[1]

# ORACLE CONTINUOUS TIME RANDOM WALKS

A. JURLEWICZ, P. KERN, M. M. MEERSCHAERT, AND H.-P. SCHEFFLER

Abstract. In a continuous time random walk (CTRW), a random waiting time precedes each random jump. The CTRW model is useful in physics, to model diffusing particles. Its scaling limit is a time-changed process, whose densities solve an anomalous diffusion equation. Some applications require the anticipating version, an oracle continuous time random walk (OCTRW), where the next jump after any given time is also included. This paper develops limit theory and governing equations for the OCTRW, which can be quite different from the non-anticipating case if the waiting time and the subsequent jump are dependent random variables.

## 1. Introduction

The continuous time random walk (CTRW) is a model developed in physics to represent diffusing particles. A random waiting time $J_n > 0$ precedes the $n$th random jump $Y_n$ of the particle. Typically we assume that $(Y_n, J_n)$ are iid random vectors in space-time with possible dependence between the waiting time $J_n$ and the jump $Y_n$. This coupling can be used to enforce certain physical constraints, e.g., particle velocity $Y_n/J_n$ should not exceed the speed of light [37]. The jumps can represent movements of tracer particles in underground aquifers [6], downstream movements of gravel particles along river beds [36], biological cell movements [13], motion of DNA-binding proteins along a chromosome [41], or movements of animals in search of a food source [32]. In finance, the jumps represent changes in price (or log-returns) [34]. The CTRW is a random walk subordinated to a renewal process. If the space-time vector belongs to some generalized domain of attraction, then under some technical conditions [3] the CTRW scaling limit is a stable Lévy process whose time index is replaced by the hitting time process of another (possibly dependent) stable subordinator. The hitting time or first passage time process is the scaling limit of the renewal process, and adjusts the outer process for random waiting times between jumps. The probability densities $c(x,t)$ of the scaling limit solve certain pseudo-differential equations that generalize the diffusion equation $\partial_t c = b\partial_x^2 c$. Power-law jumps $P(Y_n > x) \approx x^{-\alpha}$ for $0 < \alpha < 2$ lead to a space-fractional diffusion equation $\partial_t c = b\partial_x^\alpha c$, while power-law waiting times with $P(J_n > t) \approx t^{-\beta}$ for $0 < \beta < 1$

[1] *2010 Mathematics Subject Classification.* Primary: 60G50, 60F17, 60G18 and 60G22 ; Secondary: 60H30, 82C31.

*Date*: August 24, 2010.

*Key words and phrases.* continuous time random walk, anomalous diffusion equation, central limit theory, operator stable law.

MMM was partially supported by USA NSF grants EAR-0823965 and DMS-0803360.

lead to a time-fractional diffusion equation $\partial_t^\beta c = b\partial_x^2 c + \delta(x)t^{-\beta}/\Gamma(1-\beta)$ [25, 26]. For a coupled CTRW, the scaling limit density solves a coupled pseudo-differential equation, e.g., $(\partial_t - \partial_x^2)^\beta c(x,t) = \delta(x)t^{-\beta}/\Gamma(1-\beta)$ for power law waiting times $J_n$ and mean zero conditionally Gaussian jumps with variance $2J_n$ [3, Example 5.2].

In certain applications it is useful to consider the anticipating version of a CTRW, in which one additional jump is included. Let $T(n) = J_1 + \cdots + J_n$ be the time of the $n$th jump, and $S(n) = Y_1 + \cdots + Y_n$ the particle location after $n$ jumps. Then

$$N(t) = \max\{n \geq 0 : T(n) \leq t\}, \tag{1.1}$$

is the number of jumps by time $t > 0$ and the CTRW $S(N(t))$ is the particle location at time $t > 0$. The anticipating version is simply $S(N(t)+1)$ and we will call it an oracle continuous time random walk (OCTRW). In finance, it represents the price at the next available trading time [17]. In geophysics, it could represent the accumulated energy released during the next earthquake, or volcanic eruption. In hydrology, it can model the magnitude of the next flood event. In all these cases, it is natural for the waiting time to affect the magnitude of the next jump, leading to a coupled model. For finite mean waiting times $J_n$, the OCTRW has the same asymptotics as the usual CTRW. But if $P(J_n > t) \approx t^{-\beta}$ for $0 < \beta < 1$ (regularly varying tail) then the asymptotics, and the governing equation, are usually quite different. In this paper, we develop limit theory and governing equations for the anticipating case of an OCTRW with infinite mean waiting times. We emphasize the general setting in which $(Y_n, J_n)$ are iid but there is dependence between the waiting time $J_n$ and the subsequent jump $Y_n$. In finance, coupling between log returns and waiting times is rather common [27]. Coupling can also result from a clustering of trades: The price changes by an amount $Y_1 + \cdots + Y_M$ at time $J_1 + \cdots + J_M$ where $M > 0$ is a random cluster size [17].

## 2. Preliminaries

Let $(Y_n, J_n)$ be iid with $(Y, J)$ on $\mathbb{R} \times \mathbb{R}_+$ and set

$$T(n) = \sum_{j=1}^{n} J_j \quad \text{and} \quad S(n) = \sum_{i=1}^{n} Y_i \tag{2.1}$$

so that $(S(n), T(n))$ is a random walk on $\mathbb{R} \times \mathbb{R}_+$. For $t \geq 0$ we define the continuous time random walk (CTRW)

$$X(t) = S(N(t)) = Y_1 + \cdots + Y_{N(t)} \tag{2.2}$$

where $N(t)$ is given by (1.1). The OCTRW

$$Z(t) = S(N(t)+1) = Y_1 + \cdots + Y_{N(t)} + Y_{N(t)+1} \tag{2.3}$$

involves one additional jump. In the context of finance, $Y_n$ represents a price jump (or log return) after a waiting time $J_n$, the CTRW is the price at time $t > 0$, and the OCTRW is the price at the next available trading time in the future.

Assume $(Y, J)$ belongs to the strict generalized domain of attraction of some operator stable law [24] with exponent $E = \mathrm{diag}(1/\alpha, 1/\beta)$, so that for some $b_n > 0$ and $B_n > 0$ we have

$$(B_n S(n), b_n T(n)) \Rightarrow (A, D) \tag{2.4}$$

where $D > 0$ almost surely. Here $\Rightarrow$ denotes convergence in distribution. The distribution $\mu$ of $(A, D)$ is strictly operator stable with index $E$, meaning that $\mu^t = t^E \mu$ for all $t > 0$, where $\mu^t$ is the convolution power of the infinitely divisible law $\mu$, $t^E = \exp(E \log t)$ using the usual matrix exponential, and $(t^E\mu)(dx) = \mu(t^{-E}dx)$ is the probability distribution of $t^E(A, D) = (t^{1/\alpha}A, t^{1/\beta}D)$ for $t > 0$. Then a standard result [26, Theorem 4.1] shows that

$$\{(B(c)S(ct), b(c)T(ct))\}_{t\geq 0} \Rightarrow \{(A(t), D(t))\}_{t\geq 0} \quad \text{as } c \to \infty \tag{2.5}$$

in the Skorohod space $D([0, \infty), \mathbb{R} \times \mathbb{R}_+)$ with the $J_1$ topology, where $b(t) = b_{[t]}$, $B(t) = B_{[t]}$, and $(A(t), D(t))$ is a Lévy process with $(A(1), D(1)) = (A, D)$. In view of [24, Theorem 8.3.24] we may assume without loss of generality that $B(t)$, $b(t)$ vary regularly with index $-1/\alpha$, $-1/\beta$ respectively. Then $1/b(t)$ is regularly varying with index $1/\beta > 0$ so by [38, Property 1.5.5] there exists a regularly varying function $\tilde{b}$ with index $\beta$ such that $1/b(\tilde{b}(c)) \sim c$ as $c \to \infty$. Here $f \sim g$ means that $f(c)/g(c) \to 1$ as $c \to \infty$. Define $\tilde{B}(c) = B(\tilde{b}(c))$, a regularly varying function with index $-\beta/\alpha$.

For suitable functions $g$ on $\mathbb{R} \times \mathbb{R}_+$ we define the Fourier-Laplace transform (FLT)

$$\bar{g}(k, s) = \int_{\mathbb{R}} \int_0^\infty e^{ikx} e^{-st} g(x, t) dt\, dx \tag{2.6}$$

where $(k, s) \in \mathbb{R} \times \mathbb{R}_+$. Similarly, if $\mu$ is a bounded Borel measure on $\mathbb{R} \times \mathbb{R}_+$,

$$\bar{\mu}(k, s) = \int_{\mathbb{R}} \int_0^\infty e^{ikx} e^{-st} \mu(dx, dt)$$

is the FLT of $\mu$. If $\rho$ is a probability measure on $\mathbb{R}$, the Fourier transform (FT)

$$\hat{\rho}(k) = \int_{\mathbb{R}} e^{ikx}\, \rho(dx).$$

If $\rho_t$ is a probability measure on $\mathbb{R}$ for each $t > 0$ such that $t \mapsto \hat{\rho}_t(k)$ is Borel measurable, then

$$\bar{\rho}(k, s) = \int_0^\infty \int_{\mathbb{R}} e^{-st} e^{ikx} \rho_t(dx)\, dt$$

is the FLT of $(\rho_t)_{t>0}$.

Any infinitely divisible distribution is characterized by the Lévy-Khinchin formula. This concept carries over to the FLT setting [3, Lemma 2.1] so that

$$\mathbb{E}[e^{-sD(u)+ikA(u)}] = \exp(-u\psi(k, s)) \tag{2.7}$$

for all $(k,s) \in \mathbb{R} \times \mathbb{R}_+$. We call $\psi$ the Fourier-Laplace symbol of $(A,D)$. Moreover, there exist uniquely determined $(a,b) \in \mathbb{R} \times \mathbb{R}_+$, a positive constant $\sigma^2$ and a measure $\phi$ on $\mathbb{R} \times \mathbb{R}_+ \setminus \{(0,0)\}$ such that

$$\psi(k,s) = iak + bs + \frac{1}{2}\sigma^2 k^2 + \int_{\mathbb{R}\times\mathbb{R}_+\setminus\{(0,0)\}} \Big(1 - e^{ikx}e^{-st} + \frac{ikx}{1+x^2}\Big)\phi(dx,dt). \tag{2.8}$$

The Lévy measure $\phi$ is finite outside every neighborhood of the origin and

$$\int_{0<x^2+t\le 1} (x^2+t)\phi(dx,dt) < \infty.$$

We denote by $\phi_A(dx) = \phi(dx, \mathbb{R}_+)$ the Lévy measure of the Lévy process $\{A(u)\}_{u\ge 0}$. By setting $s=0$ in the representation (2.7) we see that

$$\int_{\mathbb{R}} e^{ikx}\, P_{A(u)}(dx) = e^{-u\psi_A(k)} \tag{2.9}$$

so that

$$\psi_A(k) = iak + \frac{1}{2}\sigma^2 k^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{-ikx} + \frac{ikx}{1+x^2}\right) \phi_A(dx) \tag{2.10}$$

is the Fourier symbol of the Lévy process $\{A(u)\}$. Similarly, we let $\phi_D(dt) = \phi(\mathbb{R}, dt)$ denote the Lévy measure of $\{D(u)\}$. By setting $k=0$ in the representation (2.7) we see that

$$\int_0^\infty e^{-st}\, P_{D(u)}(dt) = e^{-u\psi_D(s)} \tag{2.11}$$

where

$$\psi_D(s) = \int_0^\infty \left(1 - e^{-sv}\right) \phi_D(dv) \tag{2.12}$$

is the Laplace symbol of the Lévy process $\{D(u)\}$. Note that $\{D(u)\}$ is a stable subordinator with drift term $b=0$ in (2.8). Since the sample paths of $D(t)$ are càdlàg and strictly increasing with $D(0)=0$ and $D(t) \to \infty$ as $t \to \infty$, the first passage time process

$$E(t) = \inf\{x : D(x) > t\} \tag{2.13}$$

is well–defined.

Given any $\lambda > 0$ let $L^1_\lambda(\mathbb{R}\times\mathbb{R}_+)$ denote the collection of real-valued measurable functions on $\mathbb{R}\times\mathbb{R}_+$ for which the integral and hence the norm

$$\|f\|_\lambda = \int_0^\infty \int_{\mathbb{R}} e^{-\lambda t}|f(x,t)|\,dx\,dt$$

exists. With this norm, $L^1_\lambda(\mathbb{R}\times\mathbb{R}_+)$ is a Banach space that contains $L^1(\mathbb{R}\times\mathbb{R}_+)$. The symbol $\psi(k,s)$ defines a pseudo-differential operator $\psi(i\partial_x, \partial_t)$ on this space, and the negative generator of the corresponding Feller semigroup, see [28] for more details. Theorem 3.2 in [2] shows that the domain of this operator contains any

$f \in L^1_\lambda(\mathbb{R}\times\mathbb{R}_+)$ whose weak first and second order spatial derivatives as well as weak first order time derivatives are in $L^1_\lambda(\mathbb{R}\times\mathbb{R}_+)$, and that in this case we have

$$\begin{aligned}\psi(i\partial_x,\partial_t)f(x,t) &= -a\partial_x f(x,t) - \frac{1}{2}\sigma^2\partial_x^2 f(x,t)\\ &- \int_{\mathbb{R}\times\mathbb{R}_+\setminus\{(0,0)\}}\Big(H(t-u)f(x-y,t-u) - f(x,t) + \frac{y\partial_x f(x,t)}{1+y^2}\Big)\phi(dy,du)\end{aligned} \tag{2.14}$$

where $H(t) = I(t\ge 0)$ is the Heaviside step function.

## 3. Limit theorems

In this section we derive the limiting distribution of the coupled OCTRW process $Z(t) = S(N(t)+1)$ and compare it to the limit of the corresponding CTRW process $X(t) = S(N(t))$. Recall from Section 2 that $\tilde B(c) = B(\tilde b(c))$. Theorem 3.1 in [3] established process convergence for the CTRW, using some continuous mapping results from the book of Whitt [43] to show that

$$\{\tilde B(c)S(N(ct))\}_{t\ge 0} \Rightarrow \{A(E(t))\}_{t\ge 0} \tag{3.1}$$

as $c\to\infty$ in the $M_1$ topology on $D([0,\infty),\mathbb{R})$ under the technical condition

$$\mathrm{Disc}(\{A(t)\}_{t\ge 0}) \cap \mathrm{Disc}(\{D(t)\}_{t\ge 0}) = \emptyset \quad \text{almost surely.} \tag{3.2}$$

Here $\mathrm{Disc}(x) = \{t\ge 0 : x(t-)\ne x(t)\}$ is the set of discontinuity points of an element $x\in D([0,\infty),\mathbb{R})$, the space of càdlàg functions $x:[0,\infty)\to\mathbb{R}$. For example, condition (3.2) holds if $Y_n$ and $J_n$ are independent, which makes $A(t)$ and $D(t)$ independent. In general, however, the limit process $A(E(t))$ is a stable Lévy motion with index $\alpha\in(0,2]$ subordinated to the inverse or hitting time $E(t)$ of another stable Lévy motion $D(t)$ with index $\beta\in(0,1)$, which is not necessarily independent of the outer process $A(t)$. We would like to verify if (3.1) holds also for processes with simultaneous jumps, and then derive an analogous limit for OCTRW. We will use some results from the very useful book of Silvestrov [39].

