Multiparameter Multifractional Brownian Motion: Local Nondeterminism and Joint Continuity of the Local Times

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Abstract

By using a wavelet method we prove that the harmonisable-type N-parameter multifractional Brownian motion (mfBm) is a locally nondeterministic Gaussian random field. This nice property then allows us to establish joint continuity of the local times of an (N, d)-mfBm and to obtain some new results concerning its sample path behavior.

Résumé

Au moyen d'une méthode d'ondelettes nous montrons que le mouvement Brownien multifractionnaire de type harmonisable à N indices (mfBm) est un champ Gaussien localement non-déterministe. Grâce à cette propriété nous établissons ensuite la bicontinuité des temps locaux d'un (N, d)-mfBm et cela nous permet d'obtenir de nouveaux résultats concernant son comportement trajectoriel.

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1 Introduction

Multifractional Brownian motions (mfBm) were introduced independently by Lévy-Véhel and Peltier [23] and Benassi, Jaffard and Roux [8] by using respectively a moving average representation and a harmonisable representation; see (2.3) and (1.1) below. A multifractional Brownian motion is governed by a Hurst function $H(t)$ with certain regularity in place of the constant Hurst parameter $H \in (0, 1)$ in ordinary fractional Brownian motion. The most

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important feature of a mfBm is that its local regularities [e.g. pointwise Hölder exponent] change as time evolves. As such, multifractional Brownian motions are useful as stochastic models for phenomena that exhibit non-stationarity [e.g., traffic in modern telecommunication networks or signal processing].

Several authors have investigated sample path and statistical properties of multifractional Brownian motions. For example, Benassi, Jaffard and Roux [8] obtained classical Kolmogorov's laws of the iterated logarithm for mfBm. Lévy-Véhel and Peltier [23] determined its pointwise Hölder exponent as well as the Hausdorff and other fractal dimensions of its graph. Ayache, Cohen and Lévy-Véhel [3] and Herbin [19] studied the covariance structure of mfBm with harmonisable representations. Recently, Boufoussi, Dozzi and Guerbaz [12, 13] studied the existence, joint continuity and the Hölder regularity of the local time of one parameter moving-average-type mfBm and established a Chung's law of the iterated logarithm for the latter process. We refer to [14, 29] for further information.

There are several ways to define N-parameter multifractional Brownian motions. First, Benassi, Jaffard and Roux [8] defined a harmonisable-type isotropic multiparameter mfBm (see (1.1) for its definition). Later, Ayache and Léger [4], and Herbin [19] introduced so-called multifractional Brownian sheets (mfBs) in terms of their moving average representations and harmonisable representations, where the constant Hurst vector of a fractional Brownian sheet is substituted by a vector valued function. Furthermore, they showed that both moving-average-type and harmonisable-type multifractional Brownian sheets have continuous modifications and determined the pointwise and local Hölder exponent of mfBs. Meerschaert, Wu and Xiao [26] considered a slightly more general class of moving-average-type multifractional Brownian sheets and proved, among other things, the joint continuity of their local times.

The methods in [12, 13, 26] depend crucially on the non-anticipating structure of the moving-average-type multifractional Brownian motions, which shows that the latter have the property of *one-sided* local nondeterminism (see Section 2 for its definition and a proof of the last statement). However, their arguments can not be applied to harmonisable-type multifractional Brownian motions. It had been an open problem to prove that harmonisable-type multifractional Brownian motions satisfy the property of local nondeterminism. There was an attempt in [14, Theorem 7.1] to solve this problem for a one-parameter multifractional Brownian motion of harmonisable-type by exploiting the local self-similarity, yet there seems to be a gap in their proof.

The main objective of this paper is to provide a method for establishing the property of local nondeterminism for multifractional Brownian motions of harmonisable-type. Our method is originated from wavelet analysis and is different from the existing methods in the literature (cf. [11, 28, 31]). Before we present the main heuristic ideas behind it let us recall the definition of a harmonisable-type isotropic multiparameter mfBm with values in $\mathbb{R}$ and its random wavelet-type series representation due to Benassi, Jaffard and Roux [8]. Such a Gaussian field $X = \{X(t) : t \in \mathbb{R}^N\}$ is defined by

$$X(t) := \int_{\mathbb{R}^N} \frac{e^{it\cdot\xi} - 1}{|\xi|^{H(t)+N/2}}\, d\widehat{W}(\xi) \qquad \text{for every } t \in \mathbb{R}^N, \tag{1.1}$$

where $t \cdot \xi$ denotes the usual inner product of t and ξ, $|\xi|$ denotes the Euclidian norm of ξ, and

- $H(\cdot)$ is a functional parameter with values in a fixed interval $[a, b] \subset (0, 1)$; we will always assume that it satisfies a uniform Hölder condition of order $\beta = \beta(I) \in (b, 1]$ on any compact cube $I \subset \mathbb{R}^N$, i.e. there is a constant $c_1 = c_1(I) > 0$, only depending on I, such that for all $t', t'' \in I$,

$$|H(t') - H(t'')| \le c_1\, |t' - t''|^{\beta}. \tag{1.2}$$

- $d\widehat{W}$ is "the Fourier transform" of the real valued white noise dW which means that for each function $f \in L^2(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f(s)\, dW(s) = \int_{\mathbb{R}^N} \widehat{f}(\xi)\, d\widehat{W}(\xi),$$

  where $\widehat{f}$ denotes the Fourier transform of f. Recall that the Fourier transform of a function $f \in L^2(\mathbb{R}^N)$ is the limit of the Fourier transforms of functions of the Schwartz class $S(\mathbb{R}^N)$ converging to f; throughout this article the Fourier transform over $S(\mathbb{R}^N)$ is defined as $(\mathcal{F}g)(\xi) = \widehat{g}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-is\cdot\xi} g(s)\, ds$ and the inverse map as $(\mathcal{F}^{-1}h)(s) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{is\cdot\xi} h(\xi)\, d\xi$, thus the Fourier transform is a bijective isometry from $L^2(\mathbb{R}^N)$ to itself.

The Gaussian field $X = \{X(t) : t \in \mathbb{R}^N\}$ can be represented as the following random wavelet-type series

$$X(t) = \sum_{l=1}^{2^N-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^N} 2^{-jH(t)} \epsilon_{l,j,k} \Big(\Psi_l(2^j t - k, H(t)) - \Psi_l(-k, H(t)) \Big), \tag{1.3}$$

where the Ψ_l's are the deterministic wavelet-type functions defined in (2.8) and the $\epsilon_{l,j,k}$'s are the independent $\mathcal{N}(0, 1)$, Gaussian random variables defined in (2.20). For every fixed $t \in \mathbb{R}^N$, the series in (1.3) is convergent in $L^2(\Omega)$, Ω being the underlying probability space (see [8]). Moreover the series in (1.3) is, with probability 1, uniformly convergent in t on each compact subset of $\mathbb{R}^N$ (see [6]).

As we will see in the next section, for proving that $X = \{X(t) : t \in \mathbb{R}^N\}$ is locally nondeterministic on a closed (bounded) rectangle $I \subset \mathbb{R}^N_+$, it is sufficient to show that for any integer $m \geq 1$, there exists a constant $c^{(m)} > 0$ such that for all $t^0, t^1, \dots, t^m \in I$ and all real numbers $\alpha_1, \dots, \alpha_m$, one has

$$\mathbb{E}\Big|X(t^0) - \sum_{n=1}^{m} \alpha_n X(t^n)\Big|^2 \geq c^{(m)} \min\Big\{|t^n - t^0|^{2H(t^0)} \,:\, 1 \leq n \leq m\Big\}. \tag{1.4}$$

For the sake of simplicity, we will assume throughout this paper that $I = [\varepsilon, 1]^N$, where ε is a positive real number. In order to explain the main intuition behind our proof of (1.4), we suppose heuristically that, for all $(t,\theta) \in \mathbb{R}^N \times [a,b]$, $\Psi_l(t,\theta) = \mathbb{1}_{[0,1)^N}(t)$, where $\mathbb{1}_A$ denotes the indicator function of $A \subset \mathbb{R}^N$, and consequently that

$$\Psi_l(2^j t - k, \theta) = \mathbb{1}_{\prod_{i=1}^{N}[\frac{k_i}{2^j}, \frac{k_i+1}{2^j})}(t); \tag{1.5}$$

of course, this is not the correct choice of wavelet functions, however it gives the main intuition behind the crucial Relations (2.10) and (2.47). Let j_0 be the unique integer such that

$$2^{-j_0-1} < 2^{-1}\varepsilon N^{-1/2} \min\big\{|t^n - t^0| \,:\, 1 \leq n \leq m\big\} \leq 2^{-j_0}. \tag{1.6}$$

[note in passing that that j_0 will be defined in a slightly different way in the next section (see (2.23)).] It follows from (1.5) and (1.6) that there is a unique $k^0 \in \mathbb{Z}^N$ which satisfies

$$\Psi_l(2^{j_0} t^0 - k_0, H(t^0)) = 1, \tag{1.7}$$

$$\Psi_l(-k_0, H(t^n)) = 0, \qquad \text{for every } \; n = 0, 1, \dots, m \tag{1.8}$$

and

$$\Psi_l(2^{j_0} t^n - k_0, H(t^n)) = 0, \qquad \text{for every } \; n = 1, \dots, m. \tag{1.9}$$

Then putting together (1.3), (1.6), (1.7), (1.8) and (1.9) we obtain that

$$\begin{aligned}
&\mathbb{E}\Big|X(t^0) - \sum_{n=1}^{m} \alpha_n X(t^n)\Big|^2 \\
&= \sum_{l=1}^{2^N-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^N} \Big| 2^{-jH(t^0)}\Big(\Psi_l(2^j t^0 - k, H(t^0)) - \Psi_l(-k, H(t^0))\Big) \\
&\qquad\qquad - \sum_{n=1}^{m} 2^{-jH(t^n)} \alpha_n \Big(\Psi_l(2^j t^n - k, H(t^n)) - \Psi_l(-k, H(t^n))\Big)\Big|^2 \\
&\geq \Big| 2^{-j_0 H(t^0)}\Big(\Psi_1(2^{j_0} t^0 - k_0, H(t^0)) - \Psi_1(-k_0, H(t^0))\Big) \\
&\qquad\qquad - \sum_{n=1}^{m} 2^{-j_0 H(t^n)} \alpha_n \Big(\Psi_1(2^{j_0} t^n - k_0, H(t^n)) - \Psi_1(-k_0, H(t^n))\Big)\Big|^2 \\
&= 2^{-2j_0 H(t^0)} \\
&\geq 2^{-2b}\varepsilon^{2b} N^{-b} \min\Big\{|t^n - t^0|^{2H(t^0)} \,:\, 1 \leq n \leq m\Big\},
\end{aligned}$$

which shows that (1.4) holds.

The property of local nondeterminism as proved in Theorem 2.1 allows us to study fine properties of the sample paths of harmonisable-type mfBm. In particular, it can be applied to establish the joint continuity of the local times of harmonisable-type mfBm (see Theorem 3.1).

The rest of the paper is organized as follows. In Section 2 we study the local nondeterminism of mfBm X. Our main result is Theorem 2.1, which is proved by using the wavelet method. In Section 3 we apply Theorem 2.1 to prove the joint continuity of the local times of an (N,d)-mfBm X (Theorem 3.1), as well as local and uniform Hölder conditions for the maximum local time of a X (Theorem 3.5). These results can be applied in turn to study the asymptotic and fractal properties of mfBm X. We give two such applications – one is to determine the modulus of non-differentiability for a real-valued mfBm (Theorem 3.6) and the other is to prove a uniform Hausdorff dimension result for the level sets of an (N,d)-mfBm (Theorem 3.8).

We end the introduction with some notation. For any integer $p \geq 1$, a parameter $t \in \mathbb{R}^p$ is written as $(t_1, \ldots, t_p)$, or as $\langle c \rangle$, if $t_1 = \cdots = t_p = c$. For any $s,\, t \in \mathbb{R}^p$ such that $s_j < t_j\, (j = 1, \ldots, p)$, we define the closed interval (or rectangle) $[s,\, t] = \prod_{j=1}^{p} [s_j,\, t_j]$. We will use $\mathcal{A}$ to denote the class of all closed intervals $T \subset \mathbb{R}^p$. The Lebesgue measure in $\mathbb{R}^p$ is denoted by λ_p.

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2 Local Nondeterminism

The concept of (one-sided) local nondeterminism (LND, in short) of a Gaussian process was first introduced by Berman [11] to unify and extend his methods for studying the existence and joint continuity of local times of Gaussian processes.

