exp K \exp(c \log^{1/2} c) + K'
\end{aligned}$$

where K', the sum of the second series for $c = 0$, is clearly finite for all values of K, thus depends only on κ and ε. Replacing c by its value gives that there is a K depending only on Q, α, κ, ε such that

$$\mathbf{E}_0 \exp \mathbf{E}\|\tilde{e}_{\cdot,b}\| \leq K(1 + \exp Kn^{1/2}(1 + \log^{1/2}(n^{1/2}))).$$

This estimate, together with 22, *implies by Schwartz's inequality that for large n,*

$$\mathbf{E}_0 \exp \mathbf{E}\|\tilde{e}_{\cdot,b} - e_{\cdot,b}\| \leq \exp(n^{3/4}). \qquad \blacksquare$$

This and the estimate 19, *in conjunction with* 18, *imply the validity of Proposition* 6, *and thus of the almost-sure exponential upper bound* 13 *on the error, with $\beta = 2\alpha/(2+\alpha)$.* $\blacksquare$

Proof of Lemma 7 By the change of variables $u^2 = \log(1/\eta^2)$,

$$\int_0^A \log^{1/2}(1/\eta^2) d\eta = \int_{\log^{1/2}(1/A^2)}^{\infty} u^2 e^{-u^2/2} du.$$

Call this quantity $F(\log^{1/2}(1/A^2))$. Let us compare $F(x)$ to $G(x) = (x + K)e^{-x^2/2}$. Since $G'(x) - F'(x) = (1 - Kx)e^{-x^2/2}$, $G - F$ first increases, then deceases with x. By picking $G(0) = F(0)$, i.e. $K = F(0) = (\pi/2)^{1/2}$, and by noting that $F(\infty) = G(\infty) = 0$, we can conclude that $G \geq F$ everywhere, which is what the lemma asserts. $\blacksquare$

Proof of Lemma 8 As noted at the start of this section, the n-th jump time T^1_n of the first component of the path $\tilde{b}$ is a sum of I.I.D. random variables $t_1, \ldots, t_n$ with distribution that of the first exit time $\tau_{\kappa,\varepsilon}$ of the standard 2κ-Brownian motion from $[-\varepsilon, \varepsilon]$. Let $F_{2\kappa,\varepsilon}(t) = \mathrm{Prob}[\tau_{\kappa,\varepsilon} < t]$ be its distribution function. By the scaling property of Brownian motion, $F_{2\kappa,\varepsilon}(t) = F_{1,1}(2\kappa, \varepsilon^{-2}t)$. We conclude the proof by using the results and notation of Proposition 9 below:

$$\begin{aligned}\mathbf{P}_0[N^1_1 = k] &= \mathbf{P}_0[t_1 + \cdots + t_k \leq 1 < t_1 + \cdots + t_{k+1}]\\ &\leq \int \cdots \int \mathbf{1}_{s_1+\cdots+s_k\leq 1}(2\kappa\varepsilon^{-2}F_{\max})^k d_{s_1} \cdots d_{s_k}(1 - F(1 - s_k))\\ &\leq (2\kappa\varepsilon^{-2}F_{\max})^k/k!\end{aligned}$$

■

PROPOSITION 9 *Let $F(t)$ be the distribution function of the first exit time of Brownian motion from the interval* $[-1, 1]$. Then $F_{\max} = \sup_{t\geq 0} F'(t) < \infty$. *Let $T_1, \ldots, T_n, \ldots$ be the successive passage times of Brownian motion on the sites of $\mathbf{Z}$, excluding the return times to the last visited sites of $\mathbf{Z}$. Then for all $t \geq 0$,*

$$\begin{aligned}&\mathbf{P}_0[T_1 \in ds_1; T_2 - T_1 \in ds_2; \ldots; T_n - T_{n-1} \in ds_n; T_{n+1} > t]\\ &\qquad = (1 - F(t - s_n))F(ds_n) \ldots F(ds_2)F(ds_1).\end{aligned}$$