**Theorem 3.1.** *Assume that $(Y_n, J_n)$ are iid $\mathbb{R}\times\mathbb{R}_+$-valued random vectors and that (2.4) holds. Then*

$$\{\tilde B(c)S(N(ct)+1)\}_{t\ge 0} \Rightarrow \{A(E(t))\}_{t\ge 0} \tag{3.3}$$

*as $c\to\infty$ in the $J_1$ topology on $D([0,\infty),\mathbb{R})$; Also*

$$\{\tilde B(c)S(N(ct))\}_{t\ge 0} \Rightarrow \{A(E(t)-)\}_{t\ge 0} \tag{3.4}$$

*as $c\to\infty$ in the $J_1$ topology on $D([0,\infty),\mathbb{R})$.*

*Proof.* Note that the CTRW scaling limit in (3.4) has to be interpreted as the right-continuous version of $\{A(E(t)-)\}_{t\ge 0}$ so that its sample paths are proper elements of $D([0,\infty),\mathbb{R})$. Theorem 4.5.6 in [39] shows that a suitably normalized random walk $\xi_\varepsilon(t)$ subordinated to a suitably normalized renewal process $\nu_\epsilon(t)$ converges to a limit $\xi(\nu(t))$ under some technical conditions, which we now check. First of all, the triangular array $(\kappa_{\varepsilon,k}, \xi_{\varepsilon,k})$ with $\varepsilon = c^{-1}$, $n_\varepsilon = \tilde{b}(c)$, $\kappa_{\varepsilon,k} = c^{-1}J_k$ and $\xi_{\varepsilon,k} = \tilde{B}(c)Y_k$ has iid rows for any $\varepsilon > 0$. Hence condition $\mathcal{T}_4$ on [39, p. 287] holds. Next define

$$\xi_\varepsilon(t) = \sum_{k=1}^{[n_\varepsilon t]} \xi_{\varepsilon,k} = \tilde{B}(c)S(\tilde{b}(c)t)$$

and

$$\kappa_\varepsilon(t) = \sum_{k=1}^{[n_\varepsilon t]} \kappa_{\varepsilon,k} = c^{-1}T(\tilde{b}(c)t).$$

Since (2.5) holds, condition $\mathcal{A}_{66}$ on [39, p. 288] also holds with $(\kappa_0(t), \xi_0(t)) = (D(t), A(t))$. Finally, note that condition $\mathcal{J}_{20}$ on [39, p. 285] holds with $\pi_1(0+) = \phi_D(0,\infty) = \infty$, by the standard convergence criteria for triangular arrays (e.g., see [24, Theorem 3.2.2]). Now define the renewal process

$$\begin{aligned}\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \le t) &= \tilde{b}(c)^{-1} \min\{n \ge 0 : \sum_{k=1}^n \kappa_{\varepsilon,k} > t\} \\ &= \tilde{b}(c)^{-1} \min\{n \ge 0 : \sum_{k=1}^n J_k > ct\} = \tilde{b}(c)^{-1}\big(N(ct)+1\big)\end{aligned}$$

and the corresponding limit process

$$\begin{aligned}\nu_0(t) &= \sup\{s \ge 0 : \kappa_0(s) \le t\} \\ &= \sup\{s \ge 0 : D(s) \le t\} = \inf\{s \ge 0 : D(s) > t\} = E(t).\end{aligned}$$

The random walk process subordinated to the renewal process is

$$\begin{aligned}\zeta_\varepsilon(t) &= \xi_\varepsilon(\nu_\varepsilon(t)) = \tilde{B}(c)S(\tilde{b}(c)\nu_\varepsilon(t)) \\ &= \tilde{B}(c)S\big(\tilde{b}(c)\big(\tilde{b}(c)^{-1}(N(ct)+1)\big)\big) = \tilde{B}(c)S(N(ct)+1)\end{aligned}$$

which is the left-hand side of (3.3). Then [39, Theorem 4.5.6] yields

$$\tilde{B}(c)(S(N(ct)+1) = \zeta_\varepsilon(t) \to \zeta_0(t) = \xi_0(\nu_0(t)) = A(E(t)) \tag{3.5}$$

as $c \to \infty$ in the $J_1$ topology on $D([0,\infty),\mathbb{R})$.

Next we consider the CTRW limit (3.4). Following [39, page 282], we consider the so-called modified renewal process

$$\nu'_\varepsilon(t) = \tilde{b}(c)^{-1} \max\{n \ge 0 : \sum_{k=1}^n \kappa_{\varepsilon,k} \le t\} = \tilde{b}(c)^{-1} N(ct)$$

and

$$\zeta'_\varepsilon(t) = \xi_\varepsilon(\nu'_\varepsilon(t)) = \tilde{B}(c)S(N(ct))$$

which is the left-hand side of (3.4). Since [39, Theorem 4.5.6] is an application of [39, Theorem 4.5.1], the remarks on [39, page 282] show that, under the same conditions we have already checked, we also get process convergence

$$\tilde{B}(c)S(N(ct)) = \zeta'_\varepsilon(t) \to \zeta'_0(t) = \xi_0(\nu_0(t)-) = A(E(t)-)$$

as $c \to \infty$ in the $J_1$ topology on $D([0,\infty),\mathbb{R})$. □

*Remark* 3.2. Under the condition (3.2), the CTRW scaling limit $A(E(t)) = A(E(t)-)$ almost surely, so that (3.4) is consistent with (3.1). Theorem 3.1 strengthens (3.1) by relaxing the simultaneous jumps condition in Equation (3.2), and replacing the $M_1$ topology with the stronger $J_1$ topology.

*Remark* 3.3. The CTRW and OCTRW convergence results in Theorem 3.1 can also be obtained from Henry and Straka [15, Theorem 3.6], which yields

$$\{\tilde{B}(c)S(N(ct)+1), b(\tilde{b}(c))T(N(ct)+1)\}_{t\geq 0} \Rightarrow \{A(E(t)), D(E(t))\}_{t\geq 0} \tag{3.6}$$

and

$$\{\tilde{B}(c)S(N(ct)), b(\tilde{b}(c))T(N(ct))\}_{t\geq 0} \Rightarrow \{A(E(t)-), D(E(t)-)\}_{t\geq 0} \tag{3.7}$$

as $c \to \infty$ in the $J_1$ topology on $D([0,\infty),\mathbb{R})$. The proof of [15, Theorem 3.6] uses a continuous mapping approach. The convergence (3.6) was also proven by Silvestrov and Teugels [40, Theorem 3.2] by arguments similar to Theorem 3.1. Further discussion of the processes $D(E(t))$ and $D(E(t)-)$ will be included in Example 5.2.

Recall that a stochastic process $\{X(t)\}_{t\geq 0}$ is self-similar with index $H$ if for any $r > 0$ $\{X(rt)\} = \{r^H X(t)\}$ in the sense of finite-dimensional distributions, e.g., see [12].

**Corollary 3.4.** *The limit processes $A(E(t))$ and $A(E(t)-)$ in Theorem 3.1 are both self-similar with index $\beta/\alpha$.*

*Proof.* Recall that $\tilde{B}(c)$ varies regularly with index $-\beta/\alpha$, i.e., $\tilde{B}(rc)\tilde{B}(c)^{-1} \to r^{-\beta/\alpha}$ as $c \to \infty$ for every $r > 0$. From (3.3) we get

$$\{\tilde{B}(c)S(N(c\cdot rt)+1)\}_{t\geq 0} \Rightarrow \{A(E(rt))\}_{t\geq 0}$$

while a continuous mapping argument along with (3.3) yields

$$\{\tilde{B}(c)S(N(crt)+1)\} = \{\tilde{B}(c)\tilde{B}(cr)^{-1}\cdot \tilde{B}(cr)S(N(crt)+1)\} \Rightarrow \{r^{\beta/\alpha}A(E(t))\}$$

so that $\{A(E(rt))\}$ and $\{r^{\beta/\alpha}A(E(t))\}$ are identically distributed as elements of $D([0,\infty),\mathbb{R})$. A similar argument using (3.4) shows that $\{A(E(rt)-)\}$ and $\{r^{\beta/\alpha}A(E(t)-)\}$ are identically distributed as elements of $D([0,\infty),\mathbb{R})$. Then we also have equality in the sense of finite dimensional distributions. □

*Remark* 3.5. Theorem 3.4 in [3] states that $\tilde{B}(c)S(N(ct)) \Rightarrow A(E(t))$ in the sense of one-dimensional distributions, which is true only when $A(E(t)) = A(E(t)-)$ almost surely. To clarify the argument in the proof of [3, Theorem 3.4], note that (here $p_t$ is the density of $E(t)$)

$$\begin{aligned}
&\lim_{h\downarrow 0} \frac{1}{h} P\{A(s) \in M, s < E(t) \le s+h\} \\
&= \lim_{h\downarrow 0} \frac{1}{h} \int_s^{s+h} P\{A(s) \in M | E(t) = u\} p_t(u) du \\
&= \lim_{h\downarrow 0} \frac{1}{h} \int_s^{s+h} P\{A(u-(u-s)) \in M | E(t) = u\} p_t(u) du \\
&= P\{A(s-) \in M | E(t) = s\} p_t(s)
\end{aligned} \tag{3.8}$$

which leads to (3.4) in the sense of one dimensional distributions. Examples 5.2–5.6 in [3] provide governing equations for the CTRW limit process $A(E(t)-)$ in some special cases with simultaneous jumps.

*Remark* 3.6. It is not hard to extend Theorem 3.1 to the more general case of triangular array convergence. Let $(J_n^{(c)}, Y_n^{(c)})$ be iid on $\mathbb{R} \times \mathbb{R}_+$ for each $c > 0$ and set

$$T^{(c)}(n) = \sum_{j=1}^{n} J_j^{(c)} \quad \text{and} \quad S^{(c)}(n) = \sum_{i=1}^{n} Y_i^{(c)} \tag{3.9}$$

and let $N^{(c)}(t) = \max\{n \ge 0 : T^{(c)}(n) \le t\}$. Assume that

$$\{(S^{(c)}(cu), T^{(c)}(cu))\}_{u\ge 0} \Rightarrow \{(A(u), D(u)\}_{u\ge 0} \quad \text{as } c \to \infty \tag{3.10}$$

in the $J_1$ topology on $D([0,\infty), \mathbb{R} \times \mathbb{R}_+)$, where $\{(A(u), D(u)\}_{u\ge 0}$ is a Lévy process on $\mathbb{R} \times \mathbb{R}_+$ such that $\phi_D(0,\infty) = \infty$ and $b = 0$ in (2.8). Triangular array convergence is useful in applications to finance, because the limit is more flexible. For example, $A(t)$ can be a Brownian motion with drift, or a Lévy process with finite variance but power-law probability tails. Let $\varepsilon = c^{-1}$, $n_\varepsilon = c$, $\kappa_{\varepsilon,k} = J_k^{(c)}$ and $\xi_{\varepsilon,k} = Y_k^{(c)}$. Then it follows exactly as in the proof of Theorem 3.1 that

$$\begin{gathered}
\{S^{(c)}(N^{(c)}(t)+1)\}_{t\ge 0} \Rightarrow \{A(E(t))\}_{t\ge 0} \\
\{S^{(c)}(N^{(c)}(t))\}_{t\ge 0} \Rightarrow \{A(E(t)-)\}_{t\ge 0}
\end{gathered} \tag{3.11}$$

as $c \to \infty$ in the $J_1$ topology on $D([0,\infty), \mathbb{R})$. This clarifies results in [28]: Use (3.8) above to see that Theorem 3.6, Corollary 3.8, and the governing equation (4.5) in [28] pertain to the CTRW limit process $M(t) = A(E(t)-)$ in general.

## 4. Governing equations

In this section we derive the governing pseudo-differential equations of the coupled OCTRW limit process $A(E(t))$ and the corresponding CTRW limit process $A(E(t)-)$ from Theorem 3.1. Our methods are based on Laplace and Fourier transforms. We begin by stating our main theorem. Recall that the Fourier symbol $\psi_A(k)$, the Laplace symbol $\psi_D(s)$, and the Fourier-Laplace symbol $\psi(k,s)$ were defined in Section 2. For any fixed $x \in \mathbb{R}$ define the translation $T_x(y) = y + x$. Since the set $\mathbb{R} \times (t,\infty)$ is bounded away from $(0,0)$ for any $t > 0$, $\phi(dy,(t,\infty))$ is a finite measure on $\mathbb{R}$. Define the image measure

$$T_x(\phi)(B,(t,\infty)) = \phi(T_x^{-1}(B),(t,\infty)) = \phi(B - x,(t,\infty))$$

for any Borel set $B \subset \mathbb{R}$. We will also use the notation

$$\hat{P}_Y(k) = \mathbb{E}\big[e^{ikY}\big] \quad k \in \mathbb{R}$$

for the Fourier transform of the distribution of a random variable $Y$ on $\mathbb{R}$,

$$\tilde{P}_J(s) = \mathbb{E}\big[e^{-sJ}\big] \quad s \geq 0$$

for the Laplace transform of a nonnegative random variable $J$, and

$$\bar{P}_{(Y,J)}(k,s) = \mathbb{E}\big[e^{-sJ+ikY}\big] \quad (k,s) \in \mathbb{R} \times \mathbb{R}_+$$

for the FLT of a random vector $(Y,J)$ on $\mathbb{R} \times \mathbb{R}_+$.

**Theorem 4.1.** *Assume that $(Y_n, J_n)$ are iid $\mathbb{R} \times \mathbb{R}_+$-valued random vectors and that* (2.4) *holds. Then*

$$\rho_t(dy) = \int_0^\infty \int_{\mathbb{R}} \int_0^t T_x(\phi)(dy,(t-\tau,\infty)) P_{(A(u),D(u))}(dx,d\tau)\, du \tag{4.1}$$

*is the distribution of the OCTRW limit $A(E(t))$ in* (3.3)*, and its FLT is given by*

$$\int_0^\infty e^{-st} \hat{P}_{A(E(t))}(k)\, dt = \frac{1}{s} \frac{\psi(k,s) - \psi_A(k)}{\psi(k,s)}. \tag{4.2}$$

*Furthermore,*

$$\eta_t(dy) = \int_0^\infty \int_0^t \phi_D((t-\tau,\infty)) P_{(A(s),D(s))}(dy,d\tau)\, ds \tag{4.3}$$

*is the distribution of the CTRW limit $A(E(t)-)$ in* (3.4)*, and its FLT is given by*

$$\int_0^\infty e^{-st} \hat{P}_{A(E(t)-)}(k)\, dt = \frac{1}{s} \frac{\psi_D(s)}{\psi(k,s)}. \tag{4.4}$$

The proof of Theorem 4.1 requires a few lemmas. Recall that $(Y_n, J_n)$ are iid with $(Y, J)$.