Let $Z = \{Z(t), t \geq 0\}$ be a separable Gaussian process with mean 0 and let $J \subset \mathbb{R}_+$ be an open interval. Assume that $\mathbb{E}[Z(t)^2] > 0$ for all $t \in J$ and there exists $\delta_0 > 0$ such that

$$\sigma^2(s,t) = \mathbb{E}\big[(Z(s) - Z(t))^2\big] > 0 \quad \text{for} \;\; s,\; t \in J \text{ with } \; 0 < |s-t| < \delta_0.$$

Recall from Berman [11] that Z is called locally nondeterministic on J if for every integer $m \geq 2$,

$$\lim_{\delta \to 0} \inf_{t^m - t^1 \leq \delta} V_m > 0, \tag{2.1}$$

where V_m is the relative prediction error

$$V_m = \frac{\mathrm{Var}\big(Z(t^m) - Z(t^{m-1}) | Z(t^1), \ldots, Z(t^{m-1})\big)}{\mathrm{Var}\big(Z(t^m) - Z(t^{m-1})\big)}$$

and the infimum is taken over all ordered points $t^1 < t^2 < \cdots < t^m$ in J with $t^m - t^1 \le \delta$. Because of this last restriction, Berman's LND is also referred to as the *one-sided* LND.

The above definition was extended by Cuzick [15] who defined local ϕ-nondeterminism by replacing the variance $\sigma^2(t^m, t^{m-1})$ by $\phi(t^m - t^{m-1})$, where ϕ is a positive function such that $\lim_{r\to 0+} \phi(r) = 0$. Pitt [28] further extended Berman's definition (2.1) of LND to the case of random fields $\{Z(t), t \in \mathbb{R}^N\}$ by introducing a way to order the points $t^1, t^2, \ldots, t^m \in \mathbb{R}^N$ (however, this ordering causes non-negligible loss of precision in estimating moments of local times). Roughly speaking, (2.1) suggests that the increments of Z are asymptotically independent so that many of the results on the local times of Brownian motion can be extended to general Gaussian processes and fields. See Geman and Horowitz [18] for an excellent survey. We should also mention that, in recent years, the properties of *strong local nondeterminism (SLND)* for Gaussian random fields have found important applications in investigating small ball probabilities, exact Hausdorff measure of functions for the trajectories and laws of the iterated logarithm for their local times. Such results can not be established based on the property of local nondeterminism defined in (2.1). We refer to Xiao [30, 31] for further information on properties of SLND and their applications.

The goal of this section is to show the following theorem. For simplicity, we take $I = [\varepsilon, 1]^N$, where $\varepsilon \in (0, 1)$ is a fixed real number.

Theorem 2.1 *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a harmonisable-type multifractional Brownian motion with values in $\mathbb{R}$ defined by (1.1). For any integer $m \ge 1$, there exists a constant $c^{(m)} > 0$, depending on a, b, c_1, β, N, m and I only, such that for all $t^0, t^1, \ldots, t^m \in I$, the following inequality for the conditional variance holds:*

$$\sqrt{\operatorname{Var}\Big(X(t^0)\Big|X(t^n) : 1 \le n \le m\Big)} \ge c^{(m)} \min\Big\{|t^n - t^0|^{H(t^0)} \,:\, 1 \le n \le m\Big\}. \tag{2.2}$$

Remark 2.2 Now we verify that when $N = 1$ (2.2) implies that mfBm X satisfies (2.1). Indeed, if (2.2) holds, then for all $\varepsilon \le t^1 < t^2 < \cdots < t^m \le 1$, we have

$$\operatorname{Var}\Big(X(t^m) - X(t^{m-1})\Big|X(t^n) : 1 \le n \le m-1\Big) \ge \big(c^{(m-1)}\big)^2\, |t^m - t^{m-1}|^{2H(t^m)}.$$

By using Lemma 2.12 below, we see that (2.1) holds. The same argument applies to the case $N > 1$ and it shows that (2.2) implies that X satisfies the local nondeterminsim in the sense of Pitt [28].

We point out that when $N = 1$, (2.2) is stronger than (2.1) because the points $t^0, t^1, \ldots, t^m$ do not have to be ordered and, in particular, $t^1, \ldots, t^m$ can be taken from both left and right side of t^0. For this reason, (2.2) is referred to as the *two-sided* local nondeterminism. When $N > 1$, the property (2.2) is more natural than Pitt's definition of local nondeterminism in

[28], again because the points $t^0, t^1, \ldots, t^m$ do not have to be ordered. Hence, in this paper, we will refer to (2.2) as the property of local nondeterminism of the mfBm X. $\square$

Let us also compare Theorem 2.1 with the one-sided LND of the multifractional Brownian motion $\{B^{H(t)}(t), t \geq 0\}$ defined by using a moving average representation. Recall from Lévy-Véhel and Peltier [23] that

$$B^{H(t)}(t) = \frac{1}{\Gamma(H(t)+\frac{1}{2})} \left\{ \int_{-\infty}^{0} \left[(t-u)_+^{H(t)-\frac{1}{2}} - (-u)_+^{H(t)-\frac{1}{2}} \right] B(du) \right. \\ \left. + \int_0^t (t-u)^{H(t)-\frac{1}{2}}\, B(du) \right\}, \qquad \forall\, t \in \mathbb{R}_+, \tag{2.3}$$

where $B = \{B(s),\ s \in \mathbb{R}\}$ is a two-sided real-valued Brownian motion. Then for any $0 < s < t$, the independence of increments of Brownian motion implies

$$\begin{aligned} \mathrm{Var}\left(B^{H(t)}(t) \Big| B^{H(u)}(u) : u \leq s \right) &\geq \mathrm{Var}\left(B^{H(t)}(t) \Big| B(u) : u \leq s \right) \\ &= \frac{1}{\left(\Gamma(H(t)+\frac{1}{2})\right)^2} \int_s^t (t-u)^{2H(t)-1}\, du \\ &\geq c\,(t-s)^{2H(t)}, \end{aligned}$$

see, e.g., [13]. Hence, the argument in Remark 2.2 shows that $\{B^{H(t)}(t), t \geq 0\}$ satisfies Berman's one-sided local nondeterminism. However, the above method does not seem to be sufficient for determining whether $\{B^{H(t)}(t), t \geq 0\}$ satisfies a two-sided local nondeterminism.

The proof of Theorem 2.1 mainly relies on the following proposition.

Proposition 2.3 *There is a constant $c_0^{(m)} > 0$, depending on a, b, c_1, β, N, m and I only, such that for all $t^0, t^1, \ldots, t^m \in I$ and all real numbers $\alpha_1, \ldots, \alpha_m$ verifying*

$$\max\{|\alpha_n| \,:\, 1 \leq n \leq m\} \leq 2, \tag{2.4}$$

one has

$$\left(\mathbb{E}\Big| X(t^0) - \sum_{n=1}^{m} \alpha_n X(t^n) \Big|^2 \right)^{1/2} \geq c_0^{(m)} \min\left\{ |t^n - t^0|^{H(t^0)} \,:\, 1 \leq n \leq m \right\}. \tag{2.5}$$

In order to show Proposition 2.3, we first introduce some notations and establish some preliminary results. We denote by $Y = \{Y(t,\theta) \,:\, (t,\theta) \in \mathbb{R}^N \times [a,b]\}$ the real valued centered Gaussian field defined for each $(t,\theta) \in \mathbb{R}^N \times [a,b]$ as

$$Y(t,\theta) := \int_{\mathbb{R}^N} \frac{e^{it\cdot\xi} - 1}{|\xi|^{\theta+N/2}}\, d\widehat{W}(\xi). \tag{2.6}$$

Observe that (1.1) and (2.6) imply that for each $t \in \mathbb{R}^N$,

$$X(t) = Y(t, H(t)). \tag{2.7}$$

We denote by

$$\Big\{ 2^{jN/2} \psi_l(2^j s - k) \,:\, 1 \leq l \leq 2^N - 1,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^N \Big\},$$

a Lemarié-Meyer wavelet basis of $L^2(\mathbb{R}^N)$ [22, 25, 16]. For each $l = 1, \ldots, 2^N - 1$ and $(t, \theta) \in \mathbb{R}^N \times \mathbb{R}$, we set

$$\Psi_l(t, \theta) := \int_{\mathbb{R}^N} e^{it\cdot\xi} \frac{\widehat{\psi}_l(\xi)}{|\xi|^{\theta + N/2}}\, d\xi; \tag{2.8}$$

the last integral exists since $\widehat{\psi}_l$ is a compactly supported C^∞-function vanishing in a neighbourhood of the origin, more precisely one has

$$\operatorname{supp} \widehat{\psi}_l \subseteq \mathcal{D} := \Big[-\frac{8\pi}{3}, \frac{8\pi}{3} \Big]^N \setminus \Big(-\frac{2\pi}{3}, \frac{2\pi}{3} \Big)^N. \tag{2.9}$$

The following lemma means that the function $t \mapsto \Psi_l(t, \theta)$ decreases at infinity, uniformly in $\theta \in [a, b]$, faster than any polynomial. It will play an important role in the sequel.

Lemma 2.4 *Ψ_l is a continuous function (and even a C^∞-function) on $\mathbb{R}^N \times \mathbb{R}$ and satisfies the following property: For each $\theta \in \mathbb{R}$, $\Psi_l(\cdot, \theta)$ belongs to $\mathcal{S}(\mathbb{R}^N)$. Moreover, for all real numbers $\lambda > 0$ and $a < b$* [1], *there exists a constant $c_2 > 0$,* only depending on a, b, N and λ, *such that the inequality*

$$\sup_{\theta \in [a,b]} |\Psi_l(t, \theta)| \leq c_2 \prod_{u=1}^{N} (3 + |t_u|)^{-\lambda} \tag{2.10}$$

holds for every $t = (t_1, \ldots, t_N) \in \mathbb{R}^N$.

The proof of Lemma 2.4 is similar to that of part (b) of Proposition 3.1 in [7]. Let us give it for the sake of completeness.

Proof First, it is convenient to introduce some notation. For any $t \in \mathbb{R}^N$, let $\delta \in \mathbb{N}^N$ be such that for every u, $\delta_u = 3$ if $t_u \geq 0$ and $\delta_u = -3$ otherwise. This implies that $\delta_u + t_u = 3 + |t_u|$ if $t_u \geq 0$ and $\delta_u + t_u = -3 - |t_u|$ otherwise. For all $\theta \in \mathbb{R}$ and $\xi \in \mathbb{R}^N \setminus \{0\}$ we set

$$f_\theta(\xi) := |\xi|^{-\theta - N/2} = \left(\sum_{l=1}^{N} \xi_l^2 \right)^{-\frac{\theta}{2} - \frac{N}{4}}, \tag{2.11}$$

$$g_{l,\delta}(\xi) := e^{-i\delta\cdot\xi} \widehat{\psi}_l(\xi), \tag{2.12}$$

and

$$h_{l,\delta,\theta}(\xi) := g_{l,\delta}(\xi) f_\theta(\xi). \tag{2.13}$$

[1] Note that only in Lemma 2.4 we do not need to assume that $a, b \in (0, 1)$.

Observe that $h_{l,\delta,\theta}$ is a C^∞-compactly supported function which has the same support as $\widehat{\psi}_l$. We denote by L a fixed integer greater than λ and for all $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{Z}_+^N$ we denote by ∂^γ the differential operator

$$\partial^\gamma = \frac{\partial^{\gamma_1+\ldots+\gamma_N}}{(\partial\xi_1)^{\gamma_1}\ldots(\partial\xi_N)^{\gamma_N}},$$

with the usual convention that ∂^0 is the identity. By using integration by parts LN times we obtain

$$\begin{aligned} |\Psi_l(t,\theta)| &= \Big|\int_{\mathcal{D}} e^{i(\delta+t)\cdot\xi} h_{l,\delta,\theta}(\xi)\,d\xi\Big| \\ &= \prod_{u=1}^{N}(3+|t_u|)^{-L}\Big|\int_{\mathcal{D}} e^{i(\delta+t)\cdot\xi}\big(\partial^{\langle L\rangle}h_{l,\delta,\theta}\big)(\xi)\,d\xi\Big|, \end{aligned}$$

where $\langle L\rangle$ denotes the multi-index of $\mathbb{N}^N$ whose components are all equal to L. It remains to show that

$$\sup\Big\{\big|\big(\partial^{\langle L\rangle}h_{l,\delta,\theta}\big)(\xi)\big| \,:\, (\xi,\theta)\in\mathcal{D}\times[a,b] \text{ and } \delta\in\{-3,3\}^N\Big\} < \infty. \tag{2.14}$$

It follows from (2.13) and from the Leibniz formula that

$$\big(\partial^{\langle L\rangle}h_{l,\delta,\theta}\big)(\xi) = \sum_{(\gamma,\tau)\in\mathbb{Z}_+^N\times\mathbb{Z}_+^N,\,\gamma+\tau=\langle L\rangle} \mathcal{C}^\gamma_{\langle L\rangle}\big(\partial^\gamma f_\theta\big)(\xi)\big(\partial^\tau g_{l,\delta}\big)(\xi), \tag{2.15}$$

where

$$\mathcal{C}^\gamma_{\langle L\rangle} := \prod_{u=1}^{N}\frac{L!}{(\gamma_u)!\times(\tau_u)!}.$$