Proof With T^+_1 and T^-_1 the first hitting times of 1 and -1 respectively, we can write:

$$\begin{aligned}\mathbf{P}_0[T_1 \in dt] &= \mathbf{P}_0[T^-_1 \in dt; T^+_1 > T^-_1] + \mathbf{P}_0[T^+_1 \in dt; T^-_1 > T^+_1]\\ &= 2\mathbf{P}_0[T^-_1 \in dt; T^+_1 > T^-_1]\\ &\leq 2\mathbf{P}_0[T^-_1 \in dt] = (2\pi t^3)^{-1/2} \exp\left(-\frac{1}{2t}\right)dt\end{aligned}$$

which is a bounded function on $\mathbf{R}_+$. That is the first assertion. The second statement is a general property of renewal processes (the $T_k - T_{k-1}$ are I.I.D. with distribution function F), and is proved by successively conditioning by the events $\{T_1 \in ds_1; \ldots; T_{k-1} \in ds_k\}$. ■

4. PROOF OF THE UPPER BOUND

By the validity of equation 13, to prove Theorem 2, it only remains to find an upper bound on the approximating Lyapunov exponent $\tilde{\gamma}$.

Recall the notation $\tilde{e}_{t,b} = \int\int \mathbf{1}_{[0,t]}(s)e^{i(\tilde{b}_t - \tilde{b}_s)\cdot\lambda}M(d\lambda, ds)$, as well as $\tilde{u}(t, 0) = \mathbf{E}_0[\exp \tilde{e}_{t,b}]$ and $\tilde{\gamma} = \lim\sup_{t\to\infty} t^{-1} \log \tilde{u}(t, 0)$.

4.1. Reduction to Gaussian Fields with Finite Dimensional Index Sets

We decompose the expectation giving $\tilde{u}$ according to the possible values of the jump times of $\tilde{b}$. Denote by $\mathbf{S}(t,k)$ the simplex set $\{s_1,\ldots,s_k$: $0\le s_1\le s_2\le \ldots s_k\le t\}$. For a multi-index $k=(k_1,\ldots,k_d)$, let $|k| = k_1 + \cdots + k_d$. The set of the first k_j jump times of the j-th component of $\tilde{b}$ is a point $(s_i^j)_{i=1}^{k_j}$ in $\mathbf{S}(t,k_j)$. Given the set of all jump times $\{s_i^j : j\in[1\cdots d]; i\in[1\ldots k_j]\}$, let $\{\tilde{s}_l : l\in[0,|k|+1]\}$ be that same set, but ordered, and with the convention $\tilde{s}_0=0$ and $\tilde{s}_{|k|+1}=t$. Lastly, let $\tilde{b}_l$ be the value of $\tilde{b}$ between the two jump times $\tilde{s}_l$ and $\tilde{s}_l+1$. Then $\tilde{u}$ can be written as:

$$\tilde{u}(t,0)=\sum_{k\in\mathbf{N}^d}\int_{\mathbf{S}(t,k_1)}\cdots\int_{\mathbf{S}(t,k_d)}\prod_{j=1}^{d}\mathbf{P}_0[T_1^j\in ds_1^j;\ldots;T_{k_j}^j\in ds_{k_j}^j; N_t^j=k_j]$$

$$\cdot\overline{\mathbf{E}}_0\left[\exp\sum_{l=0}^{|k|}W((\tilde{s}_l,\tilde{s}_{l+1}),\tilde{b}_t-\tilde{b}_l)\right]$$

where the expectation $\overline{\mathbf{E}}_0$ is conditional on the given jump times and we denote the increment $W(b,x)-W(a,x)$ by $W((a,b),x)$. Now further decompose $\cdot\tilde{u}$ over the possible sites visited by $\tilde{b}$. While the path $\tilde{b}$ is not independent from the jump times (unless $d=1$), it is still true that the path $\{\tilde{b}_t-\tilde{b}_l : l=0,\cdots,|k|+1\}$ may only follow, in reverse order, a so-called *nearest-neighbor* path started from 0. Call $\mathcal{P}_{|k|}$ the set the set of all such paths, which is identifiable to $\{1,\ldots,2d\}^{|k|}$. This simply yields