**Lemma 4.2.** *(a) For the OCTRW process $Z(t) = S(N(t)+1)$ we have for $s > 0$, $k \in \mathbb{R}$*

$$\int_0^\infty e^{-st} \hat{P}_{S(N(t)+1)}(k)\, dt = \frac{1}{s} \frac{\hat{P}_Y(k) - \bar{P}_{(Y,J)}(k,s)}{1 - \bar{P}_{(Y,J)}(k,s)}. \tag{4.5}$$

*(b) For the CTRW process $X(t) = S(N(t))$ we have for $s > 0$, $k \in \mathbb{R}$*

$$\int_0^\infty e^{-st} \hat{P}_{S(N(t))}(k)\, dt = \frac{1}{s} \frac{1 - \tilde{P}_J(s)}{1 - \bar{P}_{(Y,J)}(k,s)}. \tag{4.6}$$

*Proof.* Observe first that

$$\begin{aligned} \int_0^\infty e^{-st} \int 1_{\{T(n)\le t\}} e^{ikS(n)}\, dP\, dt &= \int \Big( \int_{T(n)}^\infty e^{-st}\, dt \Big) e^{ikS(n)}\, dP \\ &= \frac{1}{s} \int e^{-sT(n)+ikS(n)}\, dP \\ &= \frac{1}{s} \big( \bar{P}_{(Y,J)}(k,s) \big)^n. \end{aligned} \tag{4.7}$$

Note that $1_{\{N(t)=n\}} = 1_{\{T(n)\le t\}} - 1_{\{T(n+1)\le t\}}$ and hence

$$\begin{aligned} \hat{P}_{S(N(t)+1)}(k) &= \int e^{ikS(N(t)+1)}\, dP \\ &= \sum_{n=0}^\infty \int 1_{\{N(t)=n\}} e^{ikS(n+1)}\, dP \\ &= \sum_{n=0}^\infty \Big[ \int 1_{\{T(n)\le t\}} e^{ikS(n+1)}\, dP - \int 1_{\{T(n+1)\le t\}} e^{ikS(n+1)}\, dP \Big]. \end{aligned}$$

Therefore we have in view of (4.7) and independence that

$$
\begin{aligned}
&\int_0^\infty e^{-st}\hat{P}_{S(N(t)+1)}(k)\,dt\\
=&\sum_{n=0}^\infty\Big[\int_0^\infty e^{-st}\int 1_{\{T(n)\le t\}}e^{ikS(n+1)}\,dP\,dt-\int_0^\infty e^{-st}\int 1_{\{T(n+1)\le t\}}e^{ikS(n+1)}\,dP\,dt\Big]\\
=&\sum_{n=0}^\infty\Big[\int_0^\infty e^{-st}\int 1_{\{T(n)\le t\}}e^{ikS(n)}e^{ikY_{n+1}}\,dP\,dt-\frac{1}{s}\big(\bar{P}_{(Y,J)}(k,s)\big)^{n+1}\Big]\\
=&\sum_{n=0}^\infty\Big[\int_0^\infty e^{-st}\int 1_{\{T(n)\le t\}}e^{ikS(n)}\,dP\hat{P}_Y(k)\,dt-\frac{1}{s}\big(\bar{P}_{(Y,J)}(k,s)\big)^{n+1}\Big]\\
=&\sum_{n=0}^\infty\Big[\frac{1}{s}\big(\bar{P}_{(Y,J)}(k,s)\big)^{n}\hat{P}_Y(k)-\frac{1}{s}\big(\bar{P}_{(Y,J)}(k,s)\big)^{n+1}\Big]\\
=&\frac{1}{s}\big(\hat{P}_Y(k)-\bar{P}_{(Y,J)}(k,s)\big)\sum_{n=0}^\infty\big(\bar{P}_{(Y,J)}(k,s)\big)^{n}=\frac{1}{s}\frac{\hat{P}_Y(k)-\bar{P}_{(Y,J)}(k,s)}{1-\bar{P}_{(Y,J)}(k,s)}
\end{aligned}
$$

which proves (4.5).

For the proof of (4.6) note first that

$$
\begin{aligned}
\int 1_{\{T(n+1)\le t\}}e^{ikS(n)}\,dP&=\int 1_{\{T(n)+J_{n+1}\le t\}}e^{ikS(n)}\,dP\\
&=\int\int_0^t 1_{\{T(n)\le t-\tau\}}e^{ikS(n)}\,dP_J(\tau)\,dP.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\int_0^\infty e^{-st}\int 1_{\{T(n+1)\le t\}}e^{ikS(n)}\,dP\,dt\\
=&\int_0^\infty e^{-st}\int\int_0^t 1_{\{T(n)\le t-\tau\}}e^{ikS(n)}\,dP_J(\tau)\,dP\,dt\\
=&\int e^{ikS(n)}\int_0^\infty e^{-st}\int_0^t 1_{\{T(n)\le t-\tau\}}\,dP_J(\tau)\,dt\,dP\\
=&\int e^{ikS(n)}\int_0^\infty\int_\tau^\infty e^{-st}1_{\{T(n)\le t-\tau\}}\,dt\,dP_J(\tau)\,dP\\
=&\int e^{ikS(n)}\int_0^\infty\int_{T(n)+\tau}^\infty e^{-st}\,dt\,dP_J(\tau)\,dP\\
=&\frac{1}{s}\int e^{-sT(n)+ikS(n)}\,dP\int_0^\infty e^{-s\tau}\,dP_J(\tau)\\
=&\frac{1}{s}\tilde{P}_J(s)\big(\bar{P}_{(Y,J)}(k,s)\big)^{n}.
\end{aligned}
$$

In view of (4.7) we obtain

$$
\begin{aligned}
&\int_0^\infty e^{-st}\hat{P}_{S(N(t))}(k)\,dt\\
=&\sum_{n=0}^\infty\Big[\int_0^\infty e^{-st}\int 1_{\{T(n)\le t\}}e^{ikS(n)}\,dP\,dt-\int_0^\infty e^{-st}\int 1_{\{T(n+1)\le t\}}e^{ikS(n)}\,dP\,dt\Big]\\
=&\frac{1}{s}\sum_{n=0}^\infty\Big[\big(\bar{P}_{(Y,J)}(k,s)\big)^n-\tilde{P}_J(s)\big(\bar{P}_{(Y,J)}(k,s)\big)^n\Big]\\
=&\frac{1}{s}\frac{1-\tilde{P}_J(s)}{1-\bar{P}_{(Y,J)}(k,s)}
\end{aligned}
$$

and the proof is complete. □

**Lemma 4.3.** *(a) For the OCTRW process $S(N(t)+1)$ we have for all $k\in\mathbb{R}$ and $s>0$*

$$\int_0^\infty e^{-st}\hat{P}_{\tilde{B}(c)S(N(ct)+1)}(k)\,dt\to\frac{1}{s}\frac{\psi(k,s)-\psi_A(k)}{\psi(k,s)}\quad \textit{as } c\to\infty. \tag{4.8}$$

*(b) For the CTRW process $S(N(t))$ we have for all $k\in\mathbb{R}$ and $s>0$*

$$\int_0^\infty e^{-st}\hat{P}_{\tilde{B}(c)S(N(ct))}(k)\,dt\to\frac{1}{s}\frac{\psi_D(s)}{\psi(k,s)}\quad \textit{as } c\to\infty\ . \tag{4.9}$$

*Proof.* Recall from Section 2 that $\tilde{B}(c)=B(\tilde{b}(c))$ is a regularly varying function with index $-\beta/\alpha$. From (2.5) we get

$$\big(\tilde{B}(c)S(\tilde{b}(c)),c^{-1}T(\tilde{b}(c))\big)\Rightarrow(A,D)\quad\text{as } c\to\infty.$$

By the continuity theorem for the FLT for probability distributions, this is equivalent to

$$\Big(\bar{P}_{(Y,J)}(\tilde{B}(c)k,c^{-1}s)\Big)^{\tilde{b}(c)}\to\bar{P}_{(A,D)}(k,s)=e^{-\psi(k,s)}\quad\text{as } c\to\infty \tag{4.10}$$

for all $k\in\mathbb{R}$ and $s\ge 0$. Take logs and apply a Taylor expansion to see that (4.10) is equivalent to

$$\tilde{b}(c)\big(1-\bar{P}_{(Y,J)}(\tilde{B}(c)k,c^{-1}s)\big)\to\psi(k,s)\quad\text{as } c\to\infty. \tag{4.11}$$

Using $\bar{P}_{(Y,J)}(0,s)=\tilde{P}_J(s)$ and $\bar{P}_{(Y,J)}(k,0)=\hat{P}_Y(k)$ as well as $\psi(k,0)=\psi_A(k)$ in (2.10) and $\psi(0,s)=\psi_D(s)$ in (2.12), we get from (4.11)

$$
\begin{aligned}
\tilde{b}(c)\big(1-\hat{P}_Y(\tilde{B}(c)k)\big)&\to\psi_A(k)\\
\tilde{b}(c)\big(1-\tilde{P}_J(c^{-1}s)\big)&\to\psi_D(s)
\end{aligned}
\tag{4.12}
$$

as $c\to\infty$.

Proof of (a): In view of Lemma 4.2 (a) we get by a simple change of variables for all $k \in \mathbb{R}$ and $s > 0$

$$\begin{aligned}\int_0^\infty e^{-st}\hat{P}_{\tilde{B}(c)S(N(ct)+1)}(k)\,dt &= c^{-1}\int_0^\infty e^{-(sc^{-1})t}\hat{P}_{S(N(t)+1)}(\tilde{B}(c)k)\,dt\\ &= \frac{1}{s}\frac{\hat{P}_Y(\tilde{B}(c)k) - \bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)}{1-\bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)}\\ &= \frac{1}{s}\frac{\tilde{b}(c)\big(\hat{P}_Y(\tilde{B}(c)k)-1\big)+\tilde{b}(c)\big(1-\bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)\big)}{\tilde{b}(c)\big(1-\bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)\big)}\\ &\to \frac{1}{s}\frac{-\psi_A(k)+\psi(k,s)}{\psi(k,s)}\end{aligned}$$

as $c \to \infty$, using (4.11) and (4.12).

Proof of (b): Similarly, we get from Lemma (4.2) (b) that for all $k \in \mathbb{R}$ and $s > 0$

$$\begin{aligned}\int_0^\infty e^{-st}\hat{P}_{\tilde{B}(c)S(N(ct))}(k)\,dt &= c^{-1}\int_0^\infty e^{-(sc^{-1})t}\hat{P}_{S(N(t))}(\tilde{B}(c)k)\,dt\\ &= \frac{1}{s}\frac{1-\tilde{P}_J(c^{-1}s)}{1-\bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)}\\ &= \frac{1}{s}\frac{\tilde{b}(c)\big(1-\tilde{P}_J(c^{-1}s)\big)}{\tilde{b}(c)\big(1-\bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)\big)}\\ &\to \frac{1}{s}\frac{\psi_D(s)}{\psi(k,s)}\end{aligned}$$

as $c \to \infty$, using (4.11) and (4.12) again. The proof is complete. □

*Remark* 4.4. In the uncoupled case where $A, D$ are independent, we have $\psi(k,s) = \psi_A(k)+\psi_D(s)$ and hence the limits in (4.8) and (4.9) are equal. Hence it follows from Lemma 4.3 that the FLT limits of $\tilde{B}(c)S(N(ct)+1)$ and $\tilde{B}(c)S(N(ct))$ are equal if and only if $A$ and $D$ are independent.

The following Lemma provides a uniqueness theorem for FLT.

**Lemma 4.5.** *Let $(\rho_t)_{t>0}$ and $(\eta_t)_{t>0}$ be two families of probability measures on $\mathbb{R}$ such that $t \mapsto \rho_t$ and $t \mapsto \eta_t$ are weakly right-continuous. If*

$$\int_0^\infty e^{-st}\hat{\rho}_t(k)\,dt = \int_0^\infty e^{-st}\hat{\eta}_t(k)\,dt$$

*for all $s > 0$ and $k \in \mathbb{R}$, then $\rho_t = \eta_t$ for all $t > 0$.*

*Proof.* For any fixed $k \in \mathbb{R}$, the uniqueness theorem for Laplace transforms implies that $\hat{\rho}_t(k) = \hat{\eta}_t(k)$ for Lebesgue-almost all $t > 0$. By the continuity theorem for the Fourier transform, both $t \mapsto \hat{\rho}_t(k)$ and $t \mapsto \hat{\eta}_t(k)$ are right-continuous. It follows that

$\hat{\rho}_t(k) = \hat{\eta}_t(k)$ for all $t > 0$. Since $k \in \mathbb{R}$ is arbitrary, the uniqueness theorem of the Fourier transform implies $\rho_t = \eta_t$ for all $t > 0$, and the proof is complete. □

**Lemma 4.6.** *For any $t > 0$, $k \in \mathbb{R}$ and $s > 0$ we have*

$$\int_0^\infty e^{-st} \int_{\mathbb{R}} e^{ikx} \phi(dx, (t, \infty))\, dt = \frac{1}{s}\big(\psi(k, s) - \psi_A(k)\big) \tag{4.13}$$

*where $\psi(k, s)$ is the log-FLT of $(A, D)$ as in* (2.7).

*Proof.* Since $\phi(dx, (t, \infty))$ is a finite measure on $\mathbb{R}$, the Fourier-transform of $\phi(dx, (t, \infty))$ is well defined for any $t > 0$. Moreover

$$\Big|\int_{\mathbb{R}} e^{ikx} \phi(dx, (t, \infty))\Big| \le \phi(\mathbb{R}, (t, \infty)) = \phi_D(t, \infty)$$

and by [28, Eq. (3.12)] we know that

$$\int_0^\infty e^{-st} \phi_D(t, \infty)\, dt = \frac{1}{s}\psi_D(s)$$

for $s > 0$. Therefore, we can apply Fubini's theorem to get

$$\begin{aligned}
&\int_0^\infty e^{-st} \int_{\mathbb{R}} e^{ikx} \phi(dx, (t, \infty))\, dt \\
=& \int_0^\infty \int_{\mathbb{R}} e^{-st} e^{ikx} \int_0^\infty 1_{(t,\infty)}(u)\, \phi(dx, du)\, dt \\
=& \int_{\mathbb{R}} \int_0^\infty e^{ikx} \Big(\int_0^\infty 1_{(t,\infty)}(u) e^{-st}\, dt\Big)\, \phi(dx, du) \\
=& \frac{1}{s} \int_{\mathbb{R}} \int_0^\infty (1 - e^{-su}) e^{ikx}\, \phi(dx, du) \\
=& \frac{1}{s} \int_{\mathbb{R}} \int_0^\infty \Big[\Big(e^{ikx} - 1 - \frac{ikx}{1 + x^2}\Big) + \Big(1 - e^{ikx} e^{-su} + \frac{ikx}{1 + x^2}\Big)\Big]\, \phi(dx, du) \\
=& \frac{1}{s}\big(-\psi_A(k) + \psi(k, s)\big)
\end{aligned}$$

and the proof is complete. □

**Lemma 4.7.** *Equation* (4.1) *defines a probability measure $\rho_t(dy)$ on $\mathbb{R}$ such that*

$$\int_0^\infty e^{-st} \hat{\rho}_t(k)\, dt = \frac{1}{s} \frac{\psi(k, s) - \psi_A(k)}{\psi(k, s)}$$

*for any $s > 0$ and $x \in \mathbb{R}$. Moreover, the mapping $t \mapsto \rho_t$ is right continuous with respect to weak convergence.*

*Proof.* Observe first that $T_x(\phi)(\mathbb{R},(t-\tau,\infty)) = \phi_D(t-\tau,\infty)$ and hence

$$\begin{aligned}\rho_t(\mathbb{R}) &= \int_0^\infty \int_{\mathbb{R}} \int_0^t T_x(\phi)(\mathbb{R},(t-\tau,\infty))\, P_{(A(u),D(u))}(dx,d\tau)\,du \\ &= \int_0^\infty \int_0^t \phi_D(t-\tau,\infty)\, P_{(A(u),D(u))}(\mathbb{R},d\tau)\,du \\ &= \int_0^\infty \int_0^t \phi_D(t-\tau,\infty)\, P_{D(u)}(d\tau)\,du = 1\end{aligned}$$

by [28, Theorem 3.1], so that $\rho_t$ is a probability measure on $\mathbb{R}$ for any $t>0$. Observe that for $k\in\mathbb{R}$ we have using Fubini that

$$\begin{aligned}\hat{\rho}_t(k) &= \int_{u=0}^\infty \int_{x\in\mathbb{R}} \int_{\tau=0}^t \int_{y\in\mathbb{R}} e^{iky} T_x(\phi)(dy,(t-\tau,\infty))\, P_{(A(u),D(u))}(dx,d\tau)\,du \\ &= \int_{u=0}^\infty \int_{x\in\mathbb{R}} \int_{\tau=0}^t \int_{y\in\mathbb{R}} e^{ikT_x(y)} \phi(dy,(t-\tau,\infty))\, P_{(A(u),D(u))}(dx,d\tau)\,du \\ &= \int_{u=0}^\infty \int_{x\in\mathbb{R}} \int_{\tau=0}^t \int_{y\in\mathbb{R}} e^{ik(x+y)} \phi(dy,(t-\tau,\infty))\, P_{(A(u),D(u))}(dx,d\tau)\,du\end{aligned} \tag{4.14}$$