Let us now show by induction on $|\gamma| = \gamma_1+\ldots+\gamma_N$ that for all $\theta\in\mathbb{R}$ and $\xi\in\mathbb{R}^N\setminus\{0\}$,

$$\big(\partial^\gamma f_\theta\big)(\xi) = \sum_{p=0}^{|\gamma|} Q_{\gamma,p}(\xi,\theta)|\xi|^{-\theta-N/2-2p}, \tag{2.16}$$

where the $Q_{\gamma,p}$'s are polynomials on $\mathbb{R}^{N+1}$ only depending on γ. It is clear that (2.16) is satisfied when $|\gamma| = 0$, we just have to take $Q_{0,0}(\xi,\theta) = 1$. Next, we assume that (2.16) holds for all $\gamma\in\mathbb{Z}_+^N$ satisfying $|\gamma|\le n$, where n is an arbitrary fixed nonnegative integer. Let us then show that this equality also holds for all $\widetilde{\gamma}\in\mathbb{Z}_+^N$ satisfying $|\widetilde{\gamma}| = n+1$. There is $u_0\in\{1,\dots,N\}$ such that $\widetilde{\gamma} = \gamma+\nu^{u_0}$ where $|\gamma|\le n$ and where $\nu^{u_0}\in\mathbb{Z}_+^N$ is such that $\nu^{u_0}_{u_0} = 1$ and $\nu^{u_0}_u = 0$ for all $u\ne u_0$. It follows from the induction hypothesis (i.e. (2.16)) and from (2.11) that

$$\big(\partial^{\widetilde{\gamma}} f_\theta\big)(\xi) = \big(\partial^{\nu^{u_0}}\partial^\gamma f_\theta\big)(\xi) = \sum_{p=0}^{|\widetilde{\gamma}|} Q_{\widetilde{\gamma},p}(\xi,\theta)|\xi|^{-\theta-N/2-2p},$$

where $Q_{\widetilde{\gamma},0}(\xi,\theta) = \big(\partial^{\nu^{u_0}} Q_{\gamma,0}\big)(\xi,\theta)$, for all $1 \le p \le |\widetilde{\gamma}|-1$, $Q_{\widetilde{\gamma},p}(\xi,\theta) = \big(\partial^{\nu^{u_0}} Q_{\gamma,p}\big)(\xi,\theta) - (\theta+N/2+2p-2)\xi_{u_0} Q_{\gamma,p-1}(\xi,\theta)$ and $Q_{\widetilde{\gamma},|\widetilde{\gamma}|}(\xi,\theta) = -(\theta+N/2+2|\widetilde{\gamma}|-2)\xi_{u_0} Q_{\gamma,|\widetilde{\gamma}|-1}(\xi,\theta)$. This proves (2.16).

It follows from (2.16) and (2.9) that

$$\begin{aligned} &\sup\big\{\big|\big(\partial^{\gamma} f_\theta\big)(\xi)\big| \,:\, (\xi,\theta)\in\mathcal{D}\times[a,b]\big\} \\ &\le c\sum_{p=0}^{|\gamma|}\sup\big\{\big|Q_{\gamma,p}(\xi,\theta)\big| \,:\, (\xi,\theta)\in\mathcal{D}\times[a,b]\big\} < \infty, \end{aligned} \tag{2.17}$$

where $c>0$ is a finite constant only depending on a, b, N and γ. Let us now show that

$$\sup\big\{\big|\big(\partial^{\tau} g_{l,\delta}\big)(\xi)\big| \,:\, \xi\in\mathcal{D} \text{ and } \delta\in\{-3,3\}^N\big\} < \infty. \tag{2.18}$$

It follows from (2.12) and the Leibniz formula that

$$\big(\partial^{\tau} g_{l,\delta}\big)(\xi) = \sum_{(\tau',\tau'')\in\mathbb{Z}_+^N\times\mathbb{Z}_+^N,\, \tau'+\tau''=\tau} \mathcal{C}_\tau^{\tau'} (-i\delta)^{\tau'} e^{-i\delta\cdot\xi}\big(\partial^{\tau''}\widehat{\psi}_l\big)(\xi),$$

where $(-i\delta)^{\tau'} := \prod_{u=1}^N (-i\delta_u)^{\tau'_u}$. Thus

$$\begin{aligned} &\sup\big\{|\big(\partial^{\tau} g_{l,\delta}\big)(\xi)| \,:\, \xi\in\mathcal{D} \text{ and } \delta\in\{-3,3\}^N\big\} \\ &\le \sum_{(\tau',\tau'')\in\mathbb{N}^N\times\mathbb{N}^N,\, \tau'+\tau''=\tau} \mathcal{C}_\tau^{\tau'} 3^{|\tau'|}\sup\Big\{\big|\big(\partial^{\tau''}\widehat{\psi}_l\big)(\xi)\big| \,:\, \xi\in\mathcal{D}\Big\} < \infty. \end{aligned}$$

Finally putting together (2.15), (2.17) and (2.18) we get (2.14). □

Let us now give a random wavelet type series representation for the field Y.

Proposition 2.5 *The field $\{Y(t,\theta) \,:\, (t,\theta)\in\mathbb{R}^N\times[a,b]\}$ defined in (2.6) can be represented as*

$$Y(t,\theta) = \sum_{l=1}^{2^N-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^N} 2^{-j\theta}\epsilon_{l,j,k}\Big(\Psi_l(2^j t-k,\theta) - \Psi_l(-k,\theta)\Big), \tag{2.19}$$

where the Ψ_l's are the deterministic functions introduced in (2.8) and where $\{\epsilon_{l,j,k} \,:\, 1\le l\le 2^N-1,\ j\in\mathbb{Z},\ k\in\mathbb{Z}^N\}$ is the sequence of independent $\mathcal{N}(0,1)$ Gaussian random variables defined as

$$\epsilon_{l,j,k} := 2^{-Nj/2}\int_{\mathbb{R}^N} e^{i2^{-j}k\cdot\xi}\,\overline{\widehat{\psi}_l(2^{-j}\xi)}\, d\widehat{W}(\xi). \tag{2.20}$$

Moreover, for every fixed $(t,\theta)\in\mathbb{R}^N\times[a,b]$, the series in (2.19) is convergent in $L^2(\Omega)$, where Ω denotes the underlying probability space.

Remark 2.6 It can be verified that, with probability 1, the series is uniformly convergent in (t,θ) on every compact subset of $\mathbb{R}^N\times[a,b]$ (see [6] for a proof when $N=1$). □

Proof of Proposition 2.5 The proof is standard. For every fixed $(t,\theta)\in\mathbb{R}^N\times[a,b]$, we expand the function $\xi\mapsto\frac{e^{it\cdot\xi}-1}{|\xi|^{\theta+N/2}}$ in the orthonormal basis of $L^2(\mathbb{R}^N)$,

$$\Big\{2^{-Nj/2}e^{i2^{-j}k\cdot\xi}\,\overline{\widehat{\psi_l}(2^{-j}\xi)}\,:\,1\le l\le 2^N-1,\,j\in\mathbb{Z},\,k\in\mathbb{Z}^N\Big\}$$

and then we use the isometry property of the stochastic integral in (2.6). The proposition follows. □

From now on, we denote Ψ_1 by Ψ and we assume that the corresponding multivariate Lemarié-Meyer mother wavelet ψ_1 is, for each $x\in\mathbb{R}^N$, of the form

$$\psi_1(x)=\tilde{\psi}(x_1)\tilde{\varphi}(x_2)\ldots\tilde{\varphi}(x_N), \tag{2.21}$$

where $\tilde{\psi}$ is a univariate Lemarié-Meyer mother wavelet and $\tilde{\varphi}$ is a univariate Lemarié-Meyer scaling function. We set

$$\rho=2^{-1}\varepsilon N^{-1/2}\min\{|t^n-t^0|\,:\,1\le n\le m\}. \tag{2.22}$$

Observe that the fact that $t^n\in[\varepsilon,1]^N$ for all $n=0,\ldots,m$, implies that $|t^n-t^0|\le N^{1/2}$ and consequently that $\rho\in[0,\varepsilon/2]$. It is clear that Theorem 2.1 holds when $\rho=0$, so in all the sequel we assume that $\rho>0$. We denote by $\delta\ge 1$ a constant whose value will be chosen more precisely later and we denote by $j_0(\rho)=j_0(\rho,\delta)$ the unique (nonnegative) integer satisfying

$$2^{-j_0(\rho)-1}<\delta^{-1}\rho\le 2^{-j_0(\rho)}. \tag{2.23}$$

Observe that for every $n=0,1,\ldots,m$ one has

$$2^{-j_0(\rho)H(t^n)}\ge\delta^{-b}\rho^{H(t^n)}\quad\text{and}\quad 2^{-j_0(\rho)H(t^n)}\le 2^b\delta^{-a}\rho^{H(t^n)}. \tag{2.24}$$

We set

$$\mathcal{I}(\rho)=\Big\{n\in\{1,\ldots,m\}\,:\,|t^n-t^0|\le\rho^{1/2}\Big\} \tag{2.25}$$

and

$$\mathcal{I}^c(\rho)=\Big\{n\in\{1,\ldots,m\}\,:\,|t^n-t^0|>\rho^{1/2}\Big\}. \tag{2.26}$$

Lemma 2.7 *For any $n\in\mathcal{I}(\rho)$ one has*

$$2^{-j_0(\rho)H(t^n)}\le c_3 2^b\delta^{-a}\rho^{H(t^0)}, \tag{2.27}$$

where the constant $c_3=\sup\{r^{-c_1r^{\beta/2}}\,:\,r\in(0,1]\}<\infty$ (recall that β and c_1 are as in (1.2)).

Proof Using (2.24), the fact that $\rho \in (0,1]$, (1.2) and (2.25) one derives that

$$\begin{aligned} 2^{-j_0(\rho)H(t^n)} &\le 2^b \delta^{-a} \rho^{-|H(t^n)-H(t^0)|} \rho^{H(t^0)} \\ &\le 2^b \delta^{-a} \rho^{-c_1|t^n-t^0|^\beta} \rho^{H(t^0)} \le 2^b \delta^{-a} \rho^{-c_1 \rho^{\beta/2}} \rho^{H(t^0)}. \end{aligned}$$

This proves (2.27). □

Lemma 2.8 *For any real $T > 0$, $s \in \mathbb{R}^N$, $j \in \mathbb{N}_0 := \mathbb{Z}_+$, $\theta \in [a,b]$ and $l \in \{1,\dots,N\}$, let*

$$R_j(s,\theta;T,l) = \sum_{k \in \mathcal{K}_j(s_l,T)} |\Psi(2^j s - k, \theta)|^2, \tag{2.28}$$

where

$$\mathcal{K}_j(s_l,T) = \{k \in \mathbb{Z}^N \ : \ |2^j s_l - k_l| \ge T\}. \tag{2.29}$$

Then for all $\lambda > 1/2$ there is a constant $c_4 > 0$, only depending on λ, a, b and N, such that for any real $T > 0$, $s \in \mathbb{R}^N$, $j \in \mathbb{N}_0$, $\theta \in [a,b]$ and $l \in \{1,\dots,N\}$,

$$R_j(s,\theta;T,l) \le c_4 T^{1-2\lambda}. \tag{2.30}$$

Proof First observe that (2.10), (2.28), (2.29) and the fact that the function $y \mapsto (3+y)^{-2\lambda}$ is decreasing on $\mathbb{R}_+$, imply that

$$R_j(s,\theta;T,l) \le c_5 R(T), \tag{2.31}$$

where

$$c_5 := 2c_2 \Big(\sup_{x \in \mathbb{R}} \sum_{q \in \mathbb{Z}} \Big(3 + |x-q|\Big)^{-2\lambda} \Big)^{N-1} < \infty$$

and

$$R(T) = \sum_{v=0}^{+\infty} (3+v+T)^{-2\lambda}. \tag{2.32}$$

Moreover, using the fact that for every fixed $T \in \mathbb{R}_+$, the function $x \mapsto (3+x+T)^{-2\lambda}$ is decreasing on $[-1,+\infty)$ one obtains that

$$\sum_{v=0}^{+\infty} (3+v+T)^{-2\lambda} \le \int_0^{+\infty} (2+x+T)^{-2\lambda}\, dx = \frac{1}{2\lambda-1}(2+T)^{-2\lambda+1}. \tag{2.33}$$

Finally, putting together (2.31), (2.32) and (2.33) proves the lemma. □

Lemma 2.9 *For any $s \in \mathbb{R}^N$ and $\theta \in [a,b]$, we set*

$$B_{j_0(\rho)}(t^0, s, \theta) = \sum_{k \in \mathcal{A}_{j_0(\rho)}(t^0,\rho)} |\Psi(2^{j_0(\rho)}s - k, \theta)|^2, \tag{2.34}$$

where

$$\mathcal{A}_{j_0(\rho)}(t^0, \rho) := \Big\{ k \in \mathbb{Z}^N : \ |t^0 - 2^{-j_0(\rho)}k| \le \frac{\rho}{2\sqrt{N}} \Big\}. \tag{2.35}$$

Then for each real $\lambda > 1/2$ there is a constant $c_6 > 0$ only depending on λ, a, b and N, such that for all $\theta \in [a,b]$ one has:

(i) For every $n \in \{1, \dots, m\}$,

$$B_{j_0(\rho)}(t^0, t^n, \theta) \le c_6 \delta^{1-2\lambda}. \tag{2.36}$$

(ii) For every $n \in \mathcal{I}^c(\rho)$ (cf. (2.26)),

$$B_{j_0(\rho)}(t^0, t^n, \theta) \le c_6\, \delta^{1-2\lambda} \rho^{\lambda - 1/2}. \tag{2.37}$$

(iii)

$$B_{j_0(\rho)}(t^0, 0, \theta) \le c_6\, \varepsilon^{1-2\lambda} \delta^{1-2\lambda} \rho^{2\lambda-1}. \tag{2.38}$$

Proof Let us first show that Part (i) holds. Fix $n \in \{1, \dots, m\}$, it follows from (2.22) there exists $l_0 \in \{1, \dots, N\}$ such that

$$|t^n_{l_0} - t^0_{l_0}| \ge \frac{2\rho}{\varepsilon} \ge \frac{\rho}{\sqrt{N}}. \tag{2.39}$$

Next observe that (2.23), (2.35) and (2.39) entail that for each $k \in \mathcal{A}_{j_0(\rho)}(t^0, \rho)$ one has,

$$|2^{j_0(\rho)} t^n_{l_0} - k_{l_0}| \ge 2^{j_0(\rho)} \Big(|t^n_{l_0} - t^0_{l_0}| - |t^0_{l_0} - 2^{-j_0(\rho)} k_{l_0}| \Big) \ge \frac{\delta}{4\sqrt{N}}. \tag{2.40}$$

Then (2.40), (2.34), (2.28) and Lemma 2.8 imply that

$$B_{j_0(\rho)}(t^0, t^n, \theta) \le R_{j_0(\rho)}(t^n, \theta; (16N)^{-1/2}\delta, l_0) \le c_4 (16N)^{\lambda - 1/2} \delta^{1-2\lambda}, \tag{2.41}$$

which proves (2.36).