$$\tilde{u}(t,0)=\sum_{k\in\mathbf{N}^d}\int_{\mathbf{S}(t,k_1)}\cdots\int_{\mathbf{S}(t,k_d)}\prod_{j=1}^{d}\mathbf{P}_0[T_1^j\in ds_1^j;\ldots;T_{k_j}^j\in ds_{k_j}^j; N_t^j=k_j]$$

$$\cdot\sum_{\bar{x}\in\mathcal{P}_{|k|}}\exp\sum_{l=0}^{|k|}W((\tilde{s}_l,\tilde{s}_{l+1}),\bar{x}_l)\bar{\mathbf{P}}_0[\tilde{b}_t-\tilde{b}_l=\bar{x}_l : l=0,\cdots,|k|+1]. \tag{25}$$

we have for the j-th component of $\tilde{b}$:

$$\begin{aligned}&\mathbf{P}_0[T_1^j\in ds_1;\ldots;T_{k_j}^j\in ds_{k_j}; N_t^j=k_j]\\ =&\mathbf{P}_0[T_1^j\in ds_1;\ldots;T_{k_j}^j\in ds_k; T_{k_j+1}^j>t]\mathbf{1}_{\{0\le s_1\le s_2\le\cdots\le s_{k_j}\le t\}}\\ =&(1-F(t-s_{k_j}))F(ds_{k_j}-s_{k_j-1})\cdots F(ds_2-s_1)F(ds_1)\mathbf{1}_{\{0\le s_1\le\cdots\le s_{k_j}\le t\}}\\ \le& C^{k_j}ds_1\cdots ds_{k_j}\mathbf{1}_{\{0\le s_1\le s_2\le\cdots\le s_{k_j}\le t\}}\end{aligned}$$

where we set $C = F_{\max}\kappa\varepsilon^{-2}$ and used the results of Proposition 9 and the scaling properties of Brownian motion. Replacing the conditional probabilities in 25 by 1, and replacing the exponential term by its supremum over all possible jump times we obtain

$$\tilde{u}(t,0) \le \sum_{k\in\mathbf{N}^d} \frac{C^{|k|}t^{|k|}}{\prod_{j=1}^d k_j!} \sum_{\bar{x}\in\mathcal{P}_{|k|}} \exp \sup_{0\le s_1\le\cdots\le s_k\le t} \sum_{l=0}^{|k|} W((\tilde{s}_l,\tilde{s}_{l+1}),\tilde{x}_l).$$

This expression is increasing in t. Denoting by $\tilde{W}_{\bar{x}}((s_1;\ldots;s_k))$ the quantity $\sum_{l=0}^{k} W((s_l,s_{l+1}),\bar{x}_l)$ we can write

$$J(n) = \sup_{t\in[n-1,n]} \tilde{u}(t,0) \le \sum_{k\in\mathbf{N}^d} \frac{(Cn)^{|k|}}{\prod_{j=1}^d k_j!} \sum_{\bar{x}\in\mathcal{P}_{|k|}} \exp \sup_{S\in\mathbf{S}(n,|k|)} \tilde{W}_{\bar{x}}(S) \tag{26}$$

For fixed n, k and fixed path $\bar{x}$, $\tilde{W}_{\bar{x}}$ is simply a Gaussian process indexed by $\mathbf{S}(n,|k|)$. Indeed it is a linear combination of components of the Gaussian field W.