Then, by Fubini's theorem we get for any $s>0$ and $k\in\mathbb{R}$, using (4.13) that

$$\begin{aligned}&\int_0^\infty e^{-st}\hat{\rho}_t(k)\,dt \\ &= \int_{t=0}^\infty e^{-st} \int_{u=0}^\infty \int_{x\in\mathbb{R}} \int_{\tau=0}^t \int_{y\in\mathbb{R}} e^{ik(x+y)} \phi(dy,(t-\tau,\infty))\, P_{(A(u),D(u))}(dx,d\tau)\,du\,dt \\ &= \int_{t=0}^\infty \int_{u=0}^\infty \int_{x\in\mathbb{R}} \int_{\tau=0}^\infty \int_{y\in\mathbb{R}} e^{-st} e^{ik(x+y)} 1_{[0,t]}(\tau) \phi(dy,(t-\tau,\infty))\, P_{(A(u),D(u))}(dx,d\tau)\,du\,dt \\ &= \int_{u=0}^\infty \int_{x\in\mathbb{R}} \int_{\tau=0}^\infty \int_{t=0}^\infty \int_{y\in\mathbb{R}} e^{-st} e^{ik(x+y)} 1_{[0,t]}(\tau) \phi(dy,(t-\tau,\infty))\,dt\, P_{(A(u),D(u))}(dx,d\tau)\,du \\ &= \int_{u=0}^\infty \int_{x\in\mathbb{R}} \int_{\tau=0}^\infty \int_{t=\tau}^\infty \int_{y\in\mathbb{R}} e^{-st} e^{ik(x+y)} \phi(dy,(t-\tau,\infty))\,dt\, P_{(A(u),D(u))}(dx,d\tau)\,du \\ &= \int_{u=0}^\infty \int_{x\in\mathbb{R}} \int_{\tau=0}^\infty \Big( \int_{v=0}^\infty \int_{y\in\mathbb{R}} e^{-s(v+\tau)} e^{ik(x+y)} \phi(dy,(v,\infty))\,dv \Big)\, P_{(A(u),D(u))}(dx,d\tau)\,du \\ &= \int_{u=0}^\infty \int_{x\in\mathbb{R}} \int_{\tau=0}^\infty e^{-s\tau} e^{ikx} \Big( \int_{v=0}^\infty e^{-sv} \int_{y\in\mathbb{R}} e^{iky} \phi(dy,(v,\infty))\,dv \Big) P_{(A(u),D(u))}(dx,d\tau)\,du \\ &= \frac{1}{s}\big(\psi(k,s)-\psi_A(k)\big) \int_{u=0}^\infty \Big( \int_{x\in\mathbb{R}} \int_{\tau=0}^\infty e^{-s\tau} e^{ikx}\, P_{(A(u),D(u))}(dx,d\tau) \Big)\,du \\ &= \frac{1}{s}\big(\psi(k,s)-\psi_A(k)\big) \int_0^\infty e^{-u\psi(k,s)}\,du = \frac{1}{s}\frac{\psi(k,s)-\psi_A(k)}{\psi(k,s)}.\end{aligned}$$

Note that the last equality is justified since $\operatorname{Re}\psi(k,s) \geq \psi_D(s) > 0$, as in [28, p. 1619].

In order to show that $t \mapsto \rho_t$ is weakly right-continuous, in view of the continuity theorem for the Fourier transform, it is enough to show that for any fixed $k \in \mathbb{R}$ the function $t \mapsto \hat{\rho}_t(k)$ is right-continuous. Using (4.14) we get for any $t > 0$ and $h > 0$ that

$$\begin{aligned}
&\hat{\rho}_t(k) - \hat{\rho}_{t+h}(k) \\
=&\int_{u=0}^{\infty}\int_{x\in\mathbb{R}}\int_{\tau=0}^{t}\int_{y\in\mathbb{R}} e^{ik(x+y)}\phi(dy,(t-\tau,\infty))\,P_{(A(u),D(u))}(dx,d\tau)\,du \\
&-\int_{u=0}^{\infty}\int_{x\in\mathbb{R}}\int_{\tau=0}^{t+h}\int_{y\in\mathbb{R}} e^{ik(x+y)}\phi(dy,(t+h-\tau,\infty))\,P_{(A(u),D(u))}(dx,d\tau)\,du \\
=&\int_{u=0}^{\infty}\int_{x\in\mathbb{R}}\int_{\tau=0}^{t}\Big(\int_{y\in\mathbb{R}} e^{ik(x+y)}\phi(dy,(t-\tau,\infty)) \\
&\qquad\qquad -\int_{y\in\mathbb{R}} e^{ik(x+y)}\phi(dy,(t+h-\tau,\infty))\Big)P_{(A(u),D(u))}(dx,d\tau)\,du \\
&-\int_{u=0}^{\infty}\int_{x\in\mathbb{R}}\int_{\tau=t}^{t+h}\int_{y\in\mathbb{R}} e^{ik(x+y)}\phi(dy,(t+h-\tau,\infty))\,P_{(A(u),D(u))}(dx,d\tau)\,du \\
=&I_h - J_h.
\end{aligned}$$

Then we get

$$\begin{aligned}
|I_h| &\leq \int_{u=0}^{\infty}\int_{\tau=0}^{t}\big[\phi(\mathbb{R},(t-\tau,\infty)) - \phi(\mathbb{R},(t+h-\tau,\infty))\big]\,P_{(A(u),D(u))}(\mathbb{R},d\tau)\,du \\
&= \int_0^{\infty}\int_0^{t}\big[\phi_D(t-\tau,\infty) - \phi_D(t+h-\tau,\infty)\big]\,P_{D(u)}(d\tau)\,du \\
&\to 0
\end{aligned}$$

as $h \downarrow 0$ by a dominated convergence argument along with [28, Eq. (3.1)], as in [28, p. 1625]. Moreover

$$\begin{aligned}
|J_h| &\leq \int_0^{\infty}\int_t^{t+h}\phi(\mathbb{R},(t+h-\tau,\infty))\,P_{(A(u),D(u))}(\mathbb{R},d\tau)\,du \\
&= \int_0^{\infty}\int_t^{t+h}\phi_D(t+h-\tau,\infty)\,P_{D(u)}(d\tau)\,du \\
&\to 0
\end{aligned}$$

as $h \downarrow 0$ using some results in Kesten [18], as in [28, pp. 1615–1616]. This concludes the proof. $\square$

The following lemma is not required for the proof of Theorem 4.1, but is included to show that the distribution $\rho_t$ of the OCTRW limit process $A(E(t))$ is (weakly) continuous.

**Lemma 4.8.** *The mapping $t \mapsto \rho_t$ is also weakly left-continuous, thus it is weakly continuous.*

*Proof.* By the continuity theorem for Fourier transforms it is enough to show that $t \mapsto \hat{\rho}_t(k)$ is left-continuous for any $k \in \mathbb{R}$. Using (4.14) we get for any $t > 0$ and $0 < h < t$

$$\begin{aligned}
&\widehat{\rho}_t(k) - \widehat{\rho}_{t-h}(k) \\
&= \int_{u=0}^{\infty} \int_{x\in\mathbb{R}} \int_{\tau=0}^{t} \int_{y\in\mathbb{R}} e^{ik(x+y)} \phi\big(dy, (t-\tau, \infty)\big)\, P_{(A(u),D(u))}(dx, d\tau)\, du \\
&\quad - \int_{u=0}^{\infty} \int_{x\in\mathbb{R}} \int_{\tau=0}^{t-h} \int_{y\in\mathbb{R}} e^{ik(x+y)} \phi\big(dy, (t-h-\tau, \infty)\big)\, P_{(A(u),D(u))}(dx, d\tau)\, du \\
&= \int_{u=0}^{\infty} \int_{x\in\mathbb{R}} \int_{\tau=0}^{t-h} \int_{y\in\mathbb{R}} e^{ik(x+y)} \big[\phi\big(dy, (t-\tau, \infty)\big) - \phi\big(dy, (t-h-\tau, \infty)\big)\big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P_{(A(u),D(u))}(dx, d\tau)\, du \\
&\quad + \int_{u=0}^{\infty} \int_{x\in\mathbb{R}} \int_{\tau=t-h}^{t} \int_{y\in\mathbb{R}} e^{ik(x+y)} \phi\big(dy, (t-\tau, \infty)\big)\, P_{(A(u),D(u))}(dx, d\tau)\, du \\
&= I_h + J_h.
\end{aligned}$$

Then by Tonelli's theorem we get

$$\begin{aligned}
|I_h| &\le \int_{u=0}^{\infty} \int_{\tau=0}^{t-h} \big[\phi\big(\mathbb{R}, (t-h-\tau, \infty)\big) - \phi\big(\mathbb{R}, (t-\tau, \infty)\big)\big]\, P_{(A(u),D(u))}(\mathbb{R}, d\tau)\, du \\
&= \int_{u=0}^{\infty} \int_{\tau=0}^{t-h} \big[\phi_D(t-h-\tau, \infty) - \phi_D(t-\tau, \infty)\big]\, P_{D(u)}(d\tau)\, du \\
&= \int_0^{t-h} \phi_D(t-h-\tau, \infty)\, W(d\tau) - \int_0^{t} \phi_D(t-\tau, \infty)\, W(d\tau) \\
&\quad + \int_{t-h}^{t} \phi_D(t-\tau, \infty)\, W(d\tau),
\end{aligned}$$

where $W(d\tau) = \int_0^\infty P_{D(u)}(d\tau)\, du$ is the occupation measure. In view of [18, Corollary 6.2] we have $\int_0^u \phi_D(u-\tau, \infty)\, W(d\tau) = 1$ for any $u > 0$ and hence we get as $h \downarrow 0$

$$|I_h| \le \int_{t-h}^{t} \phi_D(t-\tau, \infty)\, W(d\tau) \to 0.$$

Similarly, we get as $h \downarrow 0$

$$|J_h| \le \int_{u=0}^{\infty}\int_{\tau=t-h}^{t} \phi\big(\mathbb{R},(t-\tau,\infty)\big)\, P_{(A(u),D(u))}(\mathbb{R},d\tau)\,du$$
$$= \int_{u=0}^{\infty}\int_{\tau=t-h}^{t} \phi_D(t-\tau,\infty)\, P_{D(u)}(d\tau)\,du = \int_{t-h}^{t} \phi_D(t-\tau,\infty)\,W(d\tau) \to 0$$

which concludes the proof. □

*Proof of Theorem 4.1.* Lemma 4.7 shows that $\rho_t(dy)$ is right-continuous with FLT

$$\frac{1}{s}\frac{\psi(k,s)-\psi_A(k)}{\psi(k,s)}. \tag{4.15}$$

Theorem 3.1 shows that $\tilde{B}(c)S(N(ct)+1)$ converges in $J_1$ to $A(E(t))$, and Lemma 4.3 shows that the FLT of $\tilde{B}(c)S(N(ct)+1)$ converges to the same limit (4.15). Note that $J_1$ convergence implies convergence in distribution on the set of all points of stochastic continuity of the limit process, e.g. see [39, p. 44]. Moreover, all but countably many points of a càdlàg process are points of stochastic continuity, e.g. see [39, Lemma 1.6.2]. Then

$$P_{\tilde{B}(c)S(N(ct)+1)}(dx) \Rightarrow P_{A(E(t))}(dx)$$

as $c \to \infty$ for all but countably many $t > 0$. Then the continuity theorem for the Fourier transform yields

$$\hat{P}_{\tilde{B}(c)S(N(ct)+1)}(k) \to \hat{P}_{A(E(t))}(k)$$

as $c \to \infty$ for all $k \in \mathbb{R}$, for $dt$-almost every $t > 0$. Then we have for each $k \in \mathbb{R}$ that

$$\int_0^\infty e^{-st}\hat{P}_{\tilde{B}(c)S(N(ct)+1)}(k)\,dt \to \int_0^\infty e^{-st}\hat{P}_{A(E(t))}(k)\,dt$$

as $c \to \infty$, and this together with (4.8) shows that the FLT of $A(E(t))$ equals (4.15). Since $A(t)$ is càdlàg and $E(t)$ is continuous and nondecreasing, $A(E(t))$ is a càdlàg process. Then it is right-continuous almost surely, and hence it is also right-continuous in distribution. Then Lemma 4.5 implies that $\rho_t(dy)$ equals the distribution of $A(E(t))$, which finishes the proof of (a). Part (b) follows from [28, Theorem 3.6] and Remark 3.6. The arguments are similar. □

To conclude this section, we now identify the governing equation of the OCTRW limit, and contrast with the CTRW. Suppose that the OCTRW limit process $A(E(t))$ in (3.3) has a Lebesgue density $a(x,t)$, and recall from Section 2 that $\psi(k,s)\bar{a}(k,s)$ is the FLT of $\psi(i\partial_x,\partial_t)a(x,t)$. Then it follows from Theorem 4.1 and Lemma 4.6 that

$$\bar{a}(k,s) = \frac{1}{s}\frac{\psi(k,s)-\psi_A(k)}{\psi(k,s)}.$$

Rewrite in the form

$$\psi(k,s)\bar{a}(k,s) = \frac{\psi(k,s)-\psi_A(k)}{s}$$

and invert the FLT using Lemmma 4.6 to see that

$$\psi(i\partial_x, \partial_t)a(x,t) = \phi(dx,(t,\infty)) \tag{4.16}$$

is the governing equation of the OCTRW limit. If the CTRW limit process $A(E(t)-)$ in (3.4) has a Lebesgue density $c(x,t)$, then it follows from [28, Eq. (4.5)] and Remark 3.6 that

$$\psi(i\partial_x, \partial_t)c(x,t) = \delta(x)\phi_D(t,\infty) \tag{4.17}$$

with a different boundary condition on the right-hand side.

In order to avoid distributions in the boundary condition, one can impose a smooth initial condition as in [1]. Suppose that $X_0$ is a random variable with $C^\infty$ density $p(x)$, independent of $\{(A(t), D(t))\}$, that represents the particle position at time $t = 0$. Then $A(E(t)) + X_0$ has a density $a(x,t) = \int p(x-y)\rho_t(dy)$ with Fourier transform $\hat{a}(k,t) = \hat{\rho}_t(k)\hat{p}(k)$ and FLT

$$\bar{a}(k,s) = \frac{s^{-1}[\psi(k,s) - \psi_A(k)]\hat{p}(k)}{\psi(k,s)}. \tag{4.18}$$

Lemma 4.6 shows that the Fourier transform $\hat{q}(k,t) = \int e^{ikx}\phi(dx,(t,\infty))$ exists for all $t > 0$, and that the Laplace transform of $\hat{q}(k,t)$ is given by (4.13). It follows easily that the FLT of $\int p(x-y)\phi(dy,(t,\infty))$ is given by the numerator in (4.18). Inverting the FLT in (4.18) reveals the governing equation

$$\psi(i\partial_x, \partial_t)a(x,t) = \int p(x-y)\phi(dy,(t,\infty)). \tag{4.19}$$

Using the same smooth initial condition for the CTRW limit is equivalent to replacing $\delta(x)$ by $p(x)$ in (4.17).

## 5. Examples

In this section we provide some concrete examples of OCTRW convergence, and we compute the governing equation of the limit process.

**Example 5.1.** If $Y_n$ and $J_n$ are independent, then so are the limit processes $A(t)$ and $D(t)$. The FL-symbol $\psi(k,s) = \psi_A(k) + \psi_D(s)$ and $\phi(dx,(t,\infty)) = \varepsilon_0(dx)\phi_D(t,\infty)$ where $\varepsilon_0$ is the point mass at zero. Suppose that the stable Lévy motion $A(t)$ is totally positively skewed with Fourier symbol $\psi_A(k) = b(-ik)^\alpha$ for some $0 < \alpha \leq 2, \alpha \neq 1$. Suppose that $J_n$ belongs to the domain of attraction of a standard $\beta$-stable subordinator $D$ with Laplace symbol

$$\psi_D(s) = s^\beta = \int_0^\infty \left(1 - e^{-su}\right)\phi_D(du). \tag{5.1}$$

A calculation similar to [24, Lemma 7.3.7] shows that

$$\phi_D(t,\infty) = \frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{5.2}$$

Since $\delta(x) = \varepsilon_0(dx)$, the OCTRW limit governing equation (4.16) reduces to

$$\partial_t^\beta a_1(x,t) = -b\partial_x^\alpha a_1(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{5.3}$$

where $b > 0$ if $0 < \alpha < 1$ and $b < 0$ for $1 < \alpha \le 2$. In this case, the CTRW limit equation (4.17) reduces to the same form, so that the OCTRW limit and the CTRW limit have the same governing equation. Since $\psi(k,s) = \psi_A(k) + \psi_D(s)$, it follows from Theorem 4.1 that $A(E(t))$ and $A(E(t)-)$ have the same FLT in this case. The proof of [28, Theorem 3.6] shows that the limiting CTRW distribution $\eta_t(dx)$ in (4.3) is also right-continuous. Then it follows from Lemma 4.5 that $\eta_t(dx) = \rho_t(dx)$. Theorem 4.1 shows that the FLT of $A(E(t))$ and $A(E(t)-)$ are equal if and only if $A$ and $D$ are independent. Hence this is the only case in which the OCTRW and CTRW have the same limit, which shows that (3.2) is equivalent to assuming independence of $A$ and $D$.