Let us now show that Part (ii) holds. For any fixed $n' \in \mathcal{I}^c(\rho)$, it follows from (2.26) that there exists $l'_0 \in \{1, \dots, N\}$ such that

$$|t^{n'}_{l'_0} - t^0_{l'_0}| \ge \frac{\rho^{1/2}}{\sqrt{N}}. \tag{2.42}$$

Next observe that (2.23), (2.35) and (2.42) entail that for each $k \in \mathcal{A}_{j_0(\rho)}(t^0, \rho)$ one has,

$$|2^{j_0(\rho)} t^{n'}_{l'_0} - k_{l'_0}| \ge 2^{j_0(\rho)} \Big(|t^{n'}_{l'_0} - t^0_{l'_0}| - |t^0_{l'_0} - 2^{-j_0(\rho)} k_{l'_0}| \Big) \ge \frac{\delta \rho^{-1/2}}{4\sqrt{N}}. \tag{2.43}$$

Then (2.43), (2.34), (2.28) and Lemma 2.8 imply that

$$B_{j_0(\rho)}(t^0, t^{n'}, \theta) \le R_{j_0(\rho)}(t^{n'}, \theta; (16N)^{-1/2}\delta\rho^{-1/2}, l_0') \le c_4 (16N)^{\lambda-1/2}\delta^{1-2\lambda}\rho^{\lambda-1/2} \tag{2.44}$$

and (2.37) follows.

Finally, let us prove Part (iii). By using (2.23), the fact that $t^0 \in [\varepsilon,\, 1]^N$, (2.22) and (2.35) one has for each $k \in \mathcal{A}_{j_0(\rho)}(t^0, \rho)$ and each $l = 1, \dots, N$,

$$|k_{l_0}| \ge 2^{j_0(\rho)}\Big(|t_l^0| - |t_{l_0}^0 - 2^{-j_0(\rho)} k_{l_0}|\Big) \ge \frac{\varepsilon\delta\rho^{-1}}{4}. \tag{2.45}$$

Then (2.45), (2.34), (2.28) and Lemma 2.8 imply that

$$B_{j_0(\rho)}(t^0, 0, \theta) \le R_{j_0(\rho)}(0, \theta; 4^{-1}\varepsilon\delta\rho^{-1}, l) \le c_4 4^{2\lambda-1}\varepsilon^{1-2\lambda}\delta^{1-2\lambda}\rho^{2\lambda-1}.$$

Hence (2.38) holds. This finishes the proof of Lemma 2.9. □

Lemma 2.10 *For every $(x, \theta) \in \mathbb{R}^N \times [a, b]$ one sets*

$$f(x, \theta) := \sum_{k \in \mathbb{Z}^N} |\Psi(x - k, \theta)|^2. \tag{2.46}$$

- *(i) The series in (2.46) is convergent for each $(x, \theta) \in \mathbb{R}^N \times [a, b]$. Moreover f is a continuous function on $\mathbb{R}^N \times [a, b]$.*
- *(ii) One has*

$$c_7 := \inf\{f(x, \theta) \;:\; (x, \theta) \in \mathbb{R}^N \times [a, b]\} > 0. \tag{2.47}$$

Proof Let us first prove Part (i). Since Ψ is a continuous function on $\mathbb{R}^N \times [a, b]$, it is sufficient to show that the series in (2.46) is uniformly convergent on $[-d, d]^N \times [a, b]$ for each real $d > 0$. It follows from (2.10) that for any $\lambda > 1/2$ there is a constant $c_8 > 0$ only depending on λ, d, a and b, such that for any $(x, \theta) \in \mathbb{R}^N \times [a, b]$ one has

$$|\Psi(x, \theta)| \le c_8 \prod_{u=1}^N \Big(2 + d + |x_u|\Big)^{-\lambda}$$

and the latter equality implies that for any $k \in \mathbb{Z}^N$ one has

$$\begin{aligned}\sup_{(x,\theta)\in[-d,d]^N\times[a,b]} |\Psi(x - k, \theta)|^2 &\le c_8^2 \prod_{u=1}^N \sup_{x_u\in[-d,d]^N} \Big(2 + d + |x_u - k_u|\Big)^{-2\lambda} \\ &\le c_8^2 \prod_{u=1}^N \Big(2 + |k_u|\Big)^{-2\lambda}.\end{aligned}$$

Therefore one obtains that

$$\sum_{k\in\mathbb{Z}^N}\sup_{(x,\theta)\in[-d,d]^N\times[a,b]}|\Psi(x-k,\theta)|^2<\infty,$$

which entails that the series in (2.46) is uniformly convergent on $[-d,d]^N\times[a,b]$.

Next we prove Part (ii). Our proof is more or less inspired by that of Remark 3.6 in [2]. First observe that for every $(x,\theta)\in\mathbb{R}^N\times[a,b]$ and for every $k'\in\mathbb{Z}^N$,

$$f(x,\theta)=f(x+k',\theta)$$

and thus

$$c_7=\inf\{f(x,\theta)\,:\,(x,\theta)\in[0,1]^N\times[a,b]\}. \tag{2.48}$$

Since f is continuous and $[0,1]^N\times[a,b]$ is a compact subset of $\mathbb{R}^N\times[a,b]$, we derive from (2.48) that there exists $(\tilde{x},\tilde{\theta})\in[0,1]^N\times[a,b]$ such that $c_7=f(\tilde{x},\tilde{\theta})$. Suppose ad absurdum that $c_7=0$. Then (2.46) implies that for every $k\in\mathbb{Z}^N$, $\Psi(\tilde{x}-k,\tilde{\theta})=0$. In view of (2.8) and of the $(2\pi\mathbb{Z})^N$-periodicity of the function $\xi\mapsto e^{-ik\cdot\xi}$, the later equality is equivalent to

$$\int_{\mathbb{R}^N}e^{i(\tilde{x}-k)\cdot\xi}\frac{\widehat{\psi}_1(\xi)}{|\xi|^{\tilde{\theta}+N/2}}\,d\xi=\int_{[0,2\pi]^N}e^{-ik\cdot\xi}\Big(\sum_{n\in\mathbb{Z}^N}\frac{e^{i\tilde{x}\cdot(\xi+2\pi n)}\widehat{\psi}_1(\xi+2\pi n)}{|\xi+2\pi n|^{\tilde{\theta}+N/2}}\Big)\,d\xi=0. \tag{2.49}$$

Relation (2.49) means that all the Fourier coefficients of the continuous $(2\pi\mathbb{Z})^N$-periodic function

$$\xi\mapsto\sum_{n\in\mathbb{Z}^N}\frac{e^{i\tilde{x}\cdot(\xi+2\pi n)}\widehat{\psi}_1(\xi+2\pi n)}{|\xi+2\pi n|^{\tilde{\theta}+N/2}}$$

vanish. Therefore for all $\xi\in\mathbb{R}^N$,

$$\sum_{n\in\mathbb{Z}^N}\frac{e^{i\tilde{x}\cdot(\xi+2\pi n)}\widehat{\psi}_1(\xi+2\pi n)}{|\xi+2\pi n|^{\tilde{\theta}+N/2}}=0.$$

By taking in the latter equality $\xi=(\frac{4\pi}{3},0,\ldots,0)$ and by using (2.21) as well as the fact that

$$\operatorname{supp}\widehat{\widetilde{\varphi}}\subseteq\Big[-\frac{4\pi}{3},\frac{4\pi}{3}\Big]\ \text{ and }\ \operatorname{supp}\widehat{\widetilde{\psi}}\subseteq\Big[-\frac{8\pi}{3},-\frac{2\pi}{3}\Big]\cup\Big[\frac{2\pi}{3},\frac{8\pi}{3}\Big],$$

one obtains that

$$\frac{e^{i\tilde{x}_1\frac{4\pi}{3}}\widehat{\widetilde{\psi}}(\frac{4\pi}{3})(\widehat{\widetilde{\varphi}}(0))^{N-1}}{|\frac{4\pi}{3}|^{\tilde{\theta}+N/2}}=0,$$

which leads to a contradiction. Indeed one has (see [25] or [16]) $\widehat{\widetilde{\varphi}}(0)=1$ and $|\widehat{\widetilde{\psi}}(\frac{4\pi}{3})|^2=\sum_{l\in\mathbb{Z}}|\widehat{\widetilde{\psi}}(\frac{4\pi}{3}+2\pi l)|^2=1$. □

Lemma 2.11 *Set*

$$D_{j_0(\rho)}(t^0,\theta) = \sum_{k\in\mathcal{A}^c_{j_0(\rho)}(t^0,\rho)} \Big|\Psi(2^{j_0(\rho)}t^0 - k,\theta)\Big|^2, \tag{2.50}$$

where

$$\mathcal{A}^c_{j_0(\rho)}(t^0,\rho) = \Big\{k \in \mathbb{Z}^N : \ |t^0 - 2^{-j_0(\rho)}k| > \frac{\rho}{2\sqrt{N}}\Big\}. \tag{2.51}$$

Then for each real $\lambda > 1/2$ there is a constant $c_9 > 0$ only depending on λ, a, b and N, such that for all $\theta \in [a,b]$ one has,

$$D_{j_0(\rho)}(t^0,\theta) \le c_9\delta^{1-2\lambda}. \tag{2.52}$$

Proof For any $l = 1,\ldots,N$ let

$$\mathcal{A}^c_{l,j_0(\rho)}(t^0_l,\rho) = \Big\{k \in \mathbb{Z}^N : \ |t^0_l - 2^{-j_0(\rho)}k_l| > \frac{\rho}{2N}\Big\}. \tag{2.53}$$

Clearly,

$$\mathcal{A}^c_{j_0(\rho)}(t^0,\rho) \subseteq \cup_{l=1}^N \mathcal{A}^c_{l,j_0(\rho)}(t^0_l,\rho)$$

and, as a consequence, we have

$$D_{j_0(\rho)}(t^0,\theta) \le \sum_{l=1}^N \sum_{k\in\mathcal{A}^c_{l,j_0(\rho)}(t^0_l,\rho)} \Big|\Psi(2^{j_0(\rho)}t^0 - k,\theta)\Big|^2.$$

Therefore, it is sufficient to prove that for any real $\lambda > 1/2$, there exists a constant $c_{10} > 0$, depending only on λ, a, b and N, such that for every $\theta \in [a,b]$ and $l = 1,\ldots,N$, one has

$$\sum_{k\in\mathcal{A}^c_{l,j_0(\rho)}(t^0_l,\rho)} \Big|\Psi(2^{j_0(\rho)}t^0 - k,\theta)\Big|^2 \le c_{10}\delta^{1-2\lambda}. \tag{2.54}$$

It follows from (2.23) and (2.53) that any $k \in \mathcal{A}^c_{l,j_0(\rho)}(t^0_l,\rho)$ satisfies

$$|2^{j_0(\rho)}t^0_l - k_l| > \frac{\delta}{4N}. \tag{2.55}$$

Then (2.55), (2.28), (2.29) and Lemma 2.8 imply that

$$\begin{aligned}\sum_{k\in\mathcal{A}^c_{l,j_0(\rho)}(t^0_l,\rho)} \Big|\Psi(2^{j_0(\rho)}t^0 - k,\theta)\Big|^2 &\le R_{j_0(\rho)}(t^0,\theta;(4N)^{-1}\delta,l) \\ &\le c_4(4N)^{2\lambda-1}\delta^{1-2\lambda}.\end{aligned}$$

This proves (2.54) and the lemma. □

We are now in position to prove Proposition 2.3.