4.2. Using Borell's Inequality

The remainder of the Proof of Theorem 2 follows the strategy of the proof in [3]. The goal is to find a deterministic function $\lambda(n)$ such that $\mathbf{P}$-almost surely $J(n) = \sup_{t\in[n-1,n]}\tilde{u}(t,0)$ is less than $\lambda(n)$, and for which the Lyapunov exponent is tractable. By the Borel-Cantelli lemma, an excellent choice of $\lambda(n)$ is such that the series $\sum_{n=0}^{\infty}\mathbf{P}[J(n) > \lambda(n)]$ just barely converges. Since relation 26 provides an upper bound on $J(n)$ in the form of a series $\sum_{k\in\mathbf{N}^d}\sum_{\bar{x}\in\mathcal{P}_{|k|}} J(\bar{x},k,n)$, we will look for $\lambda(n)$ in the same form $\sum_{k\in\mathbf{N}^d}\sum_{\bar{x}\in\mathcal{P}_{|k|}} \lambda(\bar{x},k,n)$, and write

$$\begin{aligned}
&\mathbf{P}[J(n) > \lambda(n)]\\
&=\mathbf{P}\Bigg[\sum_{k\in\mathbf{N}^d}\sum_{\bar{x}\in\mathcal{P}_{|k|}} J(\bar{x},k,n) - \lambda(\bar{x},k,n) > 0\Bigg]\\
&\le \sum_{k\in\mathbf{N}^d}\sum_{\bar{x}\in\mathcal{P}_{|k}} \mathbf{P}[J(\bar{x},k,n) \ge \lambda(\bar{x},k,n)] \qquad (27)\\
&\le \sum_{k\in\mathbf{N}^d}\sum_{\bar{x}\in\mathcal{P}_{k|}} \mathbf{P}\Bigg[\sup_{S\in\mathbf{S}(n,|k|)} \tilde{W}_{\bar{x}}(S) > \log\Bigg[\frac{\lambda(\bar{x},k,n)}{(Cn)^{|k|}}\prod_{j=1}^d k_j!\Bigg]\Bigg].
\end{aligned}$$

We will need a slight improvement on the Borell-type inequality 17, which comes for free when one applies inequality 17 separately on each ball of an η cover of the index set (see equation (5.10) on page 119 in [1]):

LEMMA 10 *Let $\{X(t):t\in T\}$ be a Gaussian process with canonical entropy $N(\eta)$. Then for any $\eta > 0$, for any $\lambda > \mu(\eta) = c_{\text{univ}} \int_0^\eta \log^{1/2} N(\eta')d\eta'$, with $\sigma^2 = \sup_{t\in T}\mathbf{E}[X(t)^2]$,*

$$\mathbf{P}[\|X\| > \lambda] \leq 2N(\eta)\exp - (\lambda - \mu(\eta))^2/2\sigma^2.$$

We will use this inequality on each of the Gaussian processes $\tilde{W}_{\bar{x}}(S)$: $S \in \mathbf{S}(n, |k|)\}$ for each fixed k, n and $\bar{x} \in \mathcal{P}_{|k|}$. We must compute the entropy N of each process and must choose a convenient $\eta = \eta_{\bar{x},n,k}$. Let us first say that we choose $\eta^2 = 4Q(0)\ n$; note that while the Borell-type inequality 17 is just Lemma 10 with $\eta = \eta_{\max}$, in our case the δ-diameter of the index set is $\eta^2{}_{\max} = 4Q(0)\ nk$ which, for large k, is significantly larger than our choice of η; this is why the improvement of Lemma 10 is needed. For each n, k and $\bar{x}$, we have the following estimate for the canonical metric of $\tilde{W}_{\bar{x}}$:

LEMMA 11 *Let $\delta_{\bar{x}}(S, T) = (E(\tilde{W}_{\bar{x}}(S) - \tilde{W}_{\bar{x}}(T))^2)^{1/2}$. We have:*

$$\delta_{\bar{x}}(S, T)^2 \leq 4Q(0)|T - S|_1 = 4Q(0)\sum_{j=1}^{|k|}|t_j - s_j|$$

Proof Lemma 2.1 in [3].