Corollary 3.4 shows that the uncoupled limit process is self-similar with index $\beta/\alpha$. In this case, self-similarity also follows directly by a simple conditioning argument, since the stable Lévy motion $A(t)$ is self-similar with index $1/\alpha$, and $E(t)$ is self-similar with index $\beta$ by [26, Proposition 3.1]. Equation (5.3) is called the space-time fractional diffusion equation. It has been used frequently in physics, finance, and hydrology to model anomalous diffusion [4, 5, 30, 31, 34, 35]. The underlying CTRW model explains the meaning of the fractional derivatives. A fractional derivative in space with index $\alpha < 2$ models long particle jumps, while a fractional derivative in time models long resting periods between movements.

Finally we note that for non-random jumps $Y_n = 1$ we get $A(t) = t$ and hence the CTRW limit is the hitting time $E(t)$. Its FLT is $s^{\beta-1}/(s^\beta - ik)$ and its densities $c_1(x,t)$ solve

$$\partial_t^\beta c_1(x,t) = -\partial_x c_1(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}.$$

In fact one can write $c_1(x,t)$ in terms of Mittag-Leffler functions [8, 9], which leads to a useful solution method for time-fractional equations [22, 29].

The remaining examples are coupled, i.e., $A, D$ in (2.4) are dependent random variables. A general construction in [3] yields all possible coupled limit distributions. Suppose that $J_n$ are iid with $D$, a standard $\beta$-stable subordinator with Lévy measure (5.2). For any probability measure $\omega$ on $\mathbb{R}$ and any $p > \beta/2$, suppose that the conditional distribution of $Y_n$ given $J_n = t$ is $t^p\omega$. Then [3, Theorem 2.2] shows that (2.4) holds, the Lévy measure of $(A, D)$ is

$$\phi(dy, dt) = t^p\omega(dy)\phi_D(dt), \tag{5.4}$$

and furthermore, every possible non-normal coupled limit has a Lévy measure of this form. In this case, $A$ is stable with index $\alpha = \beta/p$.

**Example 5.2.** Next we consider the completely coupled case $Y_n = J_n$ as in Kotulski [20]. Suppose that $J_n$ are iid with $D$, a standard $\beta$-stable subordinator. From (5.4) with $p = 1$ and $\omega = \varepsilon_1$, we see that the Lévy measure (jump intensity)

$$\phi(dy, dt) = \varepsilon_t(dy)\phi_D(dt) \tag{5.5}$$

of $(A, D)$ is concentrated on the line $y = t$. Zolotarev [44, Lemma 2.2.1] shows that $\mathbb{E}[e^{ikD}]$ has a unique analytic extension to the complex plane with a branch cut along the ray $\arg(k) = -3\pi/4$, hence $\psi_A(k) = \psi_D(-ik)$. Then an easy computation using (5.1) shows that $\psi(k, s) = (s-ik)^\beta$ where $b = 0$, $\sigma^2 = 0$, and $a = -\int t(1+t^2)^{-1}\phi_D(dt)$ in (2.8). Since $A = D$ the joint distribution of $(A(s), D(s))$ is given by

$$P_{(A(s),D(s))}(dx, du) = \varepsilon_u(dx)P_{D(s)}(du). \tag{5.6}$$

Theorem 4.1 (b) shows that the CTRW limit $A(E(t)-) = D(E(t)-)$ in (3.4) has FLT

$$\int_0^\infty e^{-st}\hat{\eta}_t(k)\, dt = \frac{1}{s}\frac{\psi_D(s)}{\psi(k, s)} = \frac{s^{\beta-1}}{(s-ik)^\beta}. \tag{5.7}$$

Following [3, Example 5.4] we can invert the FLT in (5.7) to see that the CTRW limit distribution $\eta_t(dx)$ has a Lebesgue density

$$c_2(x, t) = \frac{x^{\beta-1}(t-x)^{-\beta}}{\Gamma(\beta)\Gamma(1-\beta)}, \quad 0 < x < t \tag{5.8}$$

that solves the coupled governing equation (4.17), which can be written in this case as

$$\big(\partial_t + \partial_x\big)^\beta c(x, t) = \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}$$

with a coupled space-time fractional derivative operator on the left-hand side. Some properties of these operators are studied in [2].

Next we show that (5.8) also follows from the general formula (4.3) for the CTRW limit distribution $\eta_t(dx)$: First note that the support of $\eta_t(dx)$ is contained in $[0, \infty)$ in this case. For $z > 0$, using (5.6), we get using Fubini

$$\begin{aligned} \eta_t([0, z]) &= \int_0^\infty \int_0^t \int_0^z \phi_D(t-u, \infty)\varepsilon_u(dx)P_{D(s)}(du)\, ds \\ &= \int_0^\infty \int_0^t \mathbf{1}_{[0,z]}(u)\phi_D(t-u, \infty)P_{D(s)}(du)\, ds. \end{aligned} \tag{5.9}$$

Hence, if $z \geq t$ then

$$\eta_t([0, z]) = \int_0^\infty \int_0^t \phi_D(t-u, \infty)P_{D(s)}(du)\, ds = 1 \tag{5.10}$$

since the inner integral on the left-hand side of (5.10) is the probability distribution of the hitting time process $E(t)$, see [28, Theorem 3.1]. Let $g_\beta$ denote the density of

$D$. Then

$$P_{D(s)}(du) = s^{-1/\beta} g_\beta(s^{-1/\beta}u)du. \tag{5.11}$$

Moreover, in view of (5.2) we have

$$\phi_D(t-u,\infty) = \frac{1}{\Gamma(1-\beta)}(t-u)^{-\beta}. \tag{5.12}$$

Hence, if $z < t$ then (5.9) reduces to

$$\begin{aligned}\eta_t([0,z]) &= \frac{1}{\Gamma(1-\beta)} \int_0^\infty \int_0^z (t-u)^{-\beta} s^{-1/\beta} g_\beta(s^{-1/\beta}u)\, du\, ds \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^z (t-u)^{-\beta} \int_0^\infty s^{-1/\beta} g_\beta(s^{-1/\beta}u)\, ds\, du\end{aligned}$$

and therefore the density of $\eta_t$, supported on $(0,t)$, is given by

$$c_2(x,t) = \frac{1}{\Gamma(1-\beta)}(t-x)^{-\beta} \int_0^\infty s^{-1/\beta} g_\beta(s^{-1/\beta}x)\, ds. \tag{5.13}$$

A simple change of variable yields

$$\int_0^\infty s^{-1/\beta} g_\beta(s^{-1/\beta}x)\, ds = \beta x^{\beta-1} \int_0^\infty y^{-\beta} g_\beta(y)\, dy = K\beta x^{\beta-1} \tag{5.14}$$

with

$$K = \int_0^\infty y^{-\beta} g_\beta(y)\, dy. \tag{5.15}$$

Then (5.13) reduces to a beta density

$$c_2(x,t) = \frac{K\beta}{\Gamma(1-\beta)}(t-x)^{-\beta} x^{\beta-1} \quad ,0 < x < t \tag{5.16}$$

which implies that

$$K = \frac{1}{\beta\Gamma(\beta)}. \tag{5.17}$$

Hence (5.16) agrees with (5.8).

The OCTRW limit $A(E(t)) = D(E(t))$ in (3.3) has distribution $\rho_t(dx)$ with FLT

$$\int_0^\infty e^{-st} \hat{\rho}_t(k)\, dt = \frac{1}{s} \frac{\psi(k,s) - \psi_A(k)}{\psi(k,s)} = \frac{1}{s} \frac{(s-ik)^\beta - (-ik)^\beta}{(s-ik)^\beta} \tag{5.18}$$

which comes from substituting $\psi_A(k) = (-ik)^\beta$ and $\psi(k,s) = (s-ik)^\beta$ into (4.2). Now we will use (4.1) to compute the density of $\rho_t$ in this case. Observe first that the support of $\rho_t$ is contained in $[0,\infty)$. In view of (5.6) and Fubini we get for $z>0$

$$
\begin{aligned}
\rho_t([0,z]) &= \int_0^\infty \int_0^t \int_{\mathbb{R}} \phi([0,z-x],(t-\tau,\infty))\varepsilon_\tau(dx) P_{D(s)}(d\tau)\,ds \\
&= \int_0^\infty \int_0^t \phi([0,z-\tau],(t-\tau,\infty)) P_{D(s)}(d\tau)\,ds.
\end{aligned}
\tag{5.19}
$$

Using (5.5) we compute

$$
\begin{aligned}
\phi([0,z-\tau],(t-\tau,\infty)) &= \int_{t-\tau}^\infty \int_0^\infty \mathbf{1}_{[0,z-\tau]}(y)\varepsilon_u(dy)\,\phi_D(du) \\
&= \int_{t-\tau}^\infty \mathbf{1}_{[0,z-\tau]}(u)\phi_D(du).
\end{aligned}
\tag{5.20}
$$

Hence $\phi([0,z-\tau],(t-\tau,\infty))=0$ if $z\le t$ and therefore $\operatorname{supp}(\rho_t)\subset[t,\infty)$. Moreover, if $z>t$ we get from (5.20) using (5.2) that

$$
\phi([0,z-\tau],(t-\tau,\infty) = \int_{t-\tau}^{z-\tau} \phi_D(du) = \frac{1}{\Gamma(1-\beta)}\big[(t-\tau)^{-\beta}-(z-\tau)^{-\beta}\big].
\tag{5.21}
$$

Using (5.21) and (5.11), we get in (5.19) for $z>t$ that

$$
\rho_t([0,z]) = \frac{1}{\Gamma(1-\beta)}\int_0^\infty \int_0^t \big[(t-\tau)^{-\beta}-(z-\tau)^{-\beta}\big] s^{-1/\beta} g_\beta(s^{-1/\beta}\tau)\,d\tau\,ds
$$

so the density of the OCTRW limit distribution $\rho_t$ for $x>t$ is given by

$$
\begin{aligned}
a_2(x,t) &= \frac{\beta}{\Gamma(1-\beta)}\int_0^\infty \int_0^t (x-\tau)^{-\beta-1} s^{-1/\beta} g_\beta(s^{-1/\beta}\tau)\,d\tau\,ds \\
&= \frac{\beta}{\Gamma(1-\beta)}\int_0^t (x-\tau)^{-\beta-1} \int_0^\infty s^{-1/\beta} g_\beta(s^{-1/\beta}\tau)\,ds\,d\tau \\
&= \frac{\beta}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^t (x-\tau)^{-\beta-1}\tau^{\beta-1}\,d\tau
\end{aligned}
\tag{5.22}
$$

using (5.14) and (5.17). A change of variables $r=\tau/(x-\tau)$ yields

$$
\begin{aligned}
a_2(x,t) &= \frac{\beta}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^t \left(\frac{\tau}{x-\tau}\right)^{\beta-1}\frac{1}{(x-\tau)^2}\,d\tau \\
&= \frac{x^{-1}\beta}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^{t/(x-t)} r^{\beta-1}\,dr \\
&= \frac{x^{-1}}{\Gamma(\beta)\Gamma(1-\beta)}\left(\frac{t}{x-t}\right)^{\beta}
\end{aligned}
$$

for $x > t$. It is easy to see that $\int_t^\infty a_2(x,t)\,dx = 1$ and hence

$$a_2(x,t) = \frac{x^{-1}}{\Gamma(\beta)\Gamma(1-\beta)}\left(\frac{t}{x-t}\right)^\beta, \quad x > t \tag{5.23}$$

is the density of $\rho_t$ in this special case. This OCTRW limit density solves the governing equation (4.16), which can be written in this case as

$$\big(\partial_t + \partial_x\big)^\beta a_2(x,t) = \frac{1}{\Gamma(1-\beta)}\int_t^\infty \varepsilon_u(dx)\beta u^{-\beta-1}\,du \tag{5.24}$$

using (5.2) and (5.5).

Observe that from (5.23) we have

$$a_2(x,t) \sim \frac{t^\beta}{\Gamma(\beta)\Gamma(1-\beta)}x^{-1-\beta} \quad \text{as } x \to \infty$$

so that the distribution $\rho_t$ of $A(E(t))$ belongs to the domain of normal attraction of a $\beta$-stable random variable. Then it follows from [23, Theorem 1] that $\mathbb{E}(|A(E(t))|^\rho)$ exists for $0 < \rho < \beta$ and diverges for $\rho \geq \beta$.

Both the CTRW limit and the OCTRW limit in this example are related to the generalized arc sine distributions. Formula (5.8) is the density of $t\mathcal{B}$, and (5.23) is the density of $t/\mathcal{B}$, where $\mathcal{B}$ has a beta distribution with parameters $\beta$ and $1-\beta$. In this example, we have $B(c) = b(c)$ in (2.5) so that $\tilde{B}(c) = 1/c$ in Theorem 3.1. Then we have $c^{-1}S(N(ct)) \Rightarrow t\mathcal{B}$ and $c^{-1}S(N(ct)+1) \Rightarrow t/\mathcal{B}$ as $c \to \infty$. Specializing to $t = 1$ it follows that

$$\frac{c - S(N(c))}{c} \Rightarrow 1 - \mathcal{B} \quad \text{and} \quad \frac{S(N(c)+1) - c}{c} \Rightarrow \frac{1}{\mathcal{B}} - 1$$

which agrees with the results in Feller [14, Theorem XIV.3] once we note that $\Gamma(\beta)\Gamma(1-\beta) = \pi/\sin(\pi\beta)$. Hence our approach provides a different proof of the classical results on the generalized arc sine distribution for residual waiting time and spent waiting time. Our approach can also be used to simplify parts of the proof of Theorems 2–4 in Dynkin [11].

Since $A(t) = D(t)$ in this case, the CTRW limit $A(E(t)-)$ is the value of the subordinator $D(t)$ at the instant before the first passage time $E(t)$ at which it exceeds $t$. It has a beta density (5.16) supported on $0 < x < t$, which agrees with the result in Bertoin [7, p. 82]. On the other hand, the OCTRW limit $A(E(t))$ is the value of the subordinator $D(t)$ at the first passage time $E(t)$. The form of its density (5.23) can also be computed from [18, Lemma 6.1]. Here we have a sharp contrast $P[A(E(t)) > t] = 1$ and $P[A(E(t)-) < t] = 1$, which agrees with [7, III, Theorem 4]. The random variable $A(E(t)) - t$ in this case is sometimes called the overshoot.