Proof of Proposition 2.3 Let us set

$$\sigma = \left(\mathbb{E}\Big|X(t^0) - \sum_{n=1}^{m} \alpha_n X(t^n)\Big|^2\right)^{1/2}.$$

Using (2.7), (2.19) and the fact that the $\varepsilon_{l,j,k}$'s are independent $\mathcal{N}(0,1)$ Gaussian random variables, one obtains that

$$\begin{aligned}\sigma \geq \Bigg[\sum_{k \in \mathcal{A}_{j_0(\rho)}(t^0,\rho)} \Big| & 2^{-j_0(\rho)H(t^0)} \Big(\Psi(2^{j_0(\rho)}t^0 - k, H(t^0)) - \Psi(-k, H(t^0))\Big) \\ & - \sum_{n=1}^{m} \alpha_n 2^{-j_0(\rho)H(t^n)} \Big(\Psi(2^{j_0(\rho)}t^n - k, H(t^n)) - \Psi(-k, H(t^n))\Big)\Big|^2 \Bigg]^{1/2}.\end{aligned} \tag{2.56}$$

Now, let us denote by $l^2(\mathbb{Z}^N)$ the Hilbert space of the complex-valued sequences $s = (s_k)_{k\in\mathbb{Z}^N}$ which satisfy

$$\|s\|_{l^2(\mathbb{Z}^N)} = \Big(\sum_{k\in\mathbb{Z}^N} |s_k|^2\Big)^{1/2} < \infty;$$

recall that $\|\cdot\|_{l^2(\mathbb{Z}^N)}$ is the canonical norm associated with the usual inner product on $l^2(\mathbb{Z}^N)$. Then, observe that the right hand side of (2.56) can be expressed as

$$\Big\| 2^{-j_0(\rho)H(t^0)} x - 2^{-j_0(\rho)H(t^0)} y - \sum_{n=1}^{m} \alpha_n 2^{-j_0(\rho)H(t^n)} u^n + \sum_{n=0}^{m} \alpha_n 2^{-j_0(\rho)H(t^n)} v^n \Big\|_{l^2(\mathbb{Z}^N)},$$

where $\alpha_0 = -1$ and where x, y, u^n $(1 \leq n \leq m)$ and v^n $(0 \leq n \leq m)$ are the sequences of $l^2(\mathbb{Z}^N)$ defined in the following way:

- for all $k \in \mathbb{Z}^N$, $x_k = \Psi(2^{j_0(\rho)}t^0 - k, H(t^0))$;
- for all $k \in \mathcal{A}^c_{j_0(\rho)}(t^0,\rho)$, $y_k = \Psi(2^{j_0(\rho)}t^0 - k, H(t^0))$, else $y_k = 0$;
- for all $n \in \{1,\dots,m\}$ and $k \in \mathcal{A}_{j_0(\rho)}(t^0,\rho)$, $u^n_k = \Psi(2^{j_0(\rho)}t^n - k, H(t^n))$, else $u^n_k = 0$;
- for all $n \in \{0,\dots,m\}$ and $k \in \mathcal{A}_{j_0(\rho)}(t^0,\rho)$, $v^n_k = \Psi(-k, H(t^n))$, else $v^n_k = 0$.

Thus using (2.56), the triangle inequality and the homogeneity property of $\|\cdot\|_{l^2(\mathbb{Z}^N)}$, one obtains that

$$\begin{aligned}\sigma &\geq 2^{-j_0(\rho)H(t^0)}\big\|x\big\|_{l^2(\mathbb{Z})} - \Big\|2^{-j_0(\rho)H(t^0)}y + \sum_{n=1}^{m}\alpha_n 2^{-j_0(\rho)H(t^n)}u^n - \sum_{n=0}^{m}\alpha_n 2^{-j_0(\rho)H(t^n)}v^n\Big\|_{l^2(\mathbb{Z}^N)} \\ &\geq 2^{-j_0(\rho)H(t^0)}\big\|x\big\|_{l^2(\mathbb{Z}^N)} - 2^{-j_0(\rho)H(t^0)}\big\|y\big\|_{l^2(\mathbb{Z}^N)} - \sum_{n=1}^{m}|\alpha_n|2^{-j_0(\rho)H(t^n)}\big\|u^n\big\|_{l^2(\mathbb{Z}^N)} \\ &\qquad - \sum_{n=0}^{m}|\alpha_n|2^{-j_0(\rho)H(t^n)}\big\|v^n\big\|_{l^2(\mathbb{Z}^N)}.\end{aligned} \tag{2.57}$$

Let us now conveniently bound each term in the right hand side of the latter inequality. One has,

$$2^{-j_0(\rho)H(t^0)}\big\|x\big\|_{l^2(\mathbb{Z}^N)} = 2^{-j_0(\rho)H(t^0)}\Big(\sum_{k\in\mathbb{Z}^N}\Big|\Psi(2^{j_0(\rho)}t^0-k,H(t^0))\Big|^2\Big)^{1/2}.$$

Therefore, it follows from (2.24), (2.46) and (2.47) that

$$2^{-j_0(\rho)H(t^0)}\big\|x\big\|_{l^2(\mathbb{Z})} \geq \delta^{-b}\rho^{H(t^0)}\sqrt{f(2^{j_0(\rho)}t^0,H(t^0))} \geq \sqrt{c_7}\delta^{-b}\rho^{H(t^0)}. \tag{2.58}$$

One has,

$$2^{-j_0(\rho)H(t^0)}\big\|y\big\|_{l^2(\mathbb{Z}^N)} = 2^{-j_0(\rho)H(t^0)}\Big(\sum_{k\in\mathcal{A}^c_{j_0(\rho)}(t^0,\rho)}\Big|\Psi(2^{j_0(\rho)}t^0-k,H(t^0))\Big|^2\Big)^{1/2}.$$

Therefore, it follows from (2.24), (2.50) and (2.52) that

$$2^{-j_0(\rho)H(t^0)}\big\|y\big\|_{l^2(\mathbb{Z}^N)} \leq 2^b\delta^{-a}\rho^{H(t^0)}\sqrt{D_{j_0(\rho)}(t^0,H(t^0))} \leq 2^b\rho^{H(t^0)}\sqrt{c_9}\,\delta^{-(\lambda+a-1/2)}. \tag{2.59}$$

One has for all $n\in\{1,\dots,m\}$,

$$2^{-j_0(\rho)H(t^n)}\big\|u^n\big\|_{l^2(\mathbb{Z}^N)} = 2^{-j_0(\rho)H(t^n)}\Big(\sum_{k\in\mathcal{A}_{j_0(\rho)}(t^0,\rho)}\Big|\Psi(2^{j_0(\rho)}t^n-k,H(t^n))\Big|^2\Big)^{1/2}.$$

Therefore, it follows from (2.34) that

$$2^{-j_0(\rho)H(t^n)}\big\|u^n\big\|_{l^2(\mathbb{Z}^N)} = 2^{-j_0(\rho)H(t^n)}\sqrt{B_{j_0(\rho)}(t^0,t^n,H(t^n))}.$$

Next, using (2.27) as well as (2.36), one gets that for all $n\in\mathcal{I}(\rho)$,

$$2^{-j_0(\rho)H(t^n)}\big\|u^n\big\|_{l^2(\mathbb{Z}^N)} \leq c_3\sqrt{c_6}\,2^b\,\rho^{H(t^0)}\delta^{-(\lambda+a-1/2)}. \tag{2.60}$$

On the other hand, combining (2.24) with the inequality $\rho^{H(t^n)}\leq\rho^a$ and (2.37), one obtains that for all $n\in\mathcal{I}^c(\rho)$,

$$2^{-j_0(\rho)H(t^n)}\big\|u^n\big\|_{l^2(\mathbb{Z}^N)} \leq \sqrt{c_6}\,2^b\,\delta^{-(\lambda+a-1/2)}\rho^{a+\lambda/2-1/4}. \tag{2.61}$$

One has for all $n \in \{0, \ldots, m\}$,

$$2^{-j_0(\rho)H(t^n)} \big\|v^n\big\|_{l^2(\mathbb{Z}^N)} = 2^{-j_0(\rho)H(t^n)} \Big(\sum_{k \in \mathcal{A}_{j_0(\rho)}(t^0,\rho)} \Big|\Psi(-k, H(t^n))\Big|^2 \Big)^{1/2}.$$

Therefore, it follows from (2.24), the inequality $\rho^{H(t^n)} \leq \rho^a$ and (2.38), that for all $n \in \{0, \ldots, m\}$,

$$2^{-j_0(\rho)H(t^n)} \big\|v^n\big\|_{l^2(\mathbb{Z}^N)} \leq 2^b \delta^{-a} \rho^a \sqrt{B_{j_0(\rho)}(t^0, 0, H(t^n))} \leq \sqrt{c_6}\, 2^b \varepsilon^{-(\lambda-1/2)} \delta^{-(\lambda+a-1/2)} \rho^{a+\lambda-1/2}. \tag{2.62}$$

Then, combining the inequalities (2.57) to (2.62), with (2.4) as well as the inequalities $\mathrm{card}(\mathcal{I}(\rho)) \leq m$ and $\mathrm{card}(\mathcal{I}^c(\rho)) \leq m$, one derives,

$$\begin{aligned}\sigma \geq \sqrt{c_7}\, \delta^{-b} \rho^{H(t^0)} &- 2^b \rho^{H(t^0)} \sqrt{c_9}\, \delta^{-(\lambda+a-1/2)} \\ &- c_3 \sqrt{c_6}\, m 2^{1+b} \rho^{H(t^0)} \delta^{-(\lambda+a-1/2)} \\ &- \sqrt{c_6}\, m 2^{1+b} \delta^{-(\lambda+a-1/2)} \rho^{a+\lambda/2-1/4} \\ &- \sqrt{c_6}\, (m+1) 2^{1+b} \varepsilon^{-(\lambda-1/2)} \delta^{-(\lambda+a-1/2)} \rho^{a+\lambda-1/2}\end{aligned}$$

and, consequently,

$$\begin{aligned}\sigma \geq \delta^{-b} \rho^{H(t^0)} \Big(\sqrt{c_7} &- 2^b \sqrt{c_9} \delta^{-(\lambda+a-b-1/2)} - c_3 \sqrt{c_6}\, m 2^{1+b} \delta^{-(\lambda+a-b-1/2)} \\ &- \sqrt{c_6}\, m 2^{1+b} \delta^{-(\lambda+a-b-1/2)} \rho^{a+\lambda/2-b-1/4} \\ &- \sqrt{c_6}\, (m+1) 2^{1+b} \varepsilon^{-(\lambda-1/2)} \delta^{-(\lambda+a-b-1/2)} \rho^{a+\lambda-b-1/2} \Big).\end{aligned} \tag{2.63}$$

Finally, by taking λ satisfies $\lambda > 2(b-a) + 1/2$, we derive from (2.63) that there exists a constant $\delta_0 > 1$, only depending on λ, a, b, ε, m, c_3, c_6 and c_9, such that

$$\sigma \geq 2^{-1} \sqrt{c_7}\, \delta_0^{-b} \rho^{H(t^0)}.$$

This proves (2.5). □

The following lemma is not only useful for proving Theorems 2.1 and 3.1 below, but also for deriving other sample path properties of mfBm.

Lemma 2.12 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be the mfBm as defined in (1.1). Then:*

(i) There is a constant $c_{11} > 0$, only depending on a, b, c_1 and N, and another constant $c'_{11} > 0$, such that for all $s, t \in [0,1]^N$,

$$c'_{11} |s-t|^{\max\{H(s),H(t)\}} \leq \Big(\mathbb{E}\big(|X(s) - X(t)|^2\big)\Big)^{1/2} \leq c_{11} |s-t|^{\max\{H(s),H(t)\}}. \tag{2.64}$$

(ii) There is a constant $c_{12} \geq 1$, only depending on a, b and N, such that

$$c_{12}^{-1} |t|^{H(t)} \leq \Big(\mathbb{E}\big(|X(t)|^2\big)\Big)^{1/2} \leq c_{12} |t|^{H(t)} \qquad \forall\, t \in [0,\, 1]^N. \tag{2.65}$$

Proof In order to prove (i), there is no loss of generality to assume $H(s) \geq H(t)$. It follows from (2.7) and the triangle inequality that

$$\begin{aligned}\Big(\mathbb{E}|X(s) - X(t)|^2\Big)^{1/2} &\leq \Big(\mathbb{E}|Y(s,H(s)) - Y(t,H(s))|^2\Big)^{1/2}\\ &\quad + \Big(\mathbb{E}|Y(t,H(s)) - Y(t,H(t))|^2\Big)^{1/2}.\end{aligned} \tag{2.66}$$

By using (2.6) and making the change of variable $\eta = |s-t|\xi$, one obtains that

$$\begin{aligned}\mathbb{E}|Y(s,H(s)) - Y(t,H(s))|^2 &= \int_{\mathbb{R}^N} \frac{|e^{i(s-t)\cdot\xi} - 1|^2}{|\xi|^{2H(s)+N}}\, d\xi\\ &= |s-t|^{2H(s)} \int_{\mathbb{R}^N} \frac{|e^{i(|s-t|^{-1}(s-t))\cdot\eta} - 1|^2}{|\eta|^{2H(s)+N}}\, d\eta\\ &= |s-t|^{2H(s)} \int_{\mathbb{R}^N} \frac{|e^{i\eta_1} - 1|^2}{|\eta|^{2H(s)+N}}\, d\eta,\end{aligned}$$

where η_1 is the first coordinate of η and the last equality is obtained by a rotation. Since $H(s) \in [a,b]$, the above implies that there exist positive and finite constants c_{13} and c_{14} such that

$$c_{13}\, |s-t|^{2H(s)} \leq \mathbb{E}|Y(s,H(s)) - Y(t,H(s))|^2 \leq c_{14}\, |s-t|^{2H(s)}. \tag{2.67}$$

To be concrete, we can take

$$c_{13} = \int_{|\eta_1|>1} \frac{|e^{i\eta_1} - 1|^2}{|\eta|^{2b+N}}\, d\eta$$

and

$$c_{14} = \int_{|\eta|\leq 1} |\eta|^{2-2b-N}\, d\eta + 4\int_{|\eta|>1} |\eta|^{-2a-N}\, d\eta.$$

On the other hand, (2.6), the Mean Value Theorem and (1.2) imply that

$$\begin{aligned}\mathbb{E}|Y(t,H(s)) - Y(t,H(t))|^2 &\leq \int_{\mathbb{R}^N} \frac{|e^{it\cdot\xi} - 1|^2}{|\xi|^N} \bigg(\frac{1}{|\xi|^{H(s)}} - \frac{1}{|\xi|^{H(t)}}\bigg)^2 d\xi\\ &\leq c_{15}\, |s-t|^{2\beta},\end{aligned} \tag{2.68}$$

where the constant

$$c_{15} = Nc_1^2 \int_{|\xi|\leq 1} |\xi|^{2-2b-N} (\log|\xi|)^2\, d\xi + 4c_1^2 \int_{|\xi|>1} |\xi|^{-2a-N} (\log|\xi|)^2\, d\xi < \infty.$$

Putting together (2.66), (2.67), (2.68) and the fact that $\beta > b \geq H(s)$ one obtains the upper bound in (2.64).

Next we show the lower bound in (2.64). It follows from (2.7) and the triangle inequality,

$$\begin{aligned}\Big(\mathbb{E}|X(s) - X(t)|^2\Big)^{1/2} &\geq \Big(\mathbb{E}|Y(s,H(s)) - Y(t,H(s))|^2\Big)^{1/2}\\ &\quad - \Big(\mathbb{E}|Y(t,H(s)) - Y(t,H(t))|^2\Big)^{1/2}.\end{aligned} \tag{2.69}$$

Let us assume that $|s-t| < \tau_0 := \min\Big\{1, \big(2^{-1}c_{15}^{-1}c_{13}\big)^{2^{-1}(\beta-b)^{-1}}\Big\}$, then putting together (2.67), (2.68), (2.69) and the fact that $\beta > b \geq H(s)$ one obtains the lower bound in (2.64). In order to show that the lower bound in (2.64) also holds when $|s-t| \geq \tau_0$ we are going to prove that

$$\inf\left\{\frac{\mathbb{E}|X(s)-X(t)|^2}{|s-t|^{\max\{H(s),H(t)\}}} \, : \, (s,t) \in [0,1]^N \times [0,1]^N \text{ and } |s-t| \geq \tau_0\right\} > 0. \tag{2.70}$$

Suppose ad absurdum that (2.70) is not true, then by using the fact that the function $(s,t) \mapsto \frac{\mathbb{E}|X(s)-X(t)|^2}{|s-t|^{\max\{H(s),H(t)\}}}$ is a continuous function over the compact set $\big\{(s,t) \in [0,1]^N \times [0,1]^N \, : \, |s-t| \geq \tau_0\big\}$, it follows that there exists $(\widetilde{s}, \widetilde{t}) \in [0,1]^N \times [0,1]^N$ satisfying $|\widetilde{s} - \widetilde{t}| \geq \tau_0$ and $\mathbb{E}|X(\widetilde{s}) - X(\widetilde{t})|^2 = 0$. Then combining the latter equality with (1.1) it follows that for all $\xi \in \mathbb{R}^N \setminus \{0\}$ one has

$$\frac{e^{i\widetilde{s}\cdot\xi} - 1}{|\xi|^{H(\widetilde{s})+N/2}} = \frac{e^{i\widetilde{t}\cdot\xi} - 1}{|\xi|^{H(\widetilde{t})+N/2}},$$

which leads to a contradiction.

Finally we prove (ii). Using (1.1) and setting $\eta = |t|\xi$ it follows that

$$\mathbb{E}|X(t)|^2 = \int_{\mathbb{R}^N} \frac{|e^{it\cdot\xi} - 1|^2}{|\xi|^{2H(t)+N}}\, d\xi = |t|^{2H(t)} \int_{\mathbb{R}^N} \frac{|e^{i(|t|^{-1}t)\cdot\eta} - 1|^2}{|\eta|^{2H(t)+N}}\, d\eta. \tag{2.71}$$

Since the last integral lies between the constants c_{13} and c_{14}, we see that (2.65) holds. $\square$

Now we are in position to prove Theorem 2.1.

Proof of Theorem 2.1 In fact it is sufficient to show that for any integer $m \geq 1$, there is constant a constant $c^{(m)} > 0$, only depending on a, b, c_1, β, N, m and I, such that for all $t^0, t^1, \ldots, t^m \in I$ and all real numbers $\alpha_1, \ldots, \alpha_m$, one has

$$\left(\mathbb{E}\Big|X(t^0) - \sum_{n=1}^{m} \alpha_n X(t^n)\Big|^2\right)^{1/2} \geq c^{(m)} \min\Big\{|t^n - t^0|^{H(t^0)} \, : \, 1 \leq n \leq m\Big\}. \tag{2.72}$$

We have already proved (see Proposition 2.3) that (2.72) is valid when $\max\{|\alpha_n| \, : \, 1 \leq n \leq m\} \leq 2$, thus it remains to show that it is also valid when $\max\{|\alpha_n| \, : \, 1 \leq n \leq m\} > 2$. We will proceed by induction on m and the method we will use is inspired by [9].

Let us first assume that $m = 1$ and we distinguish two cases: Either

$$|\alpha_1||t^1 - t^0|^{H(t^1)} \leq 2^{-1}\left(c_{11}c_{12}\right)^{-1} N^{\frac{a-b}{2}} \varepsilon^b \, |t^1 - t^0|^{H(t^0)},$$

or

$$|\alpha_1||t^1 - t^0|^{H(t^1)} > 2^{-1}(c_{11}c_{12})^{-1} N^{\frac{a-b}{2}} \varepsilon^b |t^1 - t^0|^{H(t^0)}.$$

In the above c_{11} and c_{12} are the constants in Lemma 2.12.

In the first case, we apply the triangle inequality, the inequality $|1-\alpha_1| > 1$ and Lemma 2.12 to derive

$$\begin{aligned}
\Big(\mathbb{E}|X(t^0) - \alpha_1 X(t^1)|^2\Big)^{1/2} &= \Big(\mathbb{E}|(1-\alpha_1)X(t^0) - \alpha_1(X(t^1) - X(t^0))|^2\Big)^{1/2} \\
&\geq |1-\alpha_1|\Big(E|X(t^0)|^2\Big)^{1/2} - |\alpha_1|\Big(\mathbb{E}|X(t^1) - X(t^0)|^2\Big)^{1/2} \\
&\geq c_{12}^{-1}|t^0|^{H(t^0)} - c_{11}|\alpha_1|\,|t^1 - t^0|^{\max\{H(t^1),H(t^0)\}} \qquad (2.73)\\
&\geq c_{12}^{-1}\varepsilon^b|t^1 - t^0|^{H(t^0)} - c_{11}N^{\frac{b-a}{2}}|\alpha_1|\,|t^1 - t^0|^{H(t^1)} \\
&\geq 2^{-1}c_{12}^{-1}\varepsilon^b|t^1 - t^0|^{H(t^0)}.
\end{aligned}$$

In the second case, we apply Proposition 2.3 to obtain

$$\begin{aligned}
\Big(\mathbb{E}|X(t^0) - \alpha_1 X(t^1)|^2\Big)^{1/2} &= |\alpha_1|\Big(\mathbb{E}|X(t^1) - \alpha_1^{-1}X(t^0)|^2\Big)^{1/2} \\
&\geq c_0^{(1)}|\alpha_1||t^1 - t^0|^{H(t^1)} \qquad (2.74)\\
&\geq 2^{-1}c_0^{(1)}(c_{11}c_{12})^{-1}N^{\frac{a-b}{2}}\varepsilon^b\,|t^1 - t^0|^{H(t^0)}.
\end{aligned}$$

Set $c^{(1)} = \min\{c_0^{(1)}, 2^{-1}c_{12}^{-1}\varepsilon^b, 2^{-1}c_0^{(1)}(c_{11}c_{12})^{-1}N^{\frac{a-b}{2}}\varepsilon^b\}$. It follows from Proposition 2.3, (2.73) and (2.74) that (2.72) is verified when $m=1$.

Let us now show that a sufficient condition for (2.72) holds for any $m \geq 2$, is that this inequality is satisfied when m is replaced by $m-1$. Let $n_0 \in \{1,\dots,m\}$ be such that

$$|\alpha_{n_0}| = \max\{|\alpha_n| \,:\, 1 \leq n \leq m\} \qquad (2.75)$$

and let c_{16} be the constant defined as

$$c_{16} = 2^{-1}N^{\frac{a-b}{2}}c^{(m-1)}c_{11}^{-1}. \qquad (2.76)$$

We will again argue by cases. First assume that

$$\begin{aligned}
&|\alpha_{n_0}|\min\Big\{|t^n - t^{n_0}|^{H(t^{n_0})} \,:\, 0 \leq n \leq m \text{ and } n \neq n_0\Big\} \\
&\qquad \geq c_{16}\min\Big\{|t^n - t^0|^{H(t^0)} \,:\, 1 \leq n \leq m\Big\}.
\end{aligned} \qquad (2.77)$$

Set $\alpha_0 = -1$, it follows from Proposition 2.3 and (2.77) that

$$\begin{aligned}
&\bigg(\mathbb{E}\Big|X(t^0) - \sum_{n=1}^m \alpha_n X(t^n)\Big|^2\bigg)^{1/2} \\
&= |\alpha_{n_0}|\bigg(\mathbb{E}\Big|X(t^{n_0}) - \sum_{n=0,n\neq n_0}^m(-\alpha_{n_0}^{-1})\alpha_n X(t^n)\Big|^2\bigg)^{1/2} \qquad (2.78)\\
&\geq c_0^{(m)}|\alpha_{n_0}||\min\Big\{|t^n - t^{n_0}|^{H(t^{n_0})} \,:\, 0 \leq n \leq m \text{ and } \ n \neq n_0\Big\} \\
&\geq c_{17}\min\Big\{|t^n - t^0|^{H(t^0)} \,:\, 1 \leq n \leq m\Big\},
\end{aligned}$$

where $c_{17} = c_0^{(m)} c_{16}$.

Next we assume that

$$\begin{aligned} |\alpha_{n_0}| \min\left\{ |t^n - t^{n_0}|^{H(t^{n_0})} \ : \ 0 \le n \le m \text{ and } n \ne n_0 \right\} \\ < c_{16}| \min\left\{ |t^n - t^0|^{H(t^0)} \ : \ 1 \le n \le m \right\} \end{aligned} \tag{2.79}$$

and let $n_1 \in \{0, \dots, m\} \setminus \{n_0\}$ be such that

$$|t^{n_1} - t^{n_0}| = \min\{|t^n - t^{n_0}| \ : \ 0 \le n \le m \text{ and } n \ne n_0\}. \tag{2.80}$$

We will show that

$$\begin{aligned} &\left(\mathbb{E}\Big| X(t^0) - \sum_{n=1}^m \alpha_n X(t^n) \Big|^2 \right)^{1/2} \\ &\ge c^{(m-1)} \min\left\{ |t^n - t^0|^{H(t^0)} \ : \ 1 \le n \le m \right\} \\ &\qquad - |\alpha_{n_0}| \left(\mathbb{E}\Big| X(t^{n_0}) - X(t^{n_1}) \Big|^2 \right)^{1/2}. \end{aligned} \tag{2.81}$$

We will again argue by cases. First, assume that it is possible to choose n_1 such that $n_1 \ne 0$. Thus it follows from the triangle inequality that,

$$\begin{aligned} &\left(\mathbb{E}\Big| X(t^0) - \sum_{n=1}^m \alpha_n X(t^n) \Big|^2 \right)^{1/2} \\ &\ge \left(\mathbb{E}\Big| X(t^0) - \sum_{n=1, n\ne n_0, n_1}^m \alpha_n X(t^n) - (\alpha_{n_1} + \alpha_{n_0}) X(t^{n_1}) \Big|^2 \right)^{1/2} \\ &\qquad - |\alpha_{n_0}| \left(\mathbb{E}\Big| X(t^{n_0}) - X(t^{n_1}) \Big|^2 \right)^{1/2}, \end{aligned}$$

with the convention that $\sum_{n=1, n\ne n_0, n_1}^m \alpha_n X(t^n) = 0$ when $m = 2$. Then, using the induction hypothesis we can show that (2.81) holds. Next, we assume that $n_1 = 0$. It follows from the triangle inequality, that,

$$\begin{aligned} &\left(\mathbb{E}\Big| X(t^0) - \sum_{n=1}^m \alpha_n X(t^n) \Big|^2 \right)^{1/2} \\ &\ge \left(\mathbb{E}\Big| (1 - \alpha_{n_0}) X(t^0) - \sum_{n=1, n\ne n_0}^m \alpha_n X(t^n) \Big|^2 \right)^{1/2} \\ &\qquad - |\alpha_{n_0}| \left(\mathbb{E}\Big| X(t^{n_0}) - X(t^0) \Big|^2 \right)^{1/2} \\ &= |1 - \alpha_{n_0}| \left(\mathbb{E}\Big| X(t^0) - \sum_{n=1, n\ne n_0}^m \frac{\alpha_n}{1 - \alpha_{n_0}} X(t^n) \Big|^2 \right)^{1/2} \\ &\qquad - |\alpha_{n_0}| \left(\mathbb{E}\Big| X(t^{n_0}) - X(t^0) \Big|^2 \right)^{1/2} \end{aligned}$$

Then, using the induction hypothesis and the inequality $|1-\alpha_{n_0}|>1$, we can show that (2.81) holds. Next, it follows from (2.81), Part (i) of Lemma 2.12, (2.80), (2.79) and (2.76), that

$$\begin{aligned}
&\left(\mathbb{E}\Big|X(t^0)-\sum_{n=1}^{m}\alpha_n X(t^n)\Big|^2\right)^{1/2}\\
&\geq c^{(m-1)}\,\min\Big\{|t^n-t^0|^{H(t^0)}\,:\,1\leq n\leq m\Big\}\\
&\qquad\qquad -c_{11}\,|\alpha_{n_0}|\,|t^{n_0}-t^{n_1}|^{\max\{H(t^{n_0}),H(t^{n_1})\}}.\\
&\geq c^{(m-1)}\,\min\Big\{|t^n-t^0|^{H(t^0)}\,:\,1\leq n\leq m\Big\}\\
&\qquad\qquad -c_{11}|\alpha_{n_0}|\,N^{\frac{b-a}{2}}\,|t^{n_0}-t^{n_1}|^{H(t^{n_0})}\\
&\geq 2^{-1}c^{(m-1)}\,\min\Big\{|t^n-t^0|^{H(t^0)}\,:\,1\leq n\leq m\Big\}.
\end{aligned}\tag{2.82}$$

Finally we take $c^{(m)}=\min\{c_0^{(m)},c_{17},\ 2^{-1}c^{(m-1)}\}$, then we derive from Proposition 2.3, (2.78) and (2.82) that (2.72) holds. □

3 Joint continuity of the local times

In this section, we let $X=\{X(t),t\in\mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined by

$$X(t)=(X_1(t),\ldots,X_d(t))\tag{3.1}$$

where $X_1,\ldots,X_d$ are independent copies of the real-valued mfBm with index function $H(t)$ as defined in (1.1). We call X a harmonisable-type (N,d)-multifractional Brownian motion. The main purpose of this section is to study joint continuity of the local times and other sample path properties of X.

For any Borel set $I\subseteq\mathbb{R}^N$, the occupation measure of X on I is defined as the following measure on $\mathbb{R}^d$:

$$\mu_I(\bullet)=\lambda_N\big\{t\in I:X(t)\in\bullet\big\}.$$

If μ_I is almost surely absolutely continuous with respect to λ_d, then X is said to have local times on I and its local time $L(\bullet,I)$ is defined as the Radon–Nikodým derivative of μ_I with respect to λ_d, i.e.,

$$L(x,I)=\frac{d\mu_I}{d\lambda_d}(x),\qquad \forall x\in\mathbb{R}^d.$$

In the above, x is the so-called *space variable*, and I is the *time variable.* Sometimes, we write $L(x,t)$ in place of $L(x,[0,t])$. Note that if X has local times on I then for every Borel set $J\subseteq I$, $L(x,J)$ also exists.

It can be proved (see [18], Theorem 6.4) that the local times have a measurable modification that satisfies the following *occupation density formula*: For every Borel function $f(t,x) \geq 0$ on $I \times \mathbb{R}^d$,

$$\int_I f(t, X(t))\, dt = \int_{\mathbb{R}^d} \int_I f(t,x) L(x, dt)\, dx. \tag{3.2}$$

Suppose we fix a rectangle $I = \prod_{i=1}^N [a_i, a_i + h_i]$. If there is a version of the local time, still denoted by $L(x, \prod_{i=1}^N [a_i, a_i + t_i])$, such that it is a continuous function of $(x, t_1, \cdots, t_N) \in \mathbb{R}^d \times \prod_{i=1}^N [0, h_i]$, X is said to have a *jointly continuous local time* on I. When a local time is jointly continuous, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on the level set

$$X^{-1}(x) \cap I = \{t \in I : X(t) = x\};$$

see Adler [1] for details. Hence local times are useful in studying various fractal properties of level sets and inverse images of the vector field X. We refer to Berman [10], Ehm [17], Xiao [30, 31] and the references therein for further information.

The main result of this section is the following theorem.

Theorem 3.1 *Let $X = \{X(t),\, t \in \mathbb{R}_+^N\}$ be a harmonisable-type (N,d)-multifractional Brownian motion defined by (3.1). Let $I = [\varepsilon, 1]^N$ and $\overline{H} = \max_{t \in I} H(t)$. If $N > \overline{H}d$, then*

(i) X admits an $L^2(\mathbb{R}^d)$-integrable local time $L(\cdot, I)$ almost surely.

(ii) X has a jointly continuous local time on I.

Proof of Theorem 3.1, Part (i) By Theorem 21.9 of Geman and Horowitz [18], it suffices to prove that

$$\int_I \int_I \left(\mathbb{E}\Big[X_1(t) - X_1(s)\Big]^2 \right)^{-d/2} dsdt < \infty. \tag{3.3}$$

It follows from Part (i) of Lemma 2.12 that

$$\int_I \int_I \left(\mathbb{E}\Big[X_1(t) - X_1(s)\Big]^2 \right)^{-d/2} dsdt \leq c_{18} \int_I \int_I \frac{dsdt}{|t-s|^{\overline{H}d}} < \infty$$

since $\overline{H}d < N$. This proves (3.3) and hence Part (i) of Theorem 3.1. □

In order to prove Part (ii) of Theorem 3.1, we first derive some moment estimates for the local times of X, and then apply a multiparameter version of Kolmogorov's continuity theorem. As it will be seen, the property of local nondeterminism proved in Theorem 2.1 plays an important role in proving Lemmas 3.2 and 3.3 below.

We will make use of the following identities (cf. (25.5) and (25.7) in Geman and Horowitz [18]): For all $x, y \in \mathbb{R}^d$, $T \in \mathcal{A}$ (the class of all closed intervals of $\mathbb{R}^N$) and all integers $n \geq 1$,

$$\mathbb{E}\Big[L(x,T)^n\Big] = (2\pi)^{-nd} \int_{T^n} \int_{\mathbb{R}^{nd}} \exp\Big(-i\sum_{j=1}^n u^j \cdot x\Big) \times \mathbb{E}\exp\Big(i\sum_{j=1}^n u^j \cdot X(t^j)\Big)\, d\overline{u}\; d\overline{t} \tag{3.4}$$

and for all even integers $n \geq 2$,

$$\mathbb{E}\Big[\big(L(x,T)-L(y,T)\big)^n\Big] = (2\pi)^{-nd} \int_{T^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \Big[e^{-iu^j\cdot x} - e^{-iu^j\cdot y}\Big] \times \mathbb{E}\exp\Big(i\sum_{j=1}^n u^j \cdot X(t^j)\Big)\, d\overline{u}\; d\overline{t}, \tag{3.5}$$

where $\overline{u} = (u^1, \ldots, u^n)$, $\overline{t} = (t^1, \ldots, t^n)$, and each $u^j \in \mathbb{R}^d$, $t^j \in T$. In the coordinate notation we then write $u^j = (u^j_1, \ldots, u^j_d)$.

Lemma 3.2 *Assume the conditions of Theorem 3.1 hold. Then, for every integer $n \geq 1$, there exists a positive constant $c_{19} = c_{19}(n)$, only depending on n, N, d, $\overline{H}$ and I, such that for all $x \in \mathbb{R}^d$, all subintervals $T = [\tau, \tau + \langle h\rangle] \subseteq I$ with $h > 0$ small one has*

$$\mathbb{E}[L(x,T)^n] \leq c_{19}\, h^{n(N-\overline{H}d)}. \tag{3.6}$$

Proof The main idea for proving (3.6) is similar to that of Lemma 2.5 in Xiao [30]. For completeness we provide a sketch of it.

Since the coordinate processes $X_1, \ldots, X_d$ are independent and identically distributed, we derive from (3.4) that for all integers $n \geq 1$,

$$\begin{aligned} \mathbb{E}\big[L(x,T)^n\big] &\leq (2\pi)^{-nd} \int_{T^n} \prod_{k=1}^d \Big\{ \int_{\mathbb{R}^n} \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^n u^j_k X_1(t^j)\Big)\Big]\, d\overline{u}_k \Big\}\, d\overline{t} \\ &= (2\pi)^{-nd/2} \int_{T^n} \Big[\det\mathrm{Cov}\big(X_1(t^1), \ldots, X_1(t^n)\big)\Big]^{-\frac{d}{2}}\, d\overline{t}, \end{aligned} \tag{3.7}$$

where $\overline{u}_k = (u^1_k, \ldots, u^n_k) \in \mathbb{R}^n$, $\overline{t} = (t^1, \ldots, t^n)$ and the equality follows from the fact that for any positive definite $n \times n$ matrix Γ,

$$\int_{\mathbb{R}^n} \frac{[\det(\Gamma)]^{1/2}}{(2\pi)^{n/2}} \exp\Big(-\frac{1}{2}x'\Gamma x\Big)\, dx = 1.$$

By applying the fact that for any Gaussian random vector $(Z_1, \ldots, Z_n)$,

$$\det\mathrm{Cov}(Z_1, \ldots, Z_n) = \mathrm{Var}(Z_1) \prod_{k=2}^n \mathrm{Var}(Z_k | Z_1, \ldots, Z_{k-1})$$

we derive from Theorem 2.1 and Part (ii) of Lemma 2.12 that there exists a positive constant $c(n)$ such that

$$\mathrm{detCov}\left(X_1(t^1), \ldots, X_1(t^n)\right) \geq c(n)|t^1|^{2\overline{H}} \prod_{j=2}^{n} \min\Big\{|t^j - t^i|^{2\overline{H}} \,:\, 1 \leq i \leq j-1\Big\}. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}\mathbb{E}\big[L(x,T)^n\big] &\leq c(n) \int_{T^n} \frac{1}{|t^1|^{\overline{H}d}} \prod_{j=2}^{n} \frac{1}{\min\left\{|t^j - t^i|^{\overline{H}d} \,:\, 1 \leq i \leq j-1\right\}}\, d\bar{t} \\ &\leq c_{19}(n)\, h^{n(N-\overline{H}d)},\end{aligned}$$

where the last inequality follows from the assumption $N > \overline{H}d$ and repeated use of the following inequality (Lemma 2.3 in Xiao [30]): There exists a positive and finite constant c_{20} such that for all integers $m \geq 1$ and all distinct $s^1, \cdots, s^m \in T = [\tau,\, \tau + \langle h \rangle]$, we have

$$\int_T \frac{dt}{\min\{|t - s^j|^{\overline{H}d},\ j = 1, \cdots, m\}} \leq c_{20}\, m^{\overline{H}d/N}\, h^{N-\overline{H}d}.$$

This finishes the proof of (3.6). □

Lemma 3.3 *Assume the conditions of Theorem 3.1 hold. Then, for every even integer* $n \geq 2$ *and* $\gamma \in (0,1)$ *small enough, there exists a positive and finite constant* $c_{21} = c_{21}(n)$*, only depending on* n, γ, N, d, $\overline{H}$ *and* I*, such that for all subintervals* $T = [\tau,\, \tau + \langle h \rangle] \subseteq I$ *and all* $x, y \in \mathbb{R}^d$ *with* $|x - y| \leq 1$,

$$\mathbb{E}\Big[\big(L(x,T) - L(y,T)\big)^n\Big] \leq c_{21}\, |x - y|^{n\gamma}\, h^{n(N-(\overline{H}+\gamma)d)}. \tag{3.9}$$

Proof The proof, using (3.5) and Theorem 2.1, is similar to that of Lemma 2.5 in Xiao [30]. We omit the details. □

Now we are ready to prove Part (ii) of Theorem 3.1.

Proof of Theorem 3.1, Part (ii) Note that, by Lemmas 3.2 and 3.3 we can choose the integers n large so that the powers of $h = \mathrm{diam}\, T/\sqrt{N}$ and $|x - y|$ in (3.6) and (3.9) are bigger than $N + d$ (or as large as we wish, which is the case for the proof of Theorem 3.5 below). Hence the joint continuity of the local time of X follows from a multiparameter version of Kolmogorov's continuity theorem (cf. Khoshnevisan [21]). See e.g., Xiao [31] for more details. □

Remark 3.4 Examining the proof of Lemma 3.2 we see that the following local version of (3.6) holds: If $\tau \in I$ and $N > H(\tau)d$, then for all $\delta > 0$ and $h > 0$ small enough

$$\mathbb{E}[L(x, B(\tau, h))^n] \le c_{22}\, h^{n(N-(H(\tau)+\delta)d)},$$

where $B(\tau, r) = \{t \in I : |t - \tau| \le r\}$. Similarly, for all even integers $n \ge 2$, all h, δ, $\gamma \in (0, 1)$ small enough and all x, $y \in \mathbb{R}^d$ with $|x - y| \le 1$,

$$\mathbb{E}\Big[\big(L(x, B(\tau, h)) - L(y, B(\tau, h))\big)^n\Big] \le c_{23}\, |x - y|^{n\gamma}\, h^{n(N-(H(\tau)+\delta+\gamma)d)}.$$

These two inequalities will be need for proving Theorem 3.5 below. □

By applying Lemmas 3.2, 3.3, Remark 3.4 and a chaining argument as in Ehm [17] and Xiao [30], one can prove the following local and uniform Hölder conditions for the maximum local time $L^*(T) = \sup_{x\in\mathbb{R}^d} L(x, T)$ of X.

Theorem 3.5 *Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be an (N, d)-mfBm as in Theorem 3.1.*

(i) If $\tau \in I$ and $N > H(\tau)d$, then for any $\delta > 0$, there exists a positive finite constant c_{24} such that

$$\limsup_{r\to 0} \frac{L^*(B(\tau, r))}{r^{N-(H(\tau)+\delta)d}} \le c_{24} \quad \text{a.s.} \tag{3.10}$$

(ii) If $N > \overline{H}d$, then for any $\delta > 0$, there exists a positive finite constant c_{25} such that

$$\limsup_{r\to 0} \sup_{\tau\in I} \frac{L^*(B(\tau, r))}{r^{N-(\overline{H}+\delta)d}} \le c_{25} \quad \text{a.s.} \tag{3.11}$$

We end this section with two applications of Theorem 3.5. The first is on the "modulus of non-differentiability" of mfBm.

Theorem 3.6 *Let $X = \{X(t), t \in \mathbb{R}^N_+\}$ be a mfBm with values in $\mathbb{R}$ defined by (1.1). Let $I = [\varepsilon, 1]^N$ and $\overline{H} = \max_{t\in I} H(t)$.*

(i) For any $\delta > 0$, there exists a constant $c_{26} > 0$ such that for every $\tau \in I$,

$$\liminf_{r\to 0} \sup_{t\in B(\tau,r)} \frac{|X(t) - X(\tau)|}{r^{H(\tau)+\delta}} \ge c_{26}, \quad a.s. \tag{3.12}$$

(ii) For any $\delta > 0$, there exists a constant $c_{27} > 0$ such that

$$\liminf_{r\to 0} \inf_{\tau\in I} \sup_{t\in B(\tau,r)} \frac{|X(t) - X(\tau)|}{r^{\overline{H}+\delta}} \ge c_{27}, \quad a.s. \tag{3.13}$$

In particular, the sample function $X(t)$ is almost surely nowhere differentiable in I.

Proof We apply Theorem 3.5 with $d = 1$. Similarly to Berman [10], we have that for any $\tau \in I$, there is a constant $C_N = C_N(\tau)$ such that

$$\begin{aligned} C_N\, r^N &= \int_{\overline{X(B(\tau,r))}} L(x, B(\tau, r))\, dx \\ &\le L^*(B(\tau, r)) \sup_{s,t\in B(\tau,r)} \big|X(s) - X(t)\big|. \end{aligned}$$

If τ is in the interior of I, $C_N(\tau)$ is the volume of the unit ball in $\mathbb{R}^N$; if τ lies on the boundary of I, then $C_N(\tau)$ is a deterministic portion (at least 2^{-N}) of the volume of the unit ball in $\mathbb{R}^N$. Hence (3.12) follows from (3.10) and (3.13) follows from (3.11). □

Our second application of Theorem 3.5 is to determine the Hausdorff dimension of the level sets of an (N, d)-mfBm. The fractal properties of the level sets of Gaussian random fields with stationary increments have been studied by many authors, see [1, 20, 21, 31]. Similar questions for one-parameter multifractional Brownian motion of moving average-type and multifractional Brownian sheets have been considered in [12, 26], respectively. By using standard covering and capacity arguments one can prove that, for every $x \in \mathbb{R}^d$

$$\dim_{\rm H}\big(X^{-1}(x) \cap I\big) \le N - \underline{H}d \qquad \text{a.s.} \tag{3.14}$$

and for any $\delta > 0$,

$$\dim_{\rm H}\big(X^{-1}(x) \cap I\big) \ge N - (\underline{H} + \delta)d \tag{3.15}$$

with positive probability, which may depend on x and δ. In the above, $\underline{H} = \min_{t\in I} H(t)$ and "$\dim_{\rm H}\big(X^{-1}(x) \cap I\big) < 0$" means "$X^{-1}(x) \cap I = \varnothing$" a.s. We can write (3.14) and (3.15) as

$$\big\|\dim_{\rm H}\big(X^{-1}(x) \cap I\big)\big\|_{L^\infty(\mathbb{P})} = N - \underline{H}d, \tag{3.16}$$

where $\|Z\|_{L^\infty(\mathbb{P})}$ denotes the essential supremum of a nonnegative random variable Z:

$$\|Z\|_{L^\infty(\mathbb{P})} := \inf\{\lambda > 0 : \mathbb{P}\{Z > \lambda\} = 0\} \qquad (\inf \varnothing := \infty).$$

Moreover, for every $\tau \in I$ such that $N > H(\tau)d$, we can apply the same arguments to $B(\tau, r) = \{t \in I : |t - \tau| \le r\}$ for $r > 0$ small enough to derive

$$D(x, \tau) = \lim_{r\to 0} \big\|\dim_{\rm H}\big(X^{-1}(x) \cap B(\tau, r)\big)\big\|_{L^\infty(\mathbb{P})} = N - H(\tau)d.$$

We call $D(x, \tau)$ the local Hausdorff dimension of the level set $X^{-1}(x)$ at $\tau \in I$. Similar to the singularity spectrum of a multifractal function or the local dimension of a multifractal measure, $D(x, \tau)$ gives a way to characterize the multifractal nature of $X^{-1}(x)$.

In the following, we prove a "uniform" version of (3.16). Since $\dim_{\rm H}\big(X^{-1}(x) \cap I\big)$ is determined by $\underline{H} = \min_{t\in I} H(t)$, it is natural to work with the following random set

$$\mathcal{U} = \Big\{x \in \mathbb{R}^d : L(x, B(t^0, r)) > 0 \ \text{ for all } \ r > 0\Big\}, \tag{3.17}$$

where $t^0 \in I$ satisfies $H(t^0) = \min_{t \in I} H(t)$ and $B(t^0, r) = \{t \in I : |t - t^0| \le r\}$. Since the local time $L(x, B(t^0, r))$ is continuous in x and nondecreasing in r, $\mathcal{U}$ is a G_δ set. The following result relates the random set $\mathcal{U}$ to the image set $X(I)$.

Lemma 3.7 *Assume the conditions of Theorem 3.8 hold. Then almost surely, $\mathcal{U} \subseteq X(I)$ and $X(t) \in \mathcal{U}$ for a.e. $t \in I$.*

Proof Recall that we have taken $I = [\varepsilon, 1]^N$, so $X(I)$ is a compact set. For any $y \notin X(I)$ there is a $\delta > 0$ such that $U(y, \delta) \cap X(I) = \varnothing$, where $U(y, \delta)$ denotes the open ball in $\mathbb{R}^d$ centered at y with radius δ. Take $f(t, x) = \mathbb{1}_{U(y,\delta)}(x)$ in (3.2) we obtain $L(x, I) = 0$ for a.e. $x \in U(y, \delta)$. It follows from the a.s. continuity of $x \mapsto L(x, I)$ that $L(x, I) = 0$ for every $x \in U(y, \delta)$ and, in particular, $L(y, I) = 0$. This proves $\mathcal{U} \subseteq X(I)$.

On the other hand, by using the occupation density formula (3.2) one can prove that almost surely

$$L\big(X(t), B(t, \delta)\big) > 0 \qquad \text{for all } \delta > 0 \ \text{ and a.e. } t \in I.$$

See (6.7) in Geman and Horowitz [18]. Therefore almost surely $X(t) \in \mathcal{U}$ for a.e. $t \in I$. □

Theorem 3.8 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-multifractional Brownian motion defined by (3.1). Let $I = [\varepsilon, 1]^N$ and $\overline{H} = \max_{t \in I} H(t)$. If $N > \overline{H} d$, then almost surely*

$$\dim_{\rm H}\big(X^{-1}(x) \cap I\big) = N - \underline{H} d \tag{3.18}$$

for every $x \in \mathcal{U}$ defined in (3.17).

Proof Eq. (2.64) of Lemma 2.12 implies that, for every $\delta > 0$, $X(t)$ $(t \in I)$ satisfies almost surely a uniform Hölder condition of order $\underline{H} - \delta$. By Theorem 3.1, the local time $L(x, I)$ is almost surely a bounded function of x. Hence we can apply Lemma 3.1 of Monrad and Pitt [27] to deduce that almost surely

$$\dim_{\rm H}\big(X^{-1}(x) \cap I\big) \le N - \underline{H} d, \qquad \forall\, x \in \mathbb{R}^d. \tag{3.19}$$

To prove the lower bound, it is sufficient to show that for every $\delta \in (0, \frac{N}{d} - \underline{H})$,

$$\mathbb{P}\Big\{\dim_{\rm H}\big(X^{-1}(x) \cap I\big) \ge N - (\underline{H} + \delta) d, \ \ \forall\, x \in \mathcal{U}\Big\} = 1. \tag{3.20}$$

Since $H(t^0) = \min_{t \in I} H(t)$, there exists $r_0 > 0$ such that $\max_{t \in B(t^0, r_0)} H(t) \le \underline{H} + \delta/2$. Since, for every $x \in \mathcal{U}$, $L(x, \cdot)$ is a finite and positive Borel measure on $X^{-1}(x) \cap B(t^0, r_0)$. [Recall $B(t^0, r_0) = \{t \in I : |t - t^0| \le r\}$.] Moreover, Part (ii) of Theorem 3.5 implies that almost surely for all $t \in B(t^0, r_0)$ we have

$$L(x, B(t, r)) \le c\, r^{N - (\underline{H} + \delta) d}$$

for all $r \in (0, r_0)$ small enough. Hence we can apply Frostman's lemma [20, 21] to derive that almost surely

$$\begin{aligned}\dim_{\mathrm{H}}\big(X^{-1}(x) \cap I\big) &\geq \dim_{\mathrm{H}}\big(X^{-1}(x) \cap B(t^0, r_0)\big) \\ &\geq N - (\underline{H} + \delta)d, \qquad \forall\, x \in \mathcal{U}.\end{aligned}$$

This proves (3.20) and hence (3.18). □

Remark 3.9 We believe that if $N > \max_{t \in \mathbb{R}^N} H(t)d$, then

$$\mathbb{P}\Big\{\dim_{\mathrm{H}} X^{-1}(x) = N - \underline{H}d \ \text{ for all } \ x \in \mathbb{R}^d\Big\} = 1.$$

Because of (3.19), it amounts to prove $\mathcal{O} = \mathbb{R}^d$, where $\mathcal{O}$ is the random open set defined by

$$\mathcal{O} = \bigcup_n \Big\{x : L(x, I_n) > 0\Big\},$$

where I_n is a sequence of intervals increasing to the parameter space $\mathbb{R}^N$. In the special case of $H(t) \equiv H \in (0, 1)$ (i.e., X is an (N, d) fractional Brownian motion), Monrad and Pitt [27] applied the scaling property of fBm and proved that $\mathcal{O} = \mathbb{R}^d$ almost surely.

On the other hand, there are examples of stationary Gaussian random fields with jointly continuous local time L such that the open set $\mathcal{O}$ is a proper subset of $\mathbb{R}^d$. The following example of such a stationary Gaussian process is given in Monrad and Pitt [27]:

$$Z(t) = \frac{\sqrt{8}}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \Big[\xi_n \cos\big((2n-1)t\big) + \eta_n \sin\big((2n-1)t\big)\Big], \qquad \forall t \in \mathbb{R},$$

where $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of i.i.d. standard normal random variables. The Gaussian process Z has a jointly continuous local time. However, because $Z(t)$ has period 2π, we see that $\mathcal{O} = Z([0, 2\pi])$ is almost surely bounded. More general examples of Gaussian random fields can be found in Luan and Xiao [24].

It is an interesting open question to determine for which Gaussian random fields X we have $\mathcal{O} = \mathbb{R}^d$. Monrad and Pitt [27, Theorem 2] proved a sufficient condition for a stationary Gaussian random field to have this property. However, their method is not applicable to multiparameter multifractional Brownian motion. □

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