For integer k, we may use the following covering of $\mathbf{S}(n, k)$ with balls of radius 1 in the metric $|\cdot|_1$: all the balls of radius 1 centered at a point in the set $C_{n,k}$ of points $c = (c_1, \ldots, c_k)$ of the lattice $(k^{-1}\mathbf{Z})^K$ such that $0 \leq c_1 \leq c_2 \leq \cdots \leq c_k \leq n$. Indeed, for any $s \in \mathbf{S}(n, k)$, the point c defined by $c_i = [ks]/k$ belongs to $C_{n,k}$ and since $kc_i \leq ks < kc_i + 1$, thus $|s - c| \leq \sum_{i=1}^{k} k^{-1} = 1$. Counting the set $C_{n,k}$ is like counting the number of points $(m_1, \ldots, m_k)$ with integer coordinates increasing between 0 and nk. This number is well known to be the binomial coefficent $\binom{nk+1+k}{k}$. This number is bounded above by $2.6^k n^k$ as the following explicit computation shows (for large n, the factor 2 may be removed, and the value 6 may be replaced by e, but this is irrelevant for the sequel):

$$\begin{aligned}\binom{nk+1+k}{d} &= \frac{(nk+1+k)\cdots(nk+2)}{k!}\\ &< (n+1)^k k^{k-1}(k + (n+1)^{-1})/k!\\ &< 2(n+1)^k k^k/k!\\ &< 2.3^k n^k (1 + n^{-1})^k\end{aligned}$$

where we used the Stirling-like global bound $k! > k^k 3^{-k}$. Covering the simplex $\mathbf{S}(4Q(0)\ n, |k|)$ with balls of radius less than η^2 in the metric $|\cdot|_1$, is like covering $\mathbf{S}(4Q(0)\, n/\eta^2, |k|)$ with balls of radius less than 1 if $4Q(0)\, n/\eta^2$ is an integer. If it is not, covering $\mathbf{S}\,([4Q(0)n/\eta^2] + 1, |k|)$ with balls of radius 1 is an even harder task. Thus we obtain

$$N_{x,n,|k|}(\eta) < 2.6^{|k|}\left(1 + \frac{4Q(0)n}{\eta^2}\right)^{|k|} \tag{28}$$

For each $\bar{x}, n, k$, we always have $\sigma^2 = Q(0)n$. To compute $\mu(\eta) = \mu((4Q(0)n)^{1/2})$, notice that $\eta'^2 \le 4Q(0)n$ implies $N_{\bar{x},n,k}(\eta') \le 2.6^{|k|}(\frac{8Q(0)n}{\eta'^2})^{|k|}$ and hence

$$\begin{aligned}
\mu(\eta) &\le c_{\mathrm{univ}} \int_0^{(4Q(0)n)^{1/2}} |k|^{1/2} \log^{1/2}(48Q(0)n/\eta'^2)\, d\eta' \\
&= c_{\mathrm{univ}}(48Q(0)n|k|)^{1/2} \int_0^{12^{\;1/2}} \log^{1/2}(1/\eta''^2\, d\eta'' \\
&= c'_{\mathrm{univ}}(48Q(0)n|k|)^{1/2}.
\end{aligned}$$

Since the estimate 28 is uniform in $\bar{x}$, which is not surprising since the potential W is spatially homogeneous, the optimal choice of $\lambda(\bar{x}, n, k)$ should be the same for all $\bar{x}$. Now Lemma 10 on each summand in 27 yields

$$\begin{aligned}
&\mathbf{P}[J(n) > \lambda(n)] \\
&\le \sum_{k \in \mathbf{N}^d} (2d)^{|k|} 2.2^{|k|} \exp - \frac{1}{2Q(0)n} \begin{pmatrix} \log\left[\frac{\lambda(k,n)}{(Cn)^{|k|}} \Pi_{j=1}^d k_j!\right] \\ -c'_{\mathrm{univ}}(16Q(0)n|k|^{1/2} \end{pmatrix}^2 .
\end{aligned} \tag{29}$$