**Example 5.3.** Suppose $D$ is a stable subordinator with $\mathbb{E}(e^{-sD}) = e^{-s^\beta}$, and the conditional distribution of $Y$ given $D = t$ is normal with mean zero and variance $2t$, as in Shlesinger, Klafter and Wong [37]. Then

$$\mathbb{E}(e^{ikY}) = \mathbb{E}(\mathbb{E}(e^{ikY}|D)) = \mathbb{E}(e^{-k^2 D}) = e^{-|k|^{2\beta}}$$

so that $Y$ is symmetric stable with index $\alpha = 2\beta$. If we take $(Y_n, J_n)$ iid with $(Y, D)$, then (2.4) holds, and it follows from (5.4) that the operator stable limit $(A, D)$ has Lévy measure

$$\phi(dx, dt) = t^{1/2}\omega(dx)\,\phi_D(dt) \tag{5.25}$$

where $\omega$ is a normal distribution with mean zero and variance 2. Take $a = b = \sigma^2 = 0$ in (2.8) to see that

$$\begin{aligned}\psi(k,s) &= \int_0^\infty \int_{-\infty}^\infty \left(1 - e^{ikx}e^{-st} + \frac{ikx}{1+x^2}\right) \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) dx\phi_D(dt)\\ &= \int_0^\infty \left(1 - e^{-t(s+k^2)}\right) \phi_D(dt) = (s+k^2)^\beta\end{aligned}$$

using (5.1). The CTRW limit has FLT

$$\int_0^\infty e^{-st}\hat{P}_{A(E(t)-)}(k)\,dt = \frac{s^{\beta-1}}{(s+k^2)^\beta}. \tag{5.26}$$

Inverting the FLT as in [3, Example 5.2] shows that the CTRW limit $A(E(t)-)$ has Lebesgue density

$$c_3(x,t) = \int_0^t n_{0,2u}(x)\frac{u^{\beta-1}}{\Gamma(\beta)}\frac{(t-u)^{-\beta}}{\Gamma(1-\beta)}du \tag{5.27}$$

where $n_{0,2t}$ is the density of a normal law with mean zero and variance $2t$. This density solves the governing equation

$$\left(\partial_t - \partial_x^2\right)^\beta c_3(x,t) = \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{5.28}$$

A comparison with (5.8) shows that

$$c_3(x,t) = \int_0^t n_{0,2u}(x)c_2(u,t)\,du \tag{5.29}$$

so that the CTRW limit in this case is a variance mixture of a normal density with respect to the CTRW limit density from the completely coupled case of Example 5.2.

The OCTRW limit $A(E(t))$ in (3.3) has distribution $\rho_t(dx)$ with FLT

$$\int_0^\infty e^{-st}\hat{\rho}_t(k)\,dt = \frac{1}{s}\frac{(s+k^2)^\beta - |k|^{2\beta}}{(s+k^2)^\beta}.$$

Now we will compute the density of $\rho_t$. The Lévy measure of $(A, D)$ is given by (5.25) and the density of $(A, D)$ is given by

$$p(z,u) = n_{0,2u}(z)g_\beta(u)$$

where $g_\beta$ is the density of $D$. Since $(A, D)$ is operator-stable with exponent $\mathrm{diag}(1/(2\beta), 1/\beta)$, the density of $(A(s), D(s))$ reads

$$p_s(z,u) = s^{-3/(2\beta)} p(s^{-1/(2\beta)} z, s^{-1/\beta} u) = s^{-1/\beta} n_{0,2u}(z) g_\beta(s^{-1/\beta} u). \tag{5.30}$$

Using (5.25) we get

$$\phi((-\infty, z-x], (t-\tau, \infty)) = \frac{1}{\Gamma(1-\beta)} \int_{t-\tau}^{\infty} \int_{-\infty}^{z-x} n_{0,2u}(v)\, dv\, \beta u^{-\beta-1}\, du.$$

Observe that

$$\frac{\partial}{\partial z} \phi((-\infty, z-x], (t-\tau, \infty)) = \frac{1}{\Gamma(1-\beta)} \int_{t-\tau}^{\infty} n_{0,2u}(z-x) \beta u^{-\beta-1}\, du. \tag{5.31}$$

In view of (4.1) we have for all $z \in \mathbb{R}$

$$\begin{aligned}
\rho_t(-\infty, z] &= \int_0^\infty \int_{\mathbb{R}} \int_0^t \phi((-\infty, z-x], (t-\tau, \infty)) P_{(A(s),D(s))}(dx, d\tau)\, ds \\
&= \int_0^\infty \int_0^t \int_{\mathbb{R}} \phi((-\infty, z-x], (t-\tau, \infty)) p_s(x, \tau)\, dx\, d\tau\, ds.
\end{aligned}$$

Using (5.30) and (5.31), the density of $\rho_t$ in this example is

$$\begin{aligned}
a_3(z,t) &= \int_0^\infty \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial z} \phi((-\infty, z-x], (t-\tau, \infty)) p_s(x, \tau)\, dx\, d\tau\, ds \\
&= \frac{1}{\Gamma(1-\beta)} \int_{s=0}^\infty \int_{\tau=0}^t \int_{x \in \mathbb{R}} \int_{u=t-\tau}^\infty n_{0,2u}(z-x) \beta u^{-\beta-1}\, du\, s^{-1/\beta} n_{0,2\tau}(x) g_\beta(s^{-1/\beta}\tau)\, dx\, d\tau\, ds \\
&= \frac{1}{\Gamma(1-\beta)} \int_{s=0}^\infty \int_{\tau=0}^t \int_{u=t-\tau}^\infty n_{0,2u} * n_{0,2\tau}(z) \beta u^{-\beta-1}\, du\, g_\beta(s^{-1/\beta}\tau)\, d\tau\, s^{-1/\beta}\, ds \\
&= \frac{1}{\Gamma(1-\beta)} \int_{s=0}^\infty \int_{\tau=0}^t \int_{u=t-\tau}^\infty n_{0,2(u+\tau)}(z) \beta u^{-\beta-1}\, du\, g_\beta(s^{-1/\beta}\tau)\, d\tau\, s^{-1/\beta}\, ds \\
&= \frac{1}{\Gamma(1-\beta)} \int_{\tau=0}^t \int_{u=t-\tau}^\infty n_{0,2(u+\tau)}(z) \beta u^{-\beta-1}\, du \int_0^\infty s^{-1/\beta} g_\beta(s^{-1/\beta}\tau)\, ds\, d\tau \\
&= \frac{\beta}{\Gamma(\beta)\Gamma(1-\beta)} \int_{\tau=0}^t \int_{u=t-\tau}^\infty n_{0,2(u+\tau)}(z) u^{-\beta-1}\, du\, \tau^{\beta-1}\, d\tau
\end{aligned}$$

using (5.14), (5.15) and (5.17). A simple change of variables gives the OCTRW limit density

$$\begin{aligned}
a_3(x,t) &= \frac{\beta}{\Gamma(\beta)\Gamma(1-\beta)} \int_{\tau=0}^t \int_{s=t}^\infty n_{0,2s}(x)(s-\tau)^{-\beta-1}\, ds\, \tau^{\beta-1}\, d\tau \\
&= \frac{\beta}{\Gamma(\beta)\Gamma(1-\beta)} \int_t^\infty n_{0,2s}(x) \int_0^t (s-\tau)^{-\beta-1} \tau^{\beta-1}\, d\tau\, ds.
\end{aligned}$$

A comparison with (5.22) shows that

$$a_3(x,t) = \int_t^\infty n_{0,2s}(x) a_2(s,t)\,ds \tag{5.32}$$

a variance mixture of a normal density with respect to the OCTRW limit density of Example 5.2, analogous with (5.29). The OCTRW limit density of this example solves the governing equation (4.16), which now reads

$$\left(\partial_t - \partial_x^2\right)^\beta a_3(x,t) = \frac{1}{\Gamma(1-\beta)} \int_t^\infty n_{0,2u}(x)\,dx\,\beta u^{-\beta-1}\,du. \tag{5.33}$$

The tail behavior of the CTRW limit and the OCTRW limit are very different in this example. The CTRW limit $A(E(t)-)$ has second moment

$$\begin{aligned}\int_{-\infty}^\infty x^2 c_3(x,t)\,dx &= \int_{-\infty}^\infty x^2 \int_0^t n_{0,2s}(x) c_2(s,t)\,ds\,dx \\ &= \int_0^t \int_{-\infty}^\infty x^2 n_{0,2s}(x)\,dx\, c_2(s,t)\,ds \\ &= 2\int_0^t s\, c_2(s,t)\,ds < \infty,\end{aligned}$$

since the beta density $c_2(x,t)$ has finite support on $x \in (0,t)$, and thus has a finite first moment. Hence $A(E(t)-)$ has finite variance, so it belongs to the domain of attraction of a normal law. However, the tail of the OCTRW limit is heavier.

The OCTRW limit $A(E(t))$ has tail probability

$$\begin{aligned}P\{|A(E(t))| > z\} &= 2\int_z^\infty a_3(x,t)\,dx = 2\int_z^\infty \int_t^\infty n_{0,2s}(x)\, a_2(s,t)\,ds\,dx \\ &= \frac{2t^\beta}{\Gamma(\beta)\Gamma(1-\beta)} \int_z^\infty \int_t^\infty \frac{1}{s(s-t)^\beta}\, n_{0,2s}(x)\,ds\,dx \\ &= \frac{t^\beta}{\sqrt{\pi}\,\Gamma(\beta)\Gamma(1-\beta)} \int_z^\infty \int_t^\infty \frac{1}{s^{3/2}(s-t)^\beta}\, e^{-x^2/(4s)}\,ds\,dx.\end{aligned}$$

Now a change of variables $u = x^2/(4s)$ with $du/ds = -x^2/(4s^2) = -u/s$ gives

$$\begin{aligned}
P\{|A(E(t))| > z\} &= \frac{t^\beta}{\sqrt{\pi}\,\Gamma(\beta)\Gamma(1-\beta)} \int_z^\infty \int_0^{x^2/(4t)} \frac{1}{u\left(\frac{x^2}{4u}\right)^{1/2}\left(\frac{x^2}{4u}-t\right)^\beta} e^{-u}\,du\,dx \\
&= \frac{2(4t)^\beta}{\sqrt{\pi}\,\Gamma(\beta)\Gamma(1-\beta)} \int_z^\infty \frac{1}{x} \int_0^{x^2/(4t)} u^{\beta-1/2}e^{-u}\,\frac{1}{(x^2-4ut)^\beta}\,du\,dx \\
&= \frac{2(4t)^\beta}{\sqrt{\pi}\,\Gamma(\beta)\Gamma(1-\beta)} \int_0^\infty u^{\beta-1/2}e^{-u} \int_{\max(\sqrt{4ut},z)}^\infty \frac{1}{x(x^2-4ut)^\beta}\,dx\,du \\
&= z^{-2\beta}\cdot C_\beta \int_0^\infty u^{\beta-1/2}e^{-u}\,h_z(u)\,du,
\end{aligned}$$

where

$$C_\beta = \frac{2(4t)^\beta}{\sqrt{\pi}\,\Gamma(\beta)\Gamma(1-\beta)} \quad \text{and} \quad h_z(u) = \int_{\max(\sqrt{4ut},z)}^\infty \frac{z^{2\beta}}{x(x^2-4ut)^\beta}\,dx.$$

A further change of variables $y = x^2$ with $dy/dx = 2x$ gives

$$h_z(u) = \frac{z^{2\beta}}{2}\int_{\max(4ut,z^2)}^\infty y^{-\beta-1}\left(\frac{y}{y-4ut}\right)^\beta\,dy.$$

For fixed $u > 0$ and $\varepsilon > 0$ choose $z_0 > \sqrt{4ut}$ so large that for any $y > z_0^2$ we get

$$1 \le \left(\frac{y}{y-4ut}\right)^\beta \le 1+\varepsilon.$$

Then for $z \ge z_0$ we have

$$\frac{1}{2\beta} = \frac{z^{2\beta}}{2}\int_{z^2}^\infty y^{-\beta-1}\,dy \le h_z(u) \le (1+\varepsilon)\frac{z^{2\beta}}{2}\int_{z^2}^\infty y^{-\beta-1}\,dy = \frac{1+\varepsilon}{2\beta}.$$

Since $\varepsilon > 0$ is arbitrary, for any $u > 0$ we have $h_z(u) \to 1/(2\beta)$ as $z \to \infty$. Furthermore, for any $u > 0$ the mapping $z \mapsto h_z(u)$ is continuous and obviously increasing on the interval $(0, \sqrt{4ut})$. For $z > \sqrt{4ut}$ we have

$$\begin{aligned}
\frac{d}{dz}\,h_z(u) &= 2\beta z^{2\beta-1}\int_z^\infty \frac{1}{x(x^2-4ut)^\beta}\,dx - z^{2\beta}\,\frac{1}{z(z^2-4ut)^\beta} \\
&= z^{2\beta-1}\left(\int_z^\infty \frac{2\beta}{x(x^2-4ut)^\beta}\,dx - \frac{1}{(z^2-4ut)^\beta}\right) \\
&= z^{2\beta-1}\left(\int_{z^2}^\infty \frac{\beta}{y(y-4ut)^\beta}\,dy - \frac{1}{(z^2-4ut)^\beta}\right) \\
&< z^{2\beta-1}\left(\int_{z^2}^\infty \frac{\beta}{(y-4ut)^{\beta+1}}\,dy - \frac{1}{(z^2-4ut)^\beta}\right) = 0
\end{aligned}$$

which shows that $z \mapsto h_z(u)$ is decreasing on the interval $(\sqrt{4ut}, \infty)$. Hence

$$\begin{aligned}
\sup_{z>0} h_z(u) &= (4ut)^\beta \int_{\sqrt{4ut}}^\infty \frac{1}{x(x^2-4ut)^\beta}\,dx \\
&= (4ut)^\beta \left( \int_{4ut}^{8ut} \frac{1}{2y(y-4ut)^\beta}\,dy + \int_{8ut}^\infty \frac{1}{2y(y-4ut)^\beta}\,dy \right) \\
&\le (4ut)^{\beta-1} \int_{4ut}^{8ut} \frac{1}{2(y-4ut)^\beta}\,dy + (4ut)^\beta \int_{8ut}^\infty \frac{1}{2(y-4ut)^{\beta+1}}\,dy \\
&= (4ut)^{\beta-1} \frac{1}{2(1-\beta)(4ut)^{\beta-1}} + (4ut)^\beta \frac{1}{2\beta(4ut)^\beta} = \frac{1}{2\beta(1-\beta)}
\end{aligned}$$

independent of $u > 0$. Since $\int_0^\infty u^{\beta-1/2} e^{-u}\,du = \Gamma(\beta+1/2)$, by dominated convergence we get

$$\int_0^\infty u^{\beta-1/2} e^{-u}\, h_z(u)\,du \to \frac{\Gamma(\beta+1/2)}{2\beta} \quad \text{as } z \to \infty.$$

Altogether we have $P\{|A(E(t))| > z\} = z^{-2\beta} L(z)$ with $L(z) \to C_\beta \Gamma(\beta+1/2)/(2\beta)$ as $z \to \infty$, which shows that $A(E(t))$ belongs to the domain of normal attraction of a $2\beta$-stable distribution. Then $\mathbb{E}(|A(E(t))|^\rho)$ exists for $0 < \rho < 2\beta$ and diverges for $\rho > 2\beta$ [14, XVII.5]. In particular, the second moment of $A(E(t))$ is infinite, while the second moment of $A(E(t)-)$ is finite.

Corollary 3.4 shows that both $A(E(t))$ and $A(E(t)-)$ are self-similar with scaling index $\beta/\alpha = 1/2$, hence this example provides two alternative coupled models for anomalous diffusion, that spread at the same rate as a Brownian motion.

**Example 5.4.** Suppose $D$ is a stable subordinator with $\mathbb{E}(e^{-sD}) = e^{-s^\beta}$, and the conditional distribution of $Y$ given $D = t$ is symmetric stable with distribution $\omega^t$ where $\omega$ has Fourier symbol $b|k|^\gamma$ for some $b > 0$ and $0 < \gamma \le 2$. Note that the special case $\gamma = 2, b = 1$ was considered in Example 5.3. Then

$$\mathbb{E}(e^{-sD+ikY}) = \mathbb{E}(\mathbb{E}(e^{-sD+ikY}|D)) = \mathbb{E}(e^{-sD-Db|k|^\gamma}) = e^{-(s+b|k|^\gamma)^\beta}$$

so that $Y$ is symmetric stable with index $\alpha = \gamma\beta$. If we take $(Y_n, J_n)$ iid with $(Y, D)$ then (2.4) holds, and it follows from (5.4) that the operator stable limit $(A, D)$ has Lévy measure $\phi(dx, dt) = t^{1/\gamma}\omega(dx)\,\phi_D(dt)$. The CTRW limit has FLT

$$\bar{c}_4(k,s) = \frac{s^{\beta-1}}{(s+b|k|^\gamma)^\beta}. \tag{5.34}$$

Inverting the Laplace transform gives

$$\hat{c}_4(k,t) = \int_0^t e^{-ub|k|^\gamma} \frac{u^{\beta-1}}{\Gamma(\beta)} \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} du,$$

where we have used the formulas $\mathcal{L}t^{q-1} = s^{-q}/\Gamma(q)$ for $q > 0$, $\mathcal{L}[e^{-tc}g(t)] = \mathcal{L}(g)(s+c)$, and $\mathcal{L}(f * g)(t) = \mathcal{L}f(s)\mathcal{L}g(s)$. Finally we invert the Fourier transform to get

$$c_4(x,t) = \int_0^t f_u(x) \frac{u^{\beta-1}}{\Gamma(\beta)} \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} du \tag{5.35}$$

where $f_u(x)$ is the density of $\omega^u$. This density solves the governing equation

$$\left(\partial_t - b\partial_{|x|}^{\gamma}\right)^{\beta} c_4(x,t) = \delta(x) \frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{5.36}$$

where $\partial_{|x|}^{\gamma} f(x)$ is the inverse Fourier transform of $-|k|^{\gamma}\hat{f}(k)$, also called the Riesz fractional derivative. A comparison with (5.8) shows that

$$c_4(x,t) = \int_0^t f_u(x) c_2(u,t)\, du \tag{5.37}$$

so that the CTRW limit in this case is a scale mixture of a symmetric stable density with respect to the CTRW limit density from the completely coupled case of Example 5.2.

The OCTRW limit $A(E(t))$ in (3.3) has distribution $\rho_t(dx)$ with FLT

$$\int_0^{\infty} e^{-st} \hat{\rho}_t(k)\, dt = \frac{1}{s} \frac{(s + b|k|^{\gamma})^{\beta} - b|k|^{\gamma\beta}}{(s + b|k|^{\gamma})^{\beta}}.$$

It follows from the scaling property $f_{cu}(x) = c^{-1/\gamma} f_u(c^{-1/\gamma}x)$ that $(A(s), D(s))$ has density

$$p_s(z,u) = s^{-1/\alpha} s^{-1/\beta} p(s^{-1/\alpha}z, s^{-1/\beta}u) = s^{-1/\beta} f_u(z) g_{\beta}(s^{-1/\beta}u), \tag{5.38}$$

and then an argument quite similar to Example 5.3 shows that the density of $\rho_t$ in this example is

$$a_4(x,t) = \int_t^{\infty} f_s(x) a_2(s,t)\, ds \tag{5.39}$$

a scale mixture of a symmetric stable density with respect to the OCTRW limit density of Example 5.2, analogous with (5.37). The OCTRW limit density of this example solves the governing equation (4.16), which now reads

$$\left(\partial_t - b\partial_{|x|}^{\gamma}\right)^{\beta} a_4(x,t) = \frac{1}{\Gamma(1-\beta)} \int_t^{\infty} f_u(x)\, dx\, \beta u^{-\beta-1}\, du. \tag{5.40}$$

As in Example 5.3, the tail behavior of the CTRW limit and the OCTRW limit are quite different here. For the CTRW limit $A(E(t)-)$, note that the density $f$ of the symmetric $\alpha$-stable random variable $A(1)$ fulfills $f(x) \sim C \cdot x^{-\alpha-1}$ for some $C > 0$ as

$x \to \infty$. Thus we get $\int_z^\infty f(u)\,du = z^{-\alpha}L(z)$ with $L(z) \to C/\alpha$ as $z \to \infty$. Make a change of variables $u = s^{-1/\alpha}x$ to get

$$\begin{aligned}
P\{|A(E(t)-)| > z\} &= 2\int_z^\infty c_4(x,t)\,dx = 2\int_z^\infty \int_0^t f_s(x)\,c_2(s,t)\,ds\,dx \\
&= \frac{2}{\Gamma(\beta)\Gamma(1-\beta)}\int_z^\infty \int_0^t s^{-1/\alpha} f(s^{-1/\alpha}x)\,(t-s)^{-\beta}s^{\beta-1}\,ds\,dx \\
&= \frac{2}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^t \int_z^\infty f(s^{-1/\alpha}x)s^{-1/\alpha}\,dx\,(t-s)^{-\beta}s^{\beta-1}\,ds \\
&= \frac{2}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^t \int_{s^{-1/\alpha}z}^\infty f(u)\,du\,(t-s)^{-\beta}s^{\beta-1}\,ds \\
&= \frac{2}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^t (s^{-1/\alpha}z)^{-\alpha}L(s^{-1/\alpha}z)\,(t-s)^{-\beta}s^{\beta-1}\,ds \\
&= z^{-\alpha}\frac{2}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^t \left(\frac{s}{t-s}\right)^\beta L(s^{-1/\alpha}z)\,ds,
\end{aligned}$$

Now for $\varepsilon > 0$ choose $z_0$ sufficiently large such that for all $z \ge z_0$ we have

$$\left|L(t^{-1/\alpha}z) - \tfrac{C}{\alpha}\right| < \varepsilon K_\beta^{-1}, \text{ where } K_\beta = \int_0^t \left(\frac{s}{t-s}\right)^\beta ds \in (0,\infty).$$

Then for $z \ge z_0$ we get

$$\left|\int_0^t \left(\frac{s}{t-s}\right)^\beta L(s^{-1/\alpha}z)\,ds - \frac{C}{\alpha}\int_0^t \left(\frac{s}{t-s}\right)^\beta ds\right| < \varepsilon$$

which shows that

$$P\{|A(E(t)-)| > z\} \sim z^{-\alpha}\frac{2CK_\beta}{\alpha\Gamma(\beta)\Gamma(1-\beta)}.$$

Hence the CTRW limit $A(E(t)-)$ belongs to the domain of normal attraction of an $\alpha$-stable random variable. However, the OCTRW limit $A(E(t))$ has a heavier tail.

Write

$$\begin{aligned}
P\{|A(E(t))| > z\} &= 2\int_z^\infty a_4(x,t)\,dx = 2\int_z^\infty \int_t^\infty f_s(x)\,a_2(s,t)\,ds\,dx \\
&= \frac{2t^\beta}{\Gamma(\beta)\Gamma(1-\beta)}\int_z^\infty \int_t^\infty \frac{1}{s(s-t)^\beta}\,s^{-1/\alpha}f(s^{-1/\alpha}x)\,ds\,dx,
\end{aligned}$$

where $f_s$ denotes the density of $A(s) \stackrel{d}{=} s^\alpha A(1)$ and $f = f_1$. A change of variables $u = s^{-1/\alpha}x$ gives

$$\begin{aligned} P\{|A(E(t))| > z\} &= \frac{2t^\beta \alpha}{\Gamma(\beta)\Gamma(1-\beta)} \int_z^\infty \int_0^{t^{-1/\alpha}x} \frac{1}{x\left(\left(\frac{x}{u}\right)^\alpha - t\right)^\beta} f(u)\,du\,dx \\ &= \frac{2t^\beta \alpha}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^\infty u^{\alpha\beta} f(u) \int_{\max(t^{1/\alpha}u,z)}^\infty \frac{1}{x\left(x^\alpha - tu^\alpha\right)^\beta}\,dx\,du \\ &= z^{-\alpha\beta} \cdot C_\beta \int_0^\infty u^{\alpha\beta} f(u)\,h_z(u)\,du, \end{aligned}$$

where

$$C_\beta = \frac{2t^\beta \alpha}{\Gamma(\beta)\Gamma(1-\beta)} \quad \text{and} \quad h_z(u) = \int_{\max(t^{1/\alpha}u,z)}^\infty \frac{z^{\alpha\beta}}{x\left(x^\alpha - tu^\alpha\right)^\beta}\,dx.$$

Since $A(1)$ has finite moments of any order less than $\alpha$, we have $\int_0^\infty u^{\alpha\beta} f(u)\,du < \infty$, and then a dominated convergence argument similar to Example 5.3 yields

$$\int_0^\infty u^{\alpha\beta} f(u)\,h_z(u)\,du \to \frac{1}{\alpha\beta} \int_0^\infty u^{\alpha\beta} f(u)\,du =: K_{\alpha\beta} \in (0,\infty).$$

Altogether we have $P\{|A(E(t))| > z\} = z^{-\alpha\beta}L(z)$ with $L(z) \to C_\beta K_{\alpha\beta}$ as $z \to \infty$, which shows that the OCTRW limit $A(E(t))$ belongs to the domain of normal attraction of a stable distribution with index $\alpha\beta < \alpha$, so it has a heavier tail than the CTRW limit $A(E(t)-)$.

Corollary 3.4 shows that both $A(E(t))$ and $A(E(t)-)$ in this example are self-similar with scaling index $\beta/\alpha = 1/\gamma > 1/2$, so this example provides two alternative coupled models for anomalous super-diffusion that spread faster than Brownian motion.

**Example 5.5.** Everything in Example 5.4 extends immediately to an arbitrary stable distribution $\omega$ with symbol $ap(-ik)^\gamma + aq(ik)^\gamma$ for $\gamma \neq 1$ where $p, q$ are nonnegative with $p + q = 1$ and $a > 0$ for $0 < \gamma < 1$, $a < 0$ for $1 < \gamma \leq 2$. To connect back to Example 5.4, note that $ap(-ik)^\gamma + aq(ik)^\gamma = a\cos(\pi\gamma/2)|k|^\gamma$ when $p = q = 1/2$, so that the sign of $a$ must change at $\gamma = 1$ to keep $b = a\cos(\pi\gamma/2) > 0$. Now (5.36) is replaced by

$$\left(\partial_t + ap\partial_x^\gamma + aq\partial_{-x}^\gamma\right)^\beta c_5(x,t) = \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{5.41}$$

and (5.40) is replaced by

$$\left(\partial_t + ap\partial_x^\gamma + aq\partial_{-x}^\gamma\right)^\beta a_5(x,t) = \frac{1}{\Gamma(1-\beta)} \int_t^\infty f_u(x)\,dx\,\beta u^{-\beta-1}\,du \tag{5.42}$$

where $\partial_{-x}^\gamma h(x)$ is the inverse Fourier transform of $(ik)^\gamma \hat{h}(k)$, also called the negative Riemann-Liouville fractional derivative. To illustrate, consider the 1/2-stable Lévy density $f(x) = (2\sqrt{\pi})^{-1}x^{-3/2}e^{-1/(4x)}$ for $x > 0$; see [44, p. 66]. The corresponding distribution $\omega$ has Fourier symbol $(-ik)^{1/2}$ by [44, Theorem C.3], so we are in the case $a = p = 1$ and $\gamma = 1/2$. It follows that for $u > 0$ and $x > 0$

$$f_u(x) = u^{-2} f(u^{-2}x) = \frac{u}{x} \frac{1}{\sqrt{4\pi x}} e^{-u^2/(4x)} = \frac{u}{x} n_{0,2x}(u)$$

is the density of $\omega^u$. Thus by (5.35) and (5.41) the CTRW limit density

$$c_5(x,t) = \frac{x^{-1}}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^t \left(\frac{u}{t-u}\right)^\beta n_{0,2x}(u)\, du$$

for $x > 0$ solves the governing equation

$$(\partial_t + \partial_x^{1/2})^\beta c_5(x,t) = \delta(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}.$$

By (5.23) and (5.39) the OCTRW limit density is

$$a_5(x,t) = \frac{x^{-1}}{\Gamma(\beta)\Gamma(1-\beta)} \int_t^\infty n_{0,2x}(u) \left(\frac{t}{u-t}\right)^\beta du,$$

a density mixture of the normal distribution, in contrast to the variance mixture in Example 5.3. Of course, this is also a scale mixture of the Lévy density.

As for the OCTRW governing equation, we first observe from integration by parts

$$\begin{aligned}
\int_t^\infty f_u(x)\beta u^{-\beta-1}\, du &= \frac{\beta}{x} \int_t^\infty u^{-\beta} n_{0,2x}(u)\, du \\
&= 2\beta \int_t^\infty u^{-\beta-1} \frac{u}{2x} \frac{1}{\sqrt{4\pi x}} e^{-u^2/(4x)}\, du \\
&= 2\beta t^{-\beta-1} n_{0,2x}(t) - 2\beta(\beta+1) \int_t^\infty u^{-\beta-2} n_{0,2x}(u)\, du \\
&= 2\beta(\beta+1) \int_t^\infty u^{-\beta-2} \big(n_{0,2x}(t) - n_{0,2x}(u)\big)\, du.
\end{aligned}$$

By (5.42) the OCTRW density solves the governing equation

$$(\partial_t + \partial_x^{1/2})^\beta a_5(x,t) = \frac{2\beta(\beta+1)}{\Gamma(1-\beta)} \int_t^\infty u^{-\beta-2} \big(n_{0,2x}(t) - n_{0,2x}(u)\big)\, dx\, du.$$

**Example 5.6.** Assume that $X(t)$ is any Lévy process such that $P_{X(t)} = \omega^t$ for some infinitely divisible law $\omega$. Assume further that $D(t)$ is a $\beta$-stable subordinator, independent of $X(t)$ with $E[e^{-sD(1)}] = e^{-s^\beta}$ as before. Define a triangular array with iid rows such that

$$Y_i^{(c)} \stackrel{d}{=} X(D(c^{-1})) \quad \text{and} \quad J_i^{(c)} \stackrel{d}{=} D(c^{-1}).$$

Then it is easy to see that

$$(S^{(c)}(ct), T^{(c)}(ct)) \Rightarrow (A(t), D(t))$$

where $A(t) = X(D(t))$, which introduces the coupling we have in Examples 5.3–5.5. Since $X(t)$ and $D(t)$ are independent, a simple conditioning argument yields

$$P_{A(E(t)-)}(dx) = \int_0^\infty \omega^u(dx)\, P_{D(E(t)-)}(du) = \int_0^\infty \omega^u(dx) c_2(u,t)\, du$$

as well as

$$P_{(A(E(t))}(dx) = \int_0^\infty \omega^u(dx)\, P_{D(E(t))}(du) = \int_0^\infty \omega^u(dx) a_2(u,t)\, du.$$

Both are scale mixtures with respect to the densities from Example 5.2. Let $D = D(1)$, $A = A(1)$, and write $\mathbb{E}(e^{ikX(t)}) = e^{-t\psi_0(k)}$. Then

$$\mathbb{E}(e^{-sD}e^{ikA}) = \mathbb{E}(\mathbb{E}(e^{-sD}e^{ikX(D)}|D=t)) = \mathbb{E}(e^{-sD}e^{-D\psi_0(k)}) = e^{-(s+\psi_0(k))^\beta}$$

so that $\psi(k,s) = (s+\psi_0(k))^\beta$ in this case. If $X(t)$ has a density $f_u(t)$, then $A(E(t)-)$ has a density

$$c_6(x,t) = \int_0^t f_u(x) c_2(u,t)\, du \tag{5.43}$$

that solves the coupled pseudo-differential equation

$$\big(\partial_t + \psi_0(i\partial_x)\big)^\beta c_6(x,t) = \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{5.44}$$

while $A(E(t))$ has a density

$$a_6(x,t) = \int_0^t f_u(x) a_2(u,t)\, du \tag{5.45}$$

that solves the coupled pseudo-differential equation

$$\big(\partial_t + \psi_0(i\partial_x)\big)^\beta c_6(x,t) = \frac{1}{\Gamma(1-\beta)}\int_t^\infty f_u(x)\, dx\, \beta u^{-\beta-1}\, du. \tag{5.46}$$

If $\omega$ is stable with index $\gamma$, then this extends Examples 5.3, 5.4, and 5.5, and explains the scale mixtures seen there. It also extends [3, Example 5.6] to the case where $\omega$ is not symmetric.

If $\omega$ is not stable, then the limit in this example can only be obtained from a triangular array, as discussed in Remark 3.6. To see this, note that $(A(t), D(t)) = Y(D(t))$ where $Y(t) = (X(t), t)$ is a Lévy process on $\mathbb{R}^2$. The distribution of $Y(s)$ is $\mu^s(dx, dt) = \omega^s(dx)\varepsilon_s(dt)$, and then it follows from Sato [33, Theorem 30.1] that $Y(D(t))$ is infinitely divisible with Lévy measure

$$\omega^t(dx)\phi_D(dt) = \int_0^\infty \mu^s(dx, dt)\phi_D(ds).$$

Unless $\omega$ is stable, then this does not reduce to the form (5.4), and then it follows from [3, Theorem 2.2] that $(A(t), D(t)) = Y(D(t))$ cannot occur as the limit in (2.4) for any space-time random walk.

**Example 5.7.** Suppose $D$ is a stable subordinator with $\mathbb{E}(e^{-sD}) = e^{-s^\beta}$, and the conditional distribution of $Y$ given $D = t$ is $\omega_t(dx) = \frac{1}{2}[\varepsilon_t(dx) + \varepsilon_{-t}(dx)]$ as in [19, 21] and [3, Example 5.4]. Then

$$\mathbb{E}(e^{ikY}) = \mathbb{E}(\mathbb{E}(e^{ikY}|D)) = \tfrac{1}{2}\mathbb{E}(e^{ikD} + e^{-ikD}) = \tfrac{1}{2}[e^{-(-ik)^\beta} + e^{-(ik)^\beta}].$$

If we take $(Y_n, J_n)$ iid with $(Y, D)$ then (2.4) holds, and it follows from (5.4) that the operator stable limit $(A, D)$ has Lévy measure

$$\phi(dx, dt) = \omega_t(dx)\,\phi_D(dt).$$

Take $a = b = \sigma^2 = 0$ in (2.8) to see that

$$\begin{aligned}\psi(k,s) &= \int_0^\infty \int_{-\infty}^\infty \left(1 - e^{ikx}e^{-st} + \frac{ikx}{1+x^2}\right)\omega_t(dx)\phi_D(dt)\\ &= \tfrac{1}{2}\int_0^\infty \left(1 - e^{-(t-ik)}\right)\phi_D(dt) + \tfrac{1}{2}\int_0^\infty \left(1 - e^{-(t+ik)}\right)\phi_D(dt)\\ &= \tfrac{1}{2}[(s-ik)^\beta + (s+ik)^\beta]\end{aligned}$$

using (5.1). Then

$$\psi_A(k) = \psi(k,0) = \tfrac{1}{2}[(-ik)^\beta + (ik)^\beta] = |k|^\beta \cos(\pi\beta/2)$$

so that $Y$ is symmetric stable with index $\alpha = \beta$. The CTRW limit has FLT

$$\bar{c}_7(k,s) = \frac{2s^{\beta-1}}{(s-ik)^\beta + (s+ik)^\beta}$$

and governing equation

$$\left(\partial_t + \partial_x\right)^\beta c_7(x,t) + \left(\partial_t - \partial_x\right)^\beta c_7(x,t) = \delta(x)\frac{2t^{-\beta}}{\Gamma(1-\beta)}.$$

The OCTRW limit $A(E(t))$ in (3.3) has FLT

$$\bar{a}_7(k,s) = \frac{1}{s}\frac{(s-ik)^\beta + (s+ik)^\beta - 2|k|^\beta\cos(\pi\beta/2)}{(s-ik)^\beta + (s+ik)^\beta}$$

and governing equation

$$\left(\partial_t + \partial_x\right)^\beta a_7(x,t) + \left(\partial_t - \partial_x\right)^\beta a_7(x,t) = \frac{\beta}{2\Gamma(1-\beta)}\int_t^\infty [\varepsilon_t(dx) + \varepsilon_{-t}(dx)]t^{-\beta-1}\,dt.$$

Computing the density is much harder in this example because $\omega_t$ is not infinitely divisible.

**Example 5.8.** Our last example is taken from Jurlewicz [16, 17]. Suppose that $Y_n$ and $J_n$ are independent as in Example 5.1, so that $c^{-1/\alpha}S(ct) \Rightarrow A(t)$, $c^{-1/\beta}T(ct) \Rightarrow D(t)$, and $c^{-\beta}N(ct) \Rightarrow E(t)$. A coupling will be introduced by clustering the jumps. Let

$M_n$ denote iid cluster sizes, taking values in the nonnegative integers. Let $C(n) = \sum_{i=1}^n M_i$ with $C(0) = 0$ and define

$$Y_i^M = \sum_{C(n-1)<i\le C(n)} Y_i \quad \text{and} \quad J_i^M = \sum_{C(n-1)<i\le C(n)} J_i.$$

Define the coupled CTRW $X^M(t) = S^M(N^M(t))$ where $N^M(t) = \max\{n \ge 0 : T^M(n) \le t\}$, and the coupled OCTRW $Z^M(t) = S^M(N^M(t)+1)$. Note that $S^M(n) = S(C(n))$ and $T^M(n) = T(C(n))$. Let $N_M(t) = \max\{n \ge 0 : C(n) \le t\}$ and recall that $\{N(t) \ge m\} = \{T(m) \le t\}$. Then

$$N_M(N(t)) = \max\{n \ge 0 : C(n) \le N(t)\} = \max\{n \ge 0 : T(C(n)) \le t\} = N^M(t)$$

so that we can also write the clustered CTRW $X^M(t) = S(C(N_M(N(t))))$ and the clustered OCTRW $Z^M(t) = S(C(N_M(N(t))+1))$. Suppose that $(M_n)$ belongs to the strict domain of attraction of some stable law with index $0 < \eta < 1$. An argument similar to Example 5.2 shows that

$$\frac{C(N_M(rt))}{r} \Rightarrow t\mathcal{B} \quad \text{and} \quad \frac{C(N_M(rt)+1)}{r} \Rightarrow \frac{t}{\mathcal{B}}$$

as $r \to \infty$, where $\mathcal{B}$ has beta distribution with parameters $\eta$ and $1-\eta$. Using a transfer theorem from Dobrushin [10], Jurlewicz [16] shows that

$$\frac{C(N_M(r^\beta\ r^{-\beta}N(rt)))}{r^\beta} \Rightarrow \mathcal{B}E(t) \quad \text{and} \quad \frac{C(N_M(N(rt))+1)}{r^\beta} \Rightarrow \frac{E(t)}{\mathcal{B}}$$

and finally, another application of the transfer theorem yields

$$\frac{S(C(N_M(N(rt))))}{r^{\beta/\alpha}} \Rightarrow A(\mathcal{B}E(t)) \quad \text{and} \quad \frac{S(C(N_M(N(rt))+1)}{r^{\beta/\alpha}} \Rightarrow A(E(t)/\mathcal{B}).$$

This argument uses the fact that $Y_n$, $J_n$ and $M_n$ are independent, and then $A(t)$, $\mathcal{B}$, and $E(t)$ are independent in the limit. Although these limits involve a mixture with respect to the generalized arc sine distributions, they do not fall under the scheme of Example 5.6, because of the further subordination to $E(t)$. Note that a representation of the CTRW and OCTRW limits in this example via under- and overshooting subordinators has been proposed recently [42]. According to this result we have

$$\frac{S(C(N_M(N(rt))))}{r^{\beta/\alpha}} \Rightarrow A(A_M(E_M(E(t))-))$$

and

$$\frac{S(C(N_M(N(rt))+1)}{r^{\beta/\alpha}} \Rightarrow A(A_M(E_M(E(t))))$$

where $A_M(E_M(t)-)$ and $A_M(E_M(t))$ are the CTRW and OCTRW limits obtained in Example 5.2 for a process with waiting times and jumps both equal to $M_n$. This representation is consistent with the theses of Theorem 3.1. We are currently investigating the governing equations for these interesting processes.

## References


[1] B. Baeumer and M.M. Meerschaert (2001) Stochastic solutions for fractional Cauchy problems. *Fractional Calculus and Applied Analysis* **4**, 481–500.

[2] B. Baeumer, M.M. Meerschaert and J. Mortensen (2005) Space-time fractional derivative operators. *Proc. Amer. Math. Soc.* **133**, No. 8, 2273–2282.

[3] P. Becker-Kern, M.M. Meerschaert and H.P. Scheffler (2004) Limit theorems for coupled continuous time random walks. *The Annals of Probability* **32**, No. 1B, 730–756.

[4] D. Benson, S. Wheatcraft and M. Meerschaert (2000) Application of a fractional advection-dispersion equation. *Water Resour. Res.* **36**, 1403–1412.

[5] D. Benson, R. Schumer, M. Meerschaert and S. Wheatcraft (2001) Fractional dispersion, Lévy motions, and the MADE tracer tests. *Transport in Porous Media* **42**, 211–240.

[6] B. Berkowitz, A. Cortis, M. Dentz, and H. Scher (2006), Modeling non-Fickian transport in geological formations as a continuous time random walk, *Rev. Geophys.*, 44, RG2003

[7] J. Bertoin (1996) *Lévy processes*. Cambridge University Press.

[8] N.H. Bingham (1971) Limit theorems for occupation times of Markov processes. *Z. Warsch. verw. Geb.* **17**, 1–22.

[9] L. Bondesson, G. Kristiansen, and F. Steutel (1996) Infinite divisibility of random variables and their integer parts. *Statist. Probab. Lett.* **28**, 271–278.

[10] R.L. Dobrushin, *Lemma on the limit of a composite random function*, Uspekhi Mat. Nauk 10 (1955) 157-159 (in Russian).

[11] E.B. Dynkin (1961) Some limit theorems for sums of independent random variables with infinite mathematical expectations. *Selected Translations in Math. Statist. and Probab.* **1**, IMS/AMS, Providence, 171–189.

[12] P. Embrechts and M. Maejima (2002) *Selfsimilar Processes*, Princeton University Press, Princeton, New York.

[13] S. Fedotov and A. Iomin (2007) Migration and proliferation dichotomy in tumor-cell invasion. *Phys Rev Lett.* **98**(11), 8101.

[14] W. Feller (1971) *An Introduction to Probability Theory and Its Applications.* Vol. II, 2nd Ed., Wiley, New York.

[15] B.I. Henry and P. Straka (2010) Lagging/leading coupled continuous time random walks, renewal times and their joint limits. Preprint, arXiv:1005.2369v1 [math.PR].

[16] A. Jurlewicz (2005) Limit theorems for randomly coarse grained continuous-time random walks. *Dissertationes Mathematicae* **431**, 1-45.

[17] A. Jurlewicz, A. Wyłomańska and P. Żebrowski (2009) Coupled continuous-time random walk approach to the Rachev-Rüschendorf model for financial data. *Physica A* **388**, 407–418.

[18] H. Kesten (1969) Hitting Probabilities of Single Points for Processes with Stationary Independent Increments. *Mem. Am. Math. Soc.***93**, 129 p.

[19] J. Klafter, A. Blumen and M.F. Shlesinger (1987) Stochastic pathways to anomalous diffusion. *Phys. Rev. A* **35**, 3081–3085.

[20] M. Kotulski (1995) Asymptotic distributions of the continuous time random walks: A probabilistic approach. *J. Statist. Phys.* **81** 777–792.

[21] M. Kotulski (1995) Asymptotic behavior of generalized Lévy walks. Chaos–the interplay between stochastic and deterministic behaviour (Karpacz, 1995), 471–477, *Lecture Notes in Phys.* **457**, Springer, Berlin.

[22] F. Mainardi and R. Gorenflo (2000) On Mittag-Leffer-type functions in fractional evolution processes, *J. Comput. Appl. Math.* **118** 283–299.

[23] M.M. Meerschaert (1990) Moments of random vectors which belong to some domain of normal attraction. *Ann. Probab.* **18**, 870–876.

[24] M.M. Meerschaert and H.P. Scheffler (2001) *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice.* Wiley Interscience, New York.
[25] M.M. Meerschaert, D.A. Benson, H.P. Scheffler and B. Baeumer (2002) Stochastic solution of space-time fractional diffusion equations. *Physical Review E* **65**(4), 1103–1106.
[26] M.M. Meerschaert and H.P. Scheffler (2004) Limit theorems for continuous time random walks with infinite mean waiting times. *J. Applied Probab.* **41**, No. 3, 623–638.
[27] M.M. Meerschaert and E. Scalas (2006) Coupled continuous time random walks in finance. *Physica A*, **370**, 114–118.
[28] M.M. Meerschaert and H.P. Scheffler (2008) Triangular array limits for continuous time random walks. *Stoch. Proc. Appl.***118**, 1606–1633.
[29] M.M. Meerschaert, E. Nane and P. Vellaisamy (2009) Fractional Cauchy problems on bounded domains. *Ann. Probab.* **37**, 979–1007.
[30] R. Metzler and J. Klafter (2000) The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **339**, 1–77.
[31] R. Metzler and J. Klafter (2004) The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Physics A* **37**, R161–R208.
[32] G. Ramos-Fernandez, J.L. Mateos, O. Miramontes, G. Cocho, H. Larralde and B. Ayala-Orozco (2003) Levy walk patterns in the foraging movements of spider monkeys (*Ateles geoffroyi*). *Behav. Ecol. Sociobiol.* **55**, 223–230.
[33] K.I. Sato (1999) *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press.
[34] E. Scalas (2004) Five years of Continuous-Time Random Walks in Econophysics. *Proceedings of WEHIA 2004*, A. Namatame (ed.), Kyoto.
[35] R. Schumer, D.A. Benson, M.M. Meerschaert and S. W. Wheatcraft (2001) Eulerian derivation of the fractional advection-dispersion equation. *J. Contaminant Hydrol.*, **48**, 69–88.
[36] R. Schumer, M. M. Meerschaert, and B. Baeumer (2009), Fractional advection-dispersion equations for modeling transport at the Earth surface, *Journal of Geophysical Research* **114**, F00A07.
[37] M. Shlesinger, J. Klafter and Y.M. Wong (1982) Random walks with infinite spatial and temporal moments. *J. Statist. Phys.* **27**, 499–512.
[38] E. Seneta (1976) *Regularly Varying Functions.* Lecture Notes in Mathematics **508**, Springer-Verlag, Berlin.
[39] D.S. Silvestrov (2004) *Limit Theorems for Randomly Stopped Stochastic Processes*, Springer-Verlag, London.
[40] D.S. Silvestrov and J.L. Teugels (2004) Limit theorems for mixed max-sum processes with renewal stopping. *Ann. Appl. Probab.* **14**, 1838–1868.
[41] M. Slutsky and L.A. Mirny (2004) Kinetics of protein-DNA interaction: Facilitated target location in sequence-dependent potential. *Biophys J.* **87**, 4021–4035.
[42] K. Weron, A. Jurlewicz, M. Magdziarz, A. Weron, and J. Trzmiel (2010) Over- and Undershooting Subordination Scenario for Fractional Two-Power-Law Relaxation Responses. *Physical Review E* accepted for publication
[43] W. Whitt (2002) *Stochastic Process Limits: An Introduction to Stochastic-Process Limits And their Application to Queues.* Springer, New York.
[44] V. M. Zolotarev, One-dimensional stable distributions, Translations of Mathematical Monographs, vol. 65, American Mathematical Society, Providence, RI, 1986, Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver.

Agnieszka Jurlewicz, Hugo Steinhaus Center, Institute of Mathematics and Computer Science, Wrocław University of Technology, Wrocław, Poland
*E-mail address*: Agnieszka.Jurlewicz@pwr.wroc.pl

Peter Kern, Mathematical Institute, Heinrich-Heine-Universität Düsseldorf, 40225 Düsseldorf, Germany
*E-mail address*: kern@math.uni-duesseldorf.de
*URL*: http://www.math.uni-duesseldorf.de/Personen/indiv/Kern

Mark M. Meerschaert, Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824
*E-mail address*: mcubed@stt.msu.edu
*URL*: http://www.stt.msu.edu/∼mcubed/

Hans-Peter Scheffler, Fachbereich Mathematik, Universität Siegen, 57068 Siegen, Germany
*E-mail address*: scheffler@mathematik.uni-siegen.de
*URL*: http://www.stat.math.uni-siegen.de/∼scheffler/