Exact Moduli of Continuity for Operator-Scaling Gaussian Random Fields

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Let $X = \{X(t), t \in \mathrm{R}^N\}$ be a centered real-valued operator-scaling Gaussian random field with stationary increments, introduced by Biermé, Meerschaert and Scheffler [8]. We prove that X satisfies a form of strong local nondeterminism and establish its exact uniform and local moduli of continuity. The main results are expressed in terms of the quasi-metric τ_E associated with the scaling exponent of X. Examples are provided to illustrate the subtle changes of the regularity properties.

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1. Introduction

For random fields, "anisotropy" is a distinct property from those of one-parameter processes and is not only important in probability (e.g., stochastic partial differential equations) and statistics (e.g., spatio-temporal modeling), but also for many applied areas such as economic, ecological, geophysical and medical sciences. See, for example, Benson, *et al.* [6], Bonami and Estrade [10], Chilés and Delfiner [11], Davies and Hall [12], Stein [24, 25], Wackernagel [27], Zhang [36], and their combined references for further information.

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Many anisotropic random fields $Y = \{Y(t),\, t \in \mathrm{R}^N\}$ in the literature have the following scaling property: There exists a linear operator E (which may not be unique) on R^N such that for all constants $c > 0$,

$$\big\{Y(c^E\, t),\, t \in \mathrm{R}^N\big\} \stackrel{f.d.}{=} \big\{c\, Y(t),\, t \in \mathrm{R}^N\big\}. \tag{1.1}$$

Here and in the sequel, "$\stackrel{f.d.}{=}$" means equality in all finite-dimensional distributions and, for $c > 0$, c^E is the linear operator on R^N defined by $c^E = \sum_{n=0}^{\infty} \frac{(\ln c)^n E^n}{n!}$. A random field $Y = \{Y(t),\, t \in \mathrm{R}^N\}$ which satisfies (1.1), is called operator-scaling in the time variable (or simply operating-scaling) with exponent E. Two important examples of real-valued operator-scaling Gaussian random fields are fractional Brownian sheets introduced by Kamont [14] and those with stationary increments introduced by Biermé, Meerschaert and Scheffler [8]. Multivariate random fields with operator-scaling properties in both time and space variables have been constructed by Li and Xiao [16].

Several authors have studied probabilistic and geometric properties of anisotropic Gaussian random fields. For example, Dunker [13], Mason and Shi [20], Belinski and Linde [5], Kühn and Linde [15] studied the small ball probabilities of a fractional Brownian sheet B^H, where $H = (H_1, \ldots, H_N) \in (0, 1)^N$. Mason and Shi [20] also computed the Hausdorff dimension of some exceptional sets related to the oscillation of the sample paths of B^H. Ayache and Xiao [2], Ayache, *et al.* [3], Wang [28], Wu and Xiao [29], Xiao and Zhang [35] studied uniform modulus of continuity, law of iterated logarithm, fractal properties and joint continuity of the local times of fractional Brownian sheets. Wu and Xiao [30] proved sharp uniform and local moduli of continuity for the local times of Gaussian fields which satisfy sectorial local nondeterminism. Luan and Xiao [18] determined the exact Hausdorff measure functions for the ranges of Gaussian fields which satisfy strong local nondeterminism. Meerschaert, *et al.* [22] established exact modulus of continuity for Gaussian fields which satisfy the condition of sectorial local nondeterminism. Their results and methods are applicable to fractional Brownian sheets and certain operator-scaling Gaussian random fields with stationary increments whose scaling exponent is a diagonal matrix. We remark that there are subtle differences between certain sample path properties of fractional Brownian sheets and those of anisotropic Gaussian random fields with stationary increments. This is due to their different properties of local nondeterminism; see Xiao [33] for more information.

For an operator-scaling Gaussian random field $X = \{X(t), t \in \mathrm{R}^N\}$ with stationary increments, Biermé, *et al.* [8] showed that the critical global or directional Hölder exponents are given by the real parts of the eigenvalues of the exponent matrix E. The main objective of this paper is to improve their results and to establish exact uniform and local moduli of continuity for these Gaussian fields. Our approach is an extension of the method in Meerschaert, *et al.* [22]. In particular, we prove in Theorem 3.1 that X has the property of strong local nondeterminism, which is expressed in terms of the natural quasi-metric $\tau_E(t-s)$ associated with the scaling exponent E (see Section 2 for its definition and properties). As an application of Theorem 3.1 and the method in [22], we establish exact uniform and local moduli of continuity for X (see Theorems 4.3 and 5.1 below).

It should be mentioned that Biermé, *et al.* [8] constructed a large class of operator-scaling α-stable random fields for any $\alpha \in (0, 2]$. By using a LePage-type series representation for stable random fields, Biermé and Lacaux [7] studied uniform modulus of continuity of these operator-scaling stable random fields. See also Xiao [34] for related results using a different approach based on the chaining argument. In this paper we will focus on the Gaussian case (i.e., $\alpha = 2$) and our Theorem 4.3 establishes the exact uniform modulus of continuity, which is more precise than the results in [7] and [34].

The rest of this paper is divided into five sections. In Section 2, we prove some basic properties on the quasi-metric τ_E associated with the scaling exponent E and recall from [8] the definition of an operator-scaling Gaussian field $X = \{X(t), t \in \mathrm{R}^N\}$ with stationary increments. In Section 3, we prove the strong local nondeterminism of X, and in Sections 4 and 5 we prove the exact uniform and local moduli of continuity of X, respectively. In Section 6 we provide two examples to illustrate our main theorems.

We end the Introduction with some notation. The parameter space is R^N, endowed with the Euclidean norm $\|\cdot\|$. For any given two points $s = (s_1, , \cdots, s_N)$, $t = (t_1, \cdots, t_N)$, the inner product of $s, t \in \mathrm{R}^N$ is denoted by $\langle s, t\rangle$. For $x \in \mathrm{R}_+$, let $\log x := \ln\big(x \vee e\big)$ and $\log\log x := \ln\big((\ln x) \vee e\big)$. Throughout this paper we will use C to denote an unspecified positive and finite constant which may be different in each occurrence. More specific constants are numbered as $C_1, C_2, \cdots$.

2. Preliminaries

In this section we show some basic properties of a real $N \times N$ matrix E and prove several lemmas on the quasi-metric τ_E on R^N. Then we recall from Biermé, *et al.* [8] the definition of operator-scaling Gaussian random fields with a harmonizable representation.

For a real $N \times N$ matrix E, it is well-known that E is similar to a real Jordan canonical form, i.e. there exists a real invertible $N \times N$ matrix P such that

$$E = PDP^{-1},$$

where D is a real $N \times N$ matrix of the form

$$D = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_p \end{pmatrix}. \tag{2.1}$$

and J_i, $1 \le i \le p$, is either Jordan cell matrix of the form

$$\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \tag{2.2}$$

with λ a real eigenvalue of E or blocks of the form

$$\begin{pmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ I_2 & \Lambda & 0 & \cdots & 0 \\ 0 & I_2 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda \end{pmatrix} \quad \text{with } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and } \Lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \tag{2.3}$$

where the complex numbers $a \pm ib$ $(b \neq 0)$ are complex conjugated eigenvalues of E.

Denote the size of J_k by $\tilde{l}_k$ and let a_k be the real part of the corresponding eigenvalue(s) of J_k. Throughout this paper, we always suppose that

$$1 < a_1 \le a_2 \le \cdots \le a_p.$$

Note that $p \le N$, $\tilde{l}_1 + \tilde{l}_2 + \cdots + \tilde{l}_p = N$ and $Q := \text{trace}(E) = \sum_{j=1}^{p} a_j \tilde{l}_j$.

As done in Biermé and Lacaux [7], we can construct the E-invariant subspace W_k associated with J_k by

$$W_k = \text{span}\Big\{ f_j \ : \sum_{i=1}^{k-1} \tilde{l}_i + 1 \le j \le \sum_{i=1}^{k} \tilde{l}_i \Big\},$$

where f_j is the j-th column vector of the matrix P. Then R^N has a direct sum decomposition of

$$\mathrm{R}^N = W_1 \oplus \cdots \oplus W_p.$$

It follows from Meerschaert and Scheffler [21, Chapter 6] (see also [8, Section 2]) that there exists a norm $\|\cdot\|_E$ on R^N such that for the unit sphere $S_E = \big\{ x \in \mathrm{R}^N : \|x\|_E = 1 \big\}$ the mapping $\Psi : (0,\infty) \times S_E \to \mathrm{R}^N \backslash \{0\}$ defined by $\Psi(r,\theta) = r^E \theta$ is a homeomorphism. Hence, every $x \in \mathrm{R}^N \setminus \{0\}$ can be written uniquely as $x = (\tau_{E}(x))^E l_E(x)$ for some radial part $\tau_{E}(x) > 0$ and some direction $l_E(x) \in S_E$ such that the functions $x \mapsto \tau_{E}(x)$ and $x \mapsto l_E(x)$ are continuous. For $x \in \mathrm{R}^N \backslash \{0\}$, $\big(\tau_{E}(x), l_E(x)\big)$ is referred to as its polar coordinates associated with E.

It is shown in [21] that $\tau_{E}(x) = \tau_{E}(-x)$ and $\tau_{E}(r^E x) = r \tau_{E}(x)$ for all $r > 0$ and $x \in \mathrm{R}^N \backslash \{0\}$. Moreover, $\tau_{E}(x) \to \infty$ as $x \to \infty$ and $\tau_{E}(x) \to 0$ as $x \to 0$. Hence we can extend $\tau_{E}(x)$ continuously to R^N by setting $\tau_{E}(0) = 0$.

The function $\tau_E(x)$ will play essential roles in this paper. We first recall some known facts about it.

(i) Lemma 2.2 in [8] shows that there exists a constant $C \ge 1$ such that

$$\tau_E(x+y) \le C\left(\tau_E(x) + \tau_E(y)\right), \quad \forall\, x,\, y \in \mathrm{R}^N. \tag{2.4}$$

Hence, we can regard $\tau_E(x-y)$ as a quasi-metric on R^N.

(ii) Since the norms $\|\cdot\|_E$ and $\|\cdot\|$ are equivalent, Lemma 2.1 in [8] implies that for any $0 < \delta < a_1$ there exist finite constants $C_1, C_2 > 0$, which may depend on δ, such that for all $\|x\| \le 1$ or all $\tau_E(x) \le 1$,

$$C_1 \|x\|^{1/(a_1-\delta)} \le \tau_E(x) \le C_2 \|x\|^{1/(a_p+\delta)}, \tag{2.5}$$

and, for all $\|x\| > 1$ or all $\tau_E(x) > 1$,

$$C_1 \|x\|^{1/(a_p+\delta)} \le \tau_E(x) \le C_2 \|x\|^{1/(a_1-\delta)}. \tag{2.6}$$

(iii) Biermé and Lacaux [7, Corollary 3.4] proved the following improvement of (2.5): For any $\eta \in (0,1)$, there exists a finite constant $C_3 \ge 1$ such that for all $x \in W_j \backslash \{0\}$, $1 \le j \le p$, with $\|x\| \le \eta$

$$C_3^{-1} \|x\|^{1/a_j} \big|\ln \|x\|\big|^{-(l_j-1)/a_j} \le \tau_E(x) \le C_3 \|x\|^{1/a_j} \big|\ln \|x\|\big|^{(l_j-1)/a_j}, \tag{2.7}$$

where $l_k = \tilde{l}_k$ if J_k is a Jordan cell matrix as in (2.2) or $l_k = \tilde{l}_k/2$ if J_k is of the form (2.3).

We remark that, as shown by Example 6.2 below, both the upper and lower bounds in (2.7) can be achieved and this fact makes the regularity properties of an operator-scaling Gaussian field with a general exponent E more intriguing.

For any $x \in \mathrm{R}^N$, let $x = \bar{x}_1 \oplus \bar{x}_2 \oplus \cdots \oplus \bar{x}_p$ be the direct sum decomposition of x in the E-invariant subspaces W_j, $j = 1, 2, \cdots, p$. This notation is used in Lemmas 2.1 and 2.2.

Lemma 2.1. *There exists a finite constant $C > 0$ such that for all $x \in \mathrm{R}^N$ and $j = 1, 2, \cdots, p$, we have*

$$\tau_E(\bar{x}_j) \le C\, \tau_E(x). \tag{2.8}$$

Proof. Since (2.8) holds trivially for $x = 0$. We only consider $x \in \mathrm{R}^N \backslash \{0\}$, which can be written as $x = (\tau_E(x))^E\, l_E(x)$ for some $l_E(x) \in S_E$. Denote the direct sum decomposition of $l_E(x)$ in the E-invariant subspaces W_j, $j = 1, 2, \cdots, p$, by $l_E(x) = x'_1 \oplus \cdots \oplus x'_p$. Then from the fact that $\big(\tau_E(x)\big)^E x'_j \in W_j$ for all $j = 1, 2, \cdots, p$, it follows that

$$\bar{x}_j = (\tau_E(x))^E\, x'_j.$$

Since S_E is bounded, i.e. there exists $M > 0$ such that $S_E \subset \{y \in \mathrm{R}^N : \|y\| \le M\}$, we can easily see that $x'_j \in \{y \in \mathrm{R}^N : \|y\| \le M\}$ for all $j = 1, 2, \cdots, p$. Let $C = \max_{\|x\| \le M} \tau_E(x) \in (0, \infty)$. Then for all $j = 1, 2, \cdots, p$

$$\tau_E(\bar{x}_j) = \tau_E(x)\tau_E(x'_j) \le C\tau_E(x),$$

which is the desired conclusion. □

As a consequence of (2.4) and Lemma 2.1, we have

Lemma 2.2. *There is a finite constant $C \ge 1$ such that*

$$C^{-1}\sum_{i=1}^{p} \tau_{_E}(\bar{x}_i) \le \tau_{_E}(x) \le C\sum_{i=1}^{p} \tau_{_E}(\bar{x}_i), \qquad \forall\, x \in \mathrm{R}^N. \tag{2.9}$$

The following lemma implies that the function $\tau_{_E}(x)$ is O-regular varying at both the origin and the infinity (cf. Bingham, *et al.* [9, pp.65–67]).

Lemma 2.3. *Give any constants $0 < a < b < \infty$, there exists a finite constant $C_4 \ge 1$ such that for all $x \in \mathrm{R}^N$ and $\beta \in [a, b]$,*

$$C_4^{-1}\tau_{_E}(x) \le \tau_{_E}(\beta x) \le C_4\tau_{_E}(x). \tag{2.10}$$

Proof. To prove the left inequality in (2.10), note that $\Lambda = \{\beta x : x \in S_E,\, \beta \in [a, b]\}$ is a compact set which does not contain 0. This and the continuity of $\tau_{_E}(\cdot)$ on R^N, imply $\min_{x\in\Lambda} \tau_{_E}(x) > 0$. Hence, by taking $C_4^{-1} = 1 \wedge \min_{x\in\Lambda} \tau_{_E}(x)$, we have

$$\tau_{_E}(\beta x) = \tau_{_E}(\beta\tau_{_E}^{E}(x) l_E(x)) = \tau_{_E}(x)\tau_{_E}(\beta l_E(x)) \ge C_4^{-1}\tau_{_E}(x).$$

The right inequality in (2.10) can be proved in the same way. This finishes the proof. □

Lemma 2.4. *There is a subsequence $\{n_k\}_{k\in\mathrm{N}} \subseteq \mathrm{N}$ such that $n_k \ge k$ for all $k \ge 1$ and*

$$\min_{1\le i\le 2^{n_k}} \tau_{_E}(\langle i2^{-n_k}\rangle) \ge C_4^{-1}\tau_{_E}(\langle 2^{-n_k}\rangle), \tag{2.11}$$

where $\langle c\rangle = (c, c, \cdots, c) \in \mathrm{R}^N$ for any $c \in \mathrm{R}$.

Proof. Suppose $\min_{1\le i\le 2^n} \tau_{_E}(\langle i2^{-n}\rangle)$ is attained at $i = K_n$. There is an integer $m_n \in [0\,, n]$ such that $2^{n-m_n-1} < K_n \le 2^{n-m_n}$. Therefore we can rewrite $K_n 2^{-n}$ as $\beta 2^{-m_n}$ for some $\beta \in (1/2, 1]$. Since $\{i2^{-m_n}, i = 1, \cdots, 2^{m_n}\} \subset \{i2^{-n}, i = 1, \cdots, 2^n\}$, we have

$$\min_{1\le i\le 2^{m_n}} \tau_{_E}(\langle i2^{-m_n}\rangle) \ge \min_{1\le i\le 2^n} \tau_{_E}(\langle i2^{-n}\rangle) = \tau_{_E}(\langle \beta 2^{-m_n}\rangle) \ge C_4^{-1}\tau_{_E}(\langle 2^{-m_n}\rangle),$$

where the last inequality follows from Lemma 2.3 with $[a, b] = [1/2, 1]$. Furthermore, by the fact

$$\min_{1\le i\le 2^n} \tau_E(\langle i2^{-n}\rangle) \le \tau_E(\langle 2^{-n}\rangle) \to 0,$$

as $n\to\infty$, we know that $\tau_E(\langle 2^{-m_n}\rangle)\to 0$ which implies that $m_n\to\infty$ as $n\to\infty$. Hence a desired subsequence $\{n_k\}_{k\in \mathrm{N}}$ can be selected from $\{m_n\}$. □

Let E' be the transpose of E. An E'-homogeneous function $\psi : \mathrm{R}^N \to [0,\infty)$ is a function which satisfies that $\psi(x) > 0$ and $\psi(r^{E'}x) = r\psi(x)$ for all $r > 0$ and $x \in \mathrm{R}^N\backslash\{0\}$. For any continuous E'-homogeneous function $\psi : \mathrm{R}^N \to [0,\infty)$, Biermé, *et al.* [8, Theorem 4.1] showed that the real-valued Gaussian random field $X_\psi = \{X_\psi(t), t \in \mathrm{R}^N\}$, where

$$X_\psi(t) = \mathrm{Re}\int_{\mathrm{R}^N} \big(e^{i\langle t,\xi\rangle} - 1\big)\,\frac{\widetilde{\mathcal{M}}(d\xi)}{\psi(\xi)^{1+Q/2}}, \qquad t\in \mathrm{R}^N, \tag{2.12}$$

is well defined and stochastic continuous if and only if $\min_{1\le j\le p} a_j > 1$. In the latter case, they further proved that X_ψ satisfies (1.1) and has stationary increments. Here $\widetilde{\mathcal{M}}$ is a centered complex-valued Gaussian random measure in R^N with the Lebesgue measure m_N as its control measure. Namely, $\widetilde{\mathcal{M}}$ is a centered complex-valued Gaussian process defined on the family $\mathcal{A} = \{A \subset \mathrm{R}^N : m_N(A) < \infty\}$ which satisfies

$$\mathrm{E}\big(\widetilde{\mathcal{M}}(A)\overline{\widetilde{\mathcal{M}}(B)}\big) = m_N(A\cap B) \quad \text{and} \quad \widetilde{\mathcal{M}}(-A) = \overline{\widetilde{\mathcal{M}}(A)} \tag{2.13}$$

for all $A,\ B \in \mathcal{A}$.

Remark 2.1 The following are some remarks on the Gaussian random field X_ψ.

- If, in addition, ψ is symmetric in the sense that $\psi(\xi) = \psi(-\xi)$ for all $\xi\in \mathrm{R}^N$, then because of (2.13) the Wiener-type integral in the right-hand side of (2.12) is real-valued. Thus, in this latter case, "Re" in (2.12) is not needed. For simplicity, we assume that ψ is symmetric in the rest of the paper. A large class of continuous, symmetric E'-homogeneous functions has been constructed in [8, Theorem 2.1].
- By replacing $\widetilde{\mathcal{M}}$ in (2.12) by a complex-valued isotropic α-stable random measure $\widetilde{\mathcal{M}}_\alpha$ with Lebesgue control measure (see [23, p.281]), Biermé, *et al.* [8, Theorem 3.1] obtained a class of *harmonizable* operator-scaling α-stable random fields. They also defined a class of operator-scaling α-stable fields by using moving-average representations. When $\alpha \in (0, 2)$, stable random fields with harmonizable and moving-average representations are generally different. However, for the Gaussian case of $\alpha = 2$, the Planchrel theorem implies that every Gaussian random field with a moving-average representation in [8] also has a harmonizable representation of the form (2.12).

3. Strong local nondeterminism of operator-scaling Gaussian fields

Let E be an $N\times N$ matrix such that the real parts of its eigenvalues satisfy $\min_{1\le j\le p} a_j > 1$ and let ψ be a continuous, symmetric, E'-homogeneous function with $\psi(x)>0$ for $x\neq 0$ as in Section 2. Let $X_\psi = \{X_\psi(t), t\in \mathrm{R}^N\}$ be the operator-scaling Gaussian field with scaling exponent E, defined by (2.12). For simplicity, we write X_ψ as X. Note that the assumptions on ψ imply

$$0 < m_\psi = \min_{x\in S_{E'}} \psi(x) \le M_\psi = \max_{x\in S_{E'}} \psi(x) < \infty. \tag{3.1}$$

The dependence structure of the operator-scaling Gaussian field X is complicated for a general matrix E. In order to study sample path properties and characterize the anisotropic nature of X, we prove that X has the property of "strong local nondeterminism" with respect to the quasi-metric $\tau_{_E}(s-t)$. The main result of this section is Theorem 3.2, which extends Theorem 3.2 in Xiao [33] and will play an important role in Section 4 below.

Since many sample path properties of X are determined by the canonical metric

$$d_X(s,t) = \big[\mathrm{E}\big(X(s)-X(t)\big)^2\big]^{1/2}, \qquad \forall\ s,t\in \mathrm{R}^N, \tag{3.2}$$

our first step is to establish the relations between $d_X(s,t)$ and $\tau_{_E}(s-t)$.

Lemma 3.1. *There exists a finite constant $C\ge 1$ such that*

$$C^{-1}\,\tau_{_E}^2(s-t) \le d_X^2(s,t) \le C\,\tau_{_E}^2(s-t), \qquad \forall\ s,t\in \mathrm{R}^N. \tag{3.3}$$

Proof. For all $s,t\in\mathrm{R}^N$, by (2.12), we have

$$\begin{aligned} d_X^2(s,t) &= \int_{\mathrm{R}^N} \big|\mathrm{e}^{i\langle s,x\rangle} - \mathrm{e}^{i\langle t,x\rangle}\big|^2 \,\frac{dx}{\psi(x)^{2+Q}} \\ &= 2\int_{\mathrm{R}^N} (1-\cos\langle s-t,x\rangle)\,\frac{dx}{\psi(x)^{2+Q}}. \end{aligned}$$

Let $y = \tau_{_E}^{E'}(s-t)x$. Then $dx = (1/\tau_{_E}(s-t))^Q\,dy$. Hence

$$d_X^2(s,t) = 2\tau_{_E}(s-t)^2 \int_{\mathrm{R}^N} \bigg(1-\cos\Big\langle \Big(\frac{1}{\tau_{_E}(s-t)}\Big)^E (s-t), y\Big\rangle\bigg)\,\frac{dy}{\psi(y)^{2+Q}}. \tag{3.4}$$

Since for all $s\neq t$, $\tau_{_E}\big(\big(\frac{1}{\tau_{_E}(s-t)}\big)^E(s-t)\big) = 1$. Hence the set

$$\bigg\{\Big(\frac{1}{\tau_{_E}(s-t)}\Big)^E (s-t) : s\neq t\in \mathrm{R}^N\bigg\}$$

is compact and does not contain 0. On the other hand, a slight modification of the proof of Theorem 4.1 in [8] shows that the function $\xi \mapsto \int_{\mathrm{R}^N} (1-\cos\langle \xi, y\rangle)\frac{dy}{\psi(y)^{2+Q}}$ is continuous on R^N and positive on $\mathrm{R}^N\backslash\{0\}$. Therefore, the last integral in (3.4) is bounded from below and above by positive and finite constants. This proves (3.3). □

Theorem 3.2. *There exists a constant $C_5 > 0$ such that for all $n \geq 2$ and all $t^1, \ldots, t^n \in \mathrm{R}^N$, we have*

$$\mathrm{Var}\big(\, X(t^n)\mid X(t^1), \ldots, X(t^{n-1})\big) \geq C_5 \min_{0\leq k\leq n-1} \tau_{_E}^2(t^n - t^k), \tag{3.5}$$

where $t^0 = 0$.

Proof. The proof is a modification of that of Theorem 3.2 in Xiao [33]. We denote $r = \min_{0\leq k\leq n-1}\tau_{_E}(t^n - t^k)$. Since

$$\mathrm{Var}\big(X(t^n)\big|X(t_1), \cdots, X(t^{n-1})\big) = \inf_{u_1,\cdots,u_{n-1}\in \mathrm{R}} \mathrm{E}\bigg[\bigg(X(t^n) - \sum_{k=1}^{n-1} u_k X(t^k)\bigg)^2\bigg],$$

it suffices to prove the existence of a constant $C > 0$ such that

$$\mathrm{E}\bigg[\bigg(X(t^n) - \sum_{k=1}^{n-1} u_k X(t^k)\bigg)^2\bigg] \geq C\, r^2$$

for all $u_k \in \mathrm{R}$, $k = 1, 2, \cdots, n-1$. It follows from (2.12) that

$$\mathrm{E}\bigg[\bigg(X(t^n) - \sum_{k=1}^{n-1} u_k X(t^k)\bigg)^2\bigg] = \int_{\mathrm{R}^N} \Big|\mathrm{e}^{i\langle t^n, x\rangle} - \sum_{k=0}^{n-1} u_k \mathrm{e}^{i\langle t^k,x\rangle}\Big|^2 \frac{dx}{\psi(x)^{2+Q}},$$

where $t^0 = 0$ and $u_0 = 1 - \sum_{k=1}^n u_k$. Let $\delta(\cdot) : \mathrm{R}^N \mapsto [0, 1]$ be a function in $C^\infty(\mathrm{R}^N)$ such that $\delta(0) = 1$ and it vanishes outside the open set $B = \{x : \tau_{_E}(x) < 1\}$. Denote by $\widehat{\delta}$ the Fourier transform of δ. Then $\widehat{\delta} \in C^\infty(\mathrm{R}^N)$ as well and decays rapidly as $x \to \infty$, that it, for all $\ell \geq 1$, we have $\|x\|^\ell |\widehat{\delta}(x)| \to 0$ as $x \to \infty$. This and (2.6) further imply that for all $\ell \geq 1$,

$$\tau_{_E}(x)^\ell |\widehat{\delta}(x)| \to 0 \qquad \text{as } x \to \infty. \tag{3.6}$$

Let $\delta_r(t) = r^{-Q}\delta(r^{-E}t)$. Then

$$\delta_r(t) = (2\pi)^{-N} \int_{\mathrm{R}^N} \mathrm{e}^{-i\langle t,x\rangle}\widehat{\delta}(r^{E'}x)dx.$$

Since $\min\{\tau_{_E}(t^n-t^k), 0\le k\le n-1\}=r$, we have $\delta_r(t^n-t^k)=0$ for all $k=0,1,\cdots,n-1$. Therefore

$$
\begin{aligned}
J &:= \int_{\mathrm{R}^N}\bigg(\mathrm{e}^{i\langle t^n,x\rangle}-\sum_{k=0}^{n-1}\mathrm{e}^{i\langle t^k,x\rangle}\bigg)\mathrm{e}^{-i\langle t^n,x\rangle}\widehat{\delta}(r^{E'}x)dx\\
&= (2\pi)^N\bigg(\delta_r(0)-\sum_{k=0}^{n-1}u_k\delta_r(t^n-t^k)\bigg)=(2\pi)^N r^{-Q}. \qquad (3.7)
\end{aligned}
$$

By Hölder's inequality, a change of variables, the E'-homogeneity of ψ and (3.1), we derive

$$
\begin{aligned}
J^2 &\le \int_{\mathrm{R}^N}\Big|\mathrm{e}^{i\langle t^n,x\rangle}-\sum_{k=0}^{n-1}\mathrm{e}^{i\langle t^k,x\rangle}\Big|^2\frac{dx}{\psi(x)^{2+Q}}\int_{\mathrm{R}^N}\psi(x)^{2+Q}|\widehat{\delta}(r^{E'}x)|^2dx\\
&= r^{-2Q-2}\mathrm{E}\bigg(\Big|X(t^n)-\sum_{k=1}^{n-1}u_kX(t^k)\Big|^2\bigg)\int_{\mathrm{R}^N}\psi(y)^{2+Q}|\widehat{\delta}(y)|^2dy\\
&\le r^{-2Q-2}\mathrm{E}\bigg(\Big|X(t^n)-\sum_{k=1}^{n-1}u_kX(t^k)\Big|^2\bigg)\int_{\mathrm{R}^N}\tau_{_E}(y)^{2+Q}M_\psi^{2+Q}\big|\widehat{\delta}(y)\big|^2\,dy.\\
&\le Cr^{-2Q-2}\mathrm{E}\bigg(\Big|X(t^n)-\sum_{k=1}^{n-1}u_kX(t^k)\Big|^2\bigg) \qquad (3.8)
\end{aligned}
$$

for some finite constant $C>0$, since $\int_{\mathrm{R}^N}\tau_{_E}(y)^{2+Q}|\widehat{\delta}(y)|^2dy<\infty$ which follows from (3.6). Combining (3.7) and (3.8) yields (3.6) for an appropriate constant $C_5>0$. □

The relation (3.5) is a property of strong local nondeterminism, which is more general than that in Xiao [33] and can be applied to establish many sample path properties of X.

For any $s,t\in\mathrm{R}^N$ with $s\ne t$, we decompose $s-t$ as a direct sum of elements in the E-invariant subspaces W_j, $j=1,2,\cdots,p$,

$$
s-t=(s_1-t_1)\oplus\cdots\oplus(s_p-t_p).
$$

Then (2.7) and Lemmas 3.1 and 2.2 imply

$$
\begin{aligned}
C^{-1}\sum_{j=1}^p\|s_j-t_j\|^{1/a_j}\big|\ln\|s_j-t_j\|\big|^{-(l_j-1)/a_j} &\le d_X^2(s,t)\\
&\le C\sum_{j=1}^p\|s_j-t_j\|^{1/a_j}\big|\ln\|s_j-t_j\|\big|^{(l_j-1)/a_j}. \qquad (3.9)
\end{aligned}
$$

Moreover, Theorem 3.2 implies that for all $n \geq 2$ and all $t^1, \ldots, t^n \in \mathrm{R}^N$, we have

$$\begin{aligned}&\mathrm{Var}\big(\, X(t^n) \mid X(t^1), \ldots, X(t^{n-1})\big)\\ &\qquad \geq C \min_{0 \leq k \leq n-1} \sum_{j=1}^p \|t_j^n - t_j^k\|^{1/a_j} \big| \ln \|t_j^n - t_j^k\| \big|^{-(l_j-1)/a_j},\end{aligned} \tag{3.10}$$

where $t^0 = 0$.

Inequalities (3.9) and (3.10) are similar to Condition (C1) and (C3′) in Xiao [33]. Hence many results on the Hausdorff dimensions of various random sets and joint continuity of the local times can be readily derived from those in [33], and these results can be explicitly expressed in terms of the real parts $\{a_j, 1 \leq j \leq p\}$ of the eigenvalues of the scaling exponent E.

To give some examples,we define a vector $(H_1, \ldots, H_N) \in (0,1)^N$ as follows.

For $1 \leq i \leq \tilde{l}_p$, define $H_i = a_p^{-1}$. In general, if $1 + \sum_{j=k}^p \tilde{l}_j \leq i \leq \sum_{j=k-1}^p \tilde{l}_j$ for some $2 \leq k \leq p$, then we define $H_i = a_{k-1}^{-1}$. Since $1 < a_1 \leq a_2 \leq \cdots \leq a_p$, we have

$$0 < H_1 \leq H_2 \leq \cdots \leq H_N < 1.$$

Consider a Gaussian random field $\vec{X} = \{\vec{X}(t), t \in \mathrm{R}^N\}$ with values in R^d defined by

$$\vec{X}(t) = (X_1(t), \ldots, X_d(t)),$$

where $X_1, \ldots, X_d$ are independent copies of the centered Gaussian field X in the above. Let $\vec{X}([0,1]^N)$ and $\mathrm{Gr}\vec{X}([0,1]^N) = \{(t, \vec{X}(t)),\, t \in [0,1]^N\}$ denote respectively the range and graph of $\vec{X}$, then Theorem 6.1 in [33] implies that with probability 1,

$$\dim_{\mathrm{H}} \vec{X}\big([0,1]^N\big) = \dim_{\mathrm{P}} \vec{X}\big([0,1]^N\big) = \min\left\{d;\ \sum_{j=1}^N \frac{1}{H_j}\right\}, \tag{3.11}$$

where $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$ denote Hausdorff and packing dimension respectively, and

$$\begin{aligned}&\dim_{\mathrm{H}} \mathrm{Gr}\vec{X}\big([0,1]^N\big) = \dim_{\mathrm{P}} \mathrm{Gr}\vec{X}\big([0,1]^N\big)\\ &\quad = \min\left\{\sum_{j=1}^k \frac{H_k}{H_k} + N - k + (1-H_k)d,\ 1 \leq k \leq N;\ \sum_{j=1}^N \frac{1}{H_j}\right\}\\ &\quad = \begin{cases} \sum_{j=1}^N \frac{1}{H_j}, & \text{if } \sum_{j=1}^N \frac{1}{H_j} \leq d,\\ \sum_{j=1}^k \frac{H_k}{H_j} + N - k + (1-H_k)d, & \text{if } \sum_{j=1}^{k-1} \frac{1}{H_j} \leq d < \sum_{j=1}^k \frac{1}{H_j}, \end{cases}\end{aligned} \tag{3.12}$$

where $\sum_{j=1}^0 \frac{1}{H_j} := 0$.

Similarly, it follows from Theorem 7.1 in Xiao [33] that the following hold:

(i) If $\sum_{j=1}^N \frac{1}{H_j} < d$, then for every $x \in \mathrm{R}^d$, $\vec{X}^{-1}(\{x\}) = \emptyset$ a.s.

(ii) If $\sum_{j=1}^N \frac{1}{H_j} > d$, then for every $x \in \mathrm{R}^d$,

$$\begin{aligned}\dim_{\mathrm{H}} \vec{X}^{-1}(\{x\}) &= \dim_{\mathrm{P}} \vec{X}^{-1}(\{x\}) \\ &= \min\left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d,\ 1 \le k \le N \right\} \\ &= \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d, \qquad \text{if} \quad \sum_{j=1}^{k-1} \frac{1}{H_j} \le d < \sum_{j=1}^{k} \frac{1}{H_j} \end{aligned} \tag{3.13}$$

holds with positive probability.

In light of the dimension results (3.11)–(3.13), it would be interesting to determine the exact Hausdorff (and packing) measure functions for the above random sets. In the special case of fractional Brownian motion, the corresponding problems have been investigated by Talagrand [26], Xiao [31, 32], Baraka and Mountford [4]. For anisotropic Gaussian random fields, the problems are more difficult. Only an exact Hausdorff measure function for the range has been determined by Luan and Xiao [18] for a special case of anisotropic Gaussian random fields.

4. Uniform modulus of continuity

In this section, we establish the exact modulus of continuity for X. We first rewrite Lemma 7.1.1 in Marcus and Rosen [19] as follows.

Lemma 4.1. *Let $\{G(u), u \in \mathrm{R}^N\}$ be a centered Gaussian random field. Let $\omega : \mathrm{R}_+ \to \mathrm{R}_+$ be a function with $\omega(0+) = 0$ and $\Gamma \subset \mathrm{R}^N$ be a compact set. Assume that there is a continuous map $\tau : \mathrm{R}^N \mapsto \mathrm{R}_+$ with $\tau(0) = 0$ such that d_G is continuous on τ, i.e., $\tau(u_n - v_n) \to 0$ implies $d_G(u_n, v_n) \to 0$. Then*

$$\lim_{\delta \to 0} \sup_{\substack{\tau(u-v) \le \delta \\ u,v \in \Gamma}} \frac{|G(u) - G(v)|}{\omega(\tau(u-v))} \le C, \quad \textit{a.s. for some constant } C < \infty$$

implies that

$$\lim_{\delta \to 0} \sup_{\substack{\tau(u-v) \le \delta \\ u,v \in \Gamma}} \frac{|G(u) - G(v)|}{\omega(\tau(u-v))} = C', \quad \textit{a.s. for some constant } C' < \infty.$$

This result is also valid for the local modulus of continuity of G, that is, it holds with v replaced by u_0 and with the supremum taken over $u \in \Gamma$.

Remark 4.1. *Lemma 4.1 is slightly different from Lemma 7.1.1 in Marcus and Rosen [19], where τ is assumed to be a pseudo-norm. However, by carefully checking its proof in [19], this requirement can be replaced by the conditions stated in Lemma 4.1.*

Using the above lemma, we prove the following uniform modulus of continuity theorem. For convenience, let $B_E(r) := \{x \in \mathrm{R}^N : \tau_E(x) \le r\}$ and $B(r) = \{x \in \mathrm{R}^N : \|x\| \le r\}$ for all $r \ge 0$, and $I := [0,1]^N$.

Theorem 4.2. *Let $X = \{X(t), t \in \mathrm{R}^N\}$ be a centered, real-valued Gaussian random field defined as in (2.12). Then*

$$\lim_{r\to 0} \sup_{\substack{s,t\in I\\ \tau_E(s-t)\le r}} \frac{|X(s)-X(t)|}{\tau_E(s-t)\sqrt{\log(1+\tau_E(s-t)^{-1})}} = C_6 \qquad a.s., \tag{4.1}$$

where C_6 is a positive and finite constant.

Proof. Note that due to monotonicity the limit in the left hand side of (4.1) exists almost surely, and the key point is that this limit is a positive and finite constant.

For $t, t' \in I$, let $\beta(t,t') = \tau_E(t-t')\sqrt{\log(1+\tau_E(t-t')^{-1})}$ and let

$$\mathcal{J}(r) = \sup_{\substack{t,t'\in I\\ \tau_E(t-t')\le r}} \frac{|X(t)-X(t')|}{\beta(t,t')}.$$

First we prove that $\lim_{r\to 0}\mathcal{J}(r) \le C < \infty$ almost surely. We introduce an auxiliary Gaussian field:

$$Y = \{Y(t,s),\, t \in I,\, s \in B_E(r)\}$$

defined by $Y(t,s) = X(t+s) - X(t)$, where r is sufficiently small such that $B_E(r) \subseteq [-1,\,1]^N$. Since X has stationary increments and $X(0) = 0$, $d_X(s,t) = d_X(0,t-s)$ for any $s,t \in \mathrm{R}^N$, the canonical metric d_Y on $T := I \times B_E(r)$ associated with Y satisfies the following inequality:

$$d_Y((t,s),(t',s')) \le C\min\{d_X(0,s)+d_X(0,s'), d_X(s,s')+d_X(t,t')\}$$

for some constant C. Denote the diameter of T in the metric d_Y by D. By Theorem 3.2 (i), we have that

$$D \le C \sup_{\substack{s\in B_E(r)\\ s'\in B_E(r)}} (d_X(0,s)+d_X(0,s')) \le Cr$$

for some constant C. Note that by Theorem 3.2 (i) and (2.5), for a given small $\delta > 0$, there is $C > 0$ such that

$$d_X(s,t) \le C\|t-s\|^{1/(a_p+\delta)}.$$

Therefore, there exists $C > 0$ such that for small $\varepsilon > 0$, if $\|t-t'\| < C\varepsilon^{a_p+\delta}$ and $\|s-s'\| < C\varepsilon^{a_p+\delta}$, then

$$(t',s') \in O_{d_Y}((t,s),\varepsilon) = \{(u,v) : d_Y((u,v),(t,s)) < \varepsilon\}.$$

Hence, $N_d(T,\varepsilon)$, the smallest number of open d_Y-balls of radius ε needed to cover T, satisfies

$$N_d(T,\varepsilon) \le C\varepsilon^{-2N(a_p+\delta)},$$

for some constant $C>0$. Then one can verify that for some constant $C>0$

$$\int_0^D \sqrt{\ln N_d(T,\varepsilon)}d\varepsilon \le Cr\sqrt{\log(1+r^{-1})}.$$

It follows from Lemma 2.1 in Talagrand [26] that for all $u \ge 2Cr\sqrt{\log(1+r^{-1})}$,

$$\mathrm{P}\bigg(\sup_{(t,s)\in T} |X(t+s)-X(t)| \ge u\bigg) \le \exp\bigg(-\frac{u^2}{4D^2}\bigg).$$

By a standard Borel-Cantelli argument, we have that for some positive constant $C<\infty$,

$$\limsup_{r\to 0} \sup_{\substack{t\in I\\ \tau_E(s-t)\le r}} \frac{|X(s)-X(t)|}{r\sqrt{\log(1+r^{-1})}} \le C \qquad \text{a.s.}$$

The monotonicity of the functions $r \mapsto r\sqrt{\log(1+r^{-1})}$ implies that $\lim_{r\to 0}\mathcal{J}(r) \le C$ almost surely. Hence, by Lemma 4.1, we see that (4.1) holds for a constant $C_6 \in [0,\infty)$.

In order to show $C_6>0$ it is sufficient to prove that

$$\lim_{r\to 0+} \mathcal{J}(r) \ge C_7, \qquad \text{a.s.,} \tag{4.2}$$

where $C_7 = C_4^{-1}\sqrt{2C_5 a_1}$. Recall that a_1 is the real part of eigenvalue λ_1. For any $k\ge 1$, we let

$$x_i^{(k)} = \langle i2^{-n_k}\rangle, \qquad i=0,1,2,\cdots,2^{n_k}$$

and $r_k = \tau_E(\langle 2^{-n_k}\rangle)$, where the sequence $\{n_k\}$ is taken as in Lemma 2.4. Since $0<\tau_E(\langle 2^{-n_k}\rangle)\to 0$ as $k\to\infty$, the monotonicity of $\mathcal{J}(r)$ implies that

$$\begin{aligned}\lim_{r\to 0+}\mathcal{J}(r) &= \lim_{k\to\infty} \sup_{s,t\in I,\tau_E(s-t)\le r_k} \frac{|X(s)-X(t)|}{\beta(s,t)} \\ &\ge \liminf_{k\to\infty} \max_{0\le i\le \frac12(2^{n_k}-1)} \frac{|X(x_{2i+1}^{(k)})-X(x_{2i}^{(k)})|}{r_k\sqrt{\log(1+r_k^{-1})}} \\ &=: \liminf_{k\to\infty} \mathcal{J}_k. \end{aligned} \tag{4.3}$$

For any small $\delta>0$, denote $C_8 = C_4^{-1}\sqrt{2C_5(a_1-\delta)}$. For any $\mu\in(0,1)$ we write

$$\begin{aligned}&\mathrm{P}\big(\mathcal{J}_k \le (1-\mu)C_8\big)\\ &\quad = \mathrm{P}\bigg(\bigg\{\frac{|X(\langle 1-2^{-n_k}\rangle)-X(\langle 1-2^{-n_k+1}\rangle)|}{r_k\sqrt{\log(1+r_k^{-1})}} \le (1-\mu)C_8\bigg\}\\ &\qquad \bigcap\bigg\{\max_{0\le i\le\frac12(2^{n_k}-1)-1} \frac{|X(\langle(2i+1)2^{-n_k}\rangle)-X(\langle(2i)2^{-n_k}\rangle)|}{r_k\sqrt{\log(1+r_k^{-1})}} \le (1-\mu)C_8\bigg\}\bigg). \end{aligned} \tag{4.4}$$

Let

$$P_1(k) = \mathrm{P}\Bigg(\max_{0 \le i \le \frac{1}{2}(2^{n_k}-1)-1} \frac{|X(\langle(2i+1)2^{-n_k}\rangle) - X(\langle(2i)2^{-n_k}\rangle)|}{r_k\sqrt{\log(1+r_k^{-1})}} \le (1-\mu)C_8 \Bigg) \tag{4.5}$$

and

$$P_2(k) = \mathrm{P}\Bigg(\frac{|X(\langle 1-2^{-n_k}\rangle) - X(\langle 1-2^{-n_k+1}\rangle)|}{r_k\sqrt{\log(1+r_k^{-1})}} \le (1-\mu)C_8 \Bigg| X(\langle 1-2^{-n_k+1}\rangle); \\ X(\langle(2i+1)2^{-n_k}\rangle), X(\langle(2i)2^{-n_k}\rangle), 0 \le i \le \frac{1}{2}(2^{n_k}-1)-1 \Bigg). \tag{4.6}$$

It follows from Theorem 3.2 and Lemma 2.4 that

$$\begin{aligned} &\mathrm{Var}\Big(X(\langle 1-2^{-n_k}\rangle) - X(\langle 1-2^{-n_k+1}\rangle) \Big| X(\langle 1-2^{-n_k+1}\rangle); \\ &\qquad\qquad X(\langle(2i+1)2^{-n_k}\rangle), X(\langle(2i)2^{-n_k}\rangle), 0 \le i \le \frac{1}{2}(2^{n_k}-1)-1 \Big) \\ &\ge C_5 \min_{1\le i\le 2^{n_k}} \tau_E^2(\langle i2^{-n_k}\rangle) \ge C_5 C_4^{-2} \tau_E^2(\langle 2^{-n_k}\rangle) = C_5 C_4^{-2}\, r_k^2. \end{aligned}$$

Thus by the fact that the conditional distributions of the Gaussian process is almost surely Gaussian, and by Anderson's inequality (see Anderson [1]) and the definition of C_8, we obtain

$$P_2(k) \le \mathrm{P}\Bigg(N(0,1) \le (1-\mu)\sqrt{2(a_1-\delta)\log(1+r_k^{-1})} \Bigg),$$

where $N(0,1)$ denotes a standard normal random variable. By using the following well-known inequality

$$(2\pi)^{-\frac{1}{2}}(1-x^{-2})x^{-1}\mathrm{e}^{-\frac{x^2}{2}} \le \mathrm{P}(N(0,1) > x) \le (2\pi)^{-\frac{1}{2}} x^{-1}\mathrm{e}^{-\frac{x^2}{2}}, \quad \forall\, x > 0,$$

we derive that for all k large enough

$$\begin{aligned} P_2(k) &\le 1 - \mathrm{P}\Bigg(N(0,1) > (1-\mu)\sqrt{2(a_1-\delta)\log(1+r_k^{-1})} \Bigg) \\ &\le 1 - r_k^{(1-\mu/2)^2(a_1-\delta)} \le \exp\Bigg(- r_k^{(1-\mu/2)^2(a_1-\delta)} \Bigg). \end{aligned} \tag{4.7}$$

Combining (4.4) with (4.5), (4.6) and (4.7), we have that

$$\mathrm{P}(\mathcal{J}_k \le (1-\mu)C_8) \le \exp\Bigg(- r_k^{(1-\mu/2)^2(a_1-\delta)} \Bigg) P_1(k).$$

By repeating the above argument, we obtain

$$\mathrm{P}\big(\mathcal{J}_k \le (1-\mu)C_8\big) \le \exp\left(-\frac{2^{n_k}-1}{2}\, r_k^{(1-\mu/2)^2(a_1-\delta)}\right) \le \exp\big(-C\, 2^{\mu n_k/2}\big), \quad (4.8)$$

where the last inequality follows from the estimate:

$$r_k^2 = \tau_{_E}^2(\langle 2^{-n_k}\rangle) \ge C_1^2 \|\langle 2^{-n_k}\rangle\|^{\frac{2}{a_1-\delta}} \ge C_1^2 2^{\frac{-2n_k}{a_1-\delta}}.$$

By (4.8) and the Borel-Cantelli lemma, we have $\liminf_{k\to\infty} \mathcal{J}_k \ge (1-\mu)C_8$ a.s. Letting $\mu \to 0$ and $\delta \to 0$ yields (4.2). The proof of Theorem 4.2 is completed. □

5. Laws of the iterated logarithm

For any fixed $t_0 \in \mathrm{R}^N$ and a family of neighborhoods $\{O(r): r>0\}$ of $0 \in \mathrm{R}^N$ whose diameters go to 0 as $r \to 0$, we consider in this section the corresponding local modulus of continuity of X at t_0

$$\omega(t_0, r) = \sup_{s\in O(r)} |X(t_0+s) - X(t_0)|.$$

Since X is anisotropic, the rate at which $\omega(t_0, r)$ goes to 0 as $r \to 0$ depends on the shape of $O(r)$. A natural choice of $O(r)$ is $B_E(r)$.

For specification and simplification, in this section, let E be a Jordan canonical form of (2.1), which satisfies all assumptions in Section 2. Recall that $\tilde{l}_j$ is the size of J_j. For any $i = 1, 2, \cdots, N$, if $\tilde{l}_1 + \cdots + \tilde{l}_{j-1} + 1 \le i \le \tilde{l}_1 + \cdots + \tilde{l}_j$, then

$$e_i = \{\underbrace{0, \cdots, 0, 1}_{i}, 0, \cdots, 0\} \in W_j.$$

The following theorem characterizes the exact local modulus of continuity of X.

Theorem 5.1. *There is a positive and finite constant C_9 such that for every $t_0 \in \mathrm{R}^N$ we have*

$$\lim_{r\to 0+} \sup_{s-t_0\in B_E(r)} \frac{|X(s) - X(t_0)|}{\tau_E(s-t_0)\sqrt{\log\log(1+\tau_E(s-t_0)^{-1})}} = C_9 \quad a.s. \quad (5.1)$$

In order to show this result, we will make use of the following lemmas.

Lemma 5.2. *There exist positive and finite constants u_0 and C_{10} such that for all $t_0 \in \mathrm{R}^N$, $u \ge u_0$ and sufficiently small $r > 0$,*

$$\mathrm{P}\left(\sup_{s\in B_E(r)} |X(t_0+s) - X(t_0)| \ge ur\sqrt{\log\log(1+r^{-1})}\right) \le e^{-C_{10}\, u^2 \log\log(1+r^{-1})}.$$

Proof. We introduce an auxiliary Gaussian field $Y = \{Y(s),\, s \in B_E(r)\}$ defined by $Y(s) = X(t_0+s) - X(t_0)$. Since X has stationary increments and $X(0)=0$, we have $d_Y(s,s') = d_X(s,s')$ for all $s, s' \in \mathrm{R}^N$. Denote the diameter of $B_E(r)$ in the metric d_Y by D. It follows from Lemma 3.1 that $D \le Cr$ for some finite constant C. Note that the decomposition of $x = (x_1, x_2, \cdots, x_N) \in B_E(r)$ in W_j is

$$\bar{x}_j = (0, \cdots, 0, x_{\tilde{l}_1+\cdots+\tilde{l}_{j-1}+1}, \cdots, x_{\tilde{l}_1+\cdots+\tilde{l}_j}, 0, \cdots, 0).$$

For any $j = 1, 2, \cdots, p$, let $l_j = \tilde{l}_j$ if J_j is a Jordan cell matrix as in (2.2) or $l_j = \tilde{l}_j/2$ if J_j is of the form (2.3). By Lemma 2.1 and (2.7), we have that for sufficiently small r,

$$\|\bar{x}_j\|^{1/a_j} \big| \ln \|\bar{x}_j\| \big|^{-(l_j-1)/a_j} \le C\, r.$$

This implies that there exists a constant C, which may depend on a_j, such that for all i with $\tilde{l}_1 + \cdots + \tilde{l}_{j-1} + 1 \le i \le \tilde{l}_1 + \cdots + \tilde{l}_j$,

$$|x_i| \le C r^{a_j} |\ln r|^{l_j - 1}.$$

Therefore $B_E(r) \subset [-h, h]$ for sufficiently small $r > 0$, where $h = (h_1, h_2, \cdots, h_N)$ with $h_i = C r^{a_j} |\ln r|^{l_j-1}$ as $\tilde{l}_1 + \cdots + \tilde{l}_{j-1} + 1 \le i \le \tilde{l}_1 + \cdots + \tilde{l}_j$. Furthermore, from (2.7), we have that for any $x = (x_1, x_2, \cdots, x_N) \in \mathrm{R}^N$ and sufficiently small $\varepsilon > 0$, if

$$|x_i| < \Big(\frac{\varepsilon}{N\mu}\Big)^{a_j} \Big| \ln(\frac{\varepsilon}{N\mu}) \Big|^{-(l_j-1)/a_j},$$

for $\tilde{l}_1 + \cdots + \tilde{l}_{j-1} + 1 \le i \le \tilde{l}_1 + \cdots + \tilde{l}_j$, where μ is a constant whose value will be determined later, then

$$\begin{aligned} \tau_{_E}(\vec{x}_i) &\le C\, \frac{\varepsilon}{N\mu} \Big| \ln(\frac{\varepsilon}{N\mu}) \Big|^{-(l_j-1)} \Big| \ln\Big[(\frac{\varepsilon}{N\mu})^{a_j} \big| \ln(\frac{\varepsilon}{N\mu}) \big|^{-(l_j-1)/a_j} \Big] \Big|^{(l_j-1)/a_j} \\ &\le C\, \frac{\varepsilon}{N\mu}, \end{aligned} \tag{5.2}$$

where $\vec{x}_i = (\underbrace{0, \cdots, 0, x_i}_{i}, 0, \cdots, 0) \in \mathrm{R}^N$. Then by (2.4) and (5.2), there exists a constant $C > 0$ such that

$$\tau_{_E}(x) = \tau_{_E}\Big(\sum_{i=1}^N \vec{x}_i \Big) \le C \sum_{i=1}^N \frac{\varepsilon}{N\mu} \le C \frac{\varepsilon}{\mu}.$$

By using Lemma 3.1 again, we have

$$d_Y(0, x) \le C\, \tau_{_E}(x) \le C_{11} \varepsilon/\mu.$$

Now we take $\mu > C_{11}$, then $x \in O_{d_Y}(\varepsilon)$ implies $[0, x) \subset O_{d_Y}(\varepsilon)$. Therefore the smallest number of open d_Y-balls of radius ε needed to cover $B_E(r) := T$, denoted by $N_d(T, \varepsilon)$, satisfies

$$N_d(T,\varepsilon) \le C \prod_{j=1}^{p} \bigg(\frac{r^{a_j}}{(\frac{\varepsilon}{\mu N})^{a_j}} \Big| \ln(\frac{\varepsilon}{\mu N}) \Big|^{(l_j-1)/a_j} \Big| \ln r \Big|^{(l_j-1)} \bigg)^{l_j}$$

for some constant $C > 0$. Then one can verify that

$$\int_0^D \sqrt{\ln N_d(T,\varepsilon)}\, d\varepsilon \le C\, r \sqrt{\log\log(1+r^{-1})}.$$

It follows from [26, Lemma 2.1] that for all sufficiently large u,

$$\mathrm{P}\bigg(\sup_{s\in B_E(r)} |X(t_0+s) - X(t_0)| \ge ur\sqrt{\log\log(1+r^{-1})} \bigg) \le \exp\bigg(- C_{10} u^2 \log\log(1+r^{-1}) \bigg).$$

This finishes the proof of Lemma 5.2. □

Lemma 5.3. *There is a constant $C_{12} \in [0,\infty)$ such that for every fixed $t_0 \in \mathrm{R}^N$,*

$$\lim_{\varepsilon\to 0} \sup_{s-t_0\in B_E(\varepsilon)} \frac{|X(s) - X(t_0)|}{\tau_E(s-t_0)\sqrt{\log\log(1+\tau_E(s-t_0)^{-1})}} = C_{12} \qquad a.s. \tag{5.3}$$

Proof. By Lemma 4.1, it is sufficient to prove

$$\lim_{\varepsilon\to 0} \sup_{s-t_0\in B_E(\varepsilon)} \frac{|X(s) - X(t_0)|}{\tau_E(s-t_0)\sqrt{\log\log(1+\tau_E(s-t_0)^{-1})}} \le C < \infty, \tag{5.4}$$

for some constant $C > 0$. Let $\varepsilon_n = \mathrm{e}^{-n}$, consider the event

$$E_n = \bigg\{ \sup_{s-t_0\in B_E(\varepsilon)} \frac{|X(s) - X(t_0)|}{\varepsilon_n \sqrt{\log\log(1+\varepsilon_n^{-1})}} > u \bigg\},$$

where $u > C_{10}^{-1/2}$ is a constant. By Lemma 5.2 we have $\mathrm{P}(E_n) \le \mathrm{e}^{-C_{10}u^2 \log n}$ for all sufficiently large n. Hence, the Borel-Cantelli Lemma implies

$$\limsup_{\varepsilon\to 0} \sup_{s\in I,\, s-t_0\in B_E(\varepsilon)} \frac{|X(s) - X(t_0)|}{\varepsilon\sqrt{\log\log(1+\varepsilon^{-1})}} \le u.$$

This and a monotonicity argument yield (5.4). □

We will also need the following truncation inequalities which extend a result in Luan and Xiao [18].

Lemma 5.4. *For a given $N \times N$ matrix E, there exists a constant $r_0 > 0$ such that for any $u > 0$ and any $t \in \mathrm{R}^N$ with $\tau_{_E}(t)u \le r_0$, we have*

$$\int_{\{\tau_{_{E'}}(\xi)<u\}} \langle t,\xi\rangle^2 \frac{d\xi}{\psi(\xi)^{2+Q}} \le 3\int_{\mathrm{R}^N} (1-\cos\langle t,\xi\rangle)\frac{d\xi}{\psi(\xi)^{2+Q}}. \tag{5.5}$$

Proof. Let $M = \max\{\|x\|, x \in S_E\}$, $K(r) = \max\{\|x\|, \tau_{_{E'}}(x) \le r\}$. Since S_E is compact set without 0 and $\tau_{_{E'}}(\cdot)$ is continuous, $M > 0$ and $K(r)$ continuous with $K(0) = 0$, $K(r) \to \infty$ as $r \to \infty$. Therefore, there exists $r_0 > 0$ such that $MK(r) \le 1$ for all $r < r_0$. By using the inequality $u^2 \le 3(1-\cos u)$ for all real numbers $|u| \le 1$, we derive that if $\tau_{_E}(t)u \le r_0$, then

$$\begin{aligned}
&\int_{\{\tau_{_{E'}}(\xi)<u\}} \langle t,\xi\rangle^2 \frac{d\xi}{\psi(\xi)^{2+Q}} = \int_{\{\tau_{_{E'}}(\xi)<u\}} \langle \tau_{_E}^E(t) l_E(t),\xi\rangle^2 \frac{d\xi}{\psi(\xi)^{2+Q}} \\
&= \int_{\{\tau_{_{E'}}(\xi)<u\}} \langle l_E(t), \tau_{_E}^{E'}(t)\xi\rangle^2 \frac{d\xi}{\psi(\xi)^{2+Q}} = \tau_{_E}^2(t)\int_{\{\tau_{_{E'}}(\xi)<\tau_{_E}(t)u\}} \langle l_E(t),\xi\rangle^2 \frac{d\xi}{\psi(\xi)^{2+Q}} \\
&\le 3\tau_{_E}^2(t)\int_{\{\tau_{_{E'}}(\xi)<\tau_{_E}(t)u\}} \big(1-\cos\langle l_E(t),\xi\rangle\big)\frac{d\xi}{\psi(\xi)^{2+Q}} \\
&= 3\int_{\{\tau_{_{E'}}(\xi)<u\}} \big(1-\cos\langle l_E(t),\tau_{_E}^{E'}(t)\xi\rangle\big)\frac{d\xi}{\psi(\xi)^{2+Q}},
\end{aligned}$$

which equals

$$3\int_{\{\tau_{_{E'}}(\xi)<u\}} \big(1-\cos\langle t,\xi\rangle\big)\frac{d\xi}{\psi(\xi)^{2+Q}} \le 3\int_{\mathrm{R}^N} \big(1-\cos\langle t,\xi\rangle\big)\frac{d\xi}{\psi(\xi)^{2+Q}}. \tag{5.6}$$

The proof of this