4.3. Computing $\boldsymbol{\lambda(n)}$ and its Lyapunov Exponent

We choose $\lambda(k, n)$ such that each summand in the above series reduces to $n^{-2}(|k|^2 + 1)$. Thus $\mathbf{P}[J(n) > \lambda(n)] \le n^{-2} K_{\mathrm{univ}}$ which is summable series in n so that $\mathbf{P}$-almost surely, for n large enough, $J(n) \ge \lambda(n)$ and in particular,

$$\limsup_{t \to \infty} \frac{1}{t} \log \tilde{u}(t, 0) \le \limsup_{n \to \infty} \frac{1}{n} \log \lambda(n), \quad \mathbf{P}\text{-almost-surely.} \tag{30}$$

By this choice of $\lambda(k, n)$,

$$\lambda(n) = \sum_{k \in \mathbf{N}^d} \frac{(2dCn)^{|k|}}{\Pi_{j=1}^d k_j!} \exp\left\{ \begin{matrix} (2Q(0)n|k|)^{1/2} \\ \cdot\left((c'_{\mathrm{univ}} + (\log(4d) + \frac{\log(1+|k|^2)+\log n^2}{|k|})^{1/2}\right) \end{matrix} \right\}$$

To find the Lyapunov exponent for this quantity, we choose ξ and ξ' arbitrarily small numbers and decompose the sum according to whether or not $|k| \le (\log n^2)/\xi$, treat the simple case of $|k|=0$ separately (no jumps, Borell's inequality is not used; we omit the details) and require that n be so large that for $|k|=(\log n^2)/\xi$, $(\log(1+|k|^2))/|k|$ be less than ξ'. We obtain $\lambda(n)$ as the sum of the following three terms:

$$\begin{aligned}&\exp(2nQ(0))^{1/2}\\&+\Bigg(\sum_{1\le|k|\le(\log n^2)/\xi}\frac{(2dCn)^{|k|}}{\Pi_{j=1}^d k_j!}\Bigg)\exp(Q(0)n\xi^{-1}\log n^2)^{1/2}(c''_{\text{univ}}+(2\log n^2)^{1/2})\\&+\sum_{|k|>(\log n^2)/\xi}\frac{(2dCn)^{|k|}}{\Pi_{j=1}^d k_j!}\exp(Q(0)n|k|)^{1/2}(c''_{\text{univ}}+(2(\log 4d)+\xi'+\xi)^{1/2}).\end{aligned}$$

The Lyapunov exponent of a sum of functions is the greatest of the Lyapunov exponents of the individual functions. The first term has null Lyapunov exponent. For the other two, note that

$$\sum_{|k|\in\mathbf{N}^d}\frac{(2dCn)^{|k|}}{\Pi_{j=1}^d k_j!}=(\exp 2dCn)^d.$$

The second term has Lyapunov exponent $2d^2C$. The third term, when the sum is for $|k| \ge 0$, can be expressed as an expectation with respect the law $\mathcal{P}(2dC)$ of a vector of d independent Poisson processes (N_n^j) with common intensity $2dC$:

$$\begin{aligned}\lambda'(n)&=e^{2d^2Cn}\mathbf{E}_{\mathcal{P}(2dC)}[\exp a'n^{1/2}(N_n^1+\cdots+N_n^d)^{1/2}]\\&\le e^{2d^2Cn}(\mathbf{E}_{\mathcal{P}(2dC)}[\exp a'(nN_n^1)^{1/2}])^d\end{aligned}$$

where $a'=Q(0)^{1/2}(c''_{\text{univ}}+2^{1/2}(\log 4d+\xi'+\xi)^{1/2})$. Let $a=Q(0)^{1/2}(c''_{\text{univ}}+(2\log 4d)^{1/2})$. Because ξ, ξ' are arbitraily small, and by inequality 30, we thus have, $\mathbf{P}$-almost-surely,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\tilde{u}(t,0)&\le 4d^2C\vee\lim_{n\to\infty}\frac{d}{n}\log\mathbf{E}_{\mathcal{P}(2dC)}[\exp a(nN_n^1)^{1/2}]\\&\le 4d^2C\vee\frac{da^2/4}{\log(2dC)^{-1}}.\end{aligned}\tag{31}$$

For the last estimate we refer to [3] (Lemma 2.2 and calculation following) where the above limit was evaluated in the case of small intensity using the

Varadhan lemma (see [5], Theorem 2.1.10); namely, it is shown that if $\lambda < e^{-l-a}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{E}_{\mathcal{P}(\lambda)}[\exp a(nN_n^1)^{1/2}] \leq \frac{a^2/4}{\log\lambda^{-1}} - \lambda$$

4.4. Conclusion

In view of the result (31) and of the almost-sure upper bound on the error (13), we have proved that under hypothesis 6 there are positive numbers C_0 and K_0 depending only on Q (and d) such that for any x in $\mathbf{R}^d$, if $C = F_{\max}\kappa\varepsilon^{-2} \leq C_0$, then

$$\limsup_{t\to\infty} \frac{1}{t} \log u(t,x) \leq 2d^2 C \vee \frac{da^2}{4\log(2dC)^{-1}} K_0 \varepsilon^{2\alpha/(2+\alpha)} \mathbf{P}\text{-}a.s.$$

Notice that for C small enough ($C < C_1$), the first term on the right-hand side reduces to $da^2/4\log(2dC)^{-1}$. We are now free to choose ε as a function of κ (provided the condition $C = F_{\max}\kappa\varepsilon^{-2} < C_0 \wedge C_1$ holds) to minimize the upper bound. This is easily done with $\varepsilon = \kappa^q$ with q arbitrarily small, to obtain that $\varepsilon^{2\alpha/(2+\alpha)} = \kappa^{2\alpha q/(2+\alpha)}$ is negligeable next to $\log^{-1}(\varepsilon^2/\kappa) = \log^{-1}(\kappa^{-1+4\alpha q/(2+\alpha)})$. Hence we obtain the upper bound in Theorem 2: there is a universal constant c and a positive number κ_0 depending on Q such that for $\kappa < \kappa_0$ and for any x, $\mathbf{P}$-amlost-surely,

$$\limsup_{t\to\infty} \frac{1}{t} \log u(t,x) \leq \frac{Q(0)d(c + (2\log 4d)^{1/2})^2}{\log \kappa^{-1}}. \qquad \blacksquare$$

References

[1] Adler, R. (1990). An Introduction to Continuity, Extrema and Related Topics for General Gaussian Processes, *Instit. Math. Stat., Hayward*, CA, **12**.

[2] Carmona, R.A. and Molchanov, S.A. (1994). Parabolic Anderson Model and Intermittency, *Memoirs of the Amer. Math. Soc.*, **518**.

[3] Carmona, R. A., Molchanov, S. A. and Viens, F. G. (1996). Sharp Upper Bound on Exponential Behavior of a Stochastic Partial Differential Equation, *Random Operators and Stochastic Equations*, **4**(1), 43–49.

[4] Dawson, D. A. and Salehi, H. (1980). Spatially Homogeneous Random Evolutions, *J. Multivariate Analysis*, **10**, 141–180.

[5] Deuschel, J.-D. and Strook, D. (1989). *Large Deviations*, Academic Press.

[6] Karatzas, I. and Shreve, S. E. (1989). *Brownian Motion and Stochastic Calculus*, Graduate Texts in Math. Springer Verlag.

[7] Major, P. (1981). *Multiple Wiener-Itô Integrals, Lecture Notes in Math.*, **849**, Springer Verlag.
[8] Shiga, T. (1992). Ergodic theorems and exponential decay of sample paths for certain interacting diffusion systems, *Osaka J. Math.*, **29**, 789–807.
[9] Viens, F. (1996). Ph.D. Dissertation, *University of California at Irvine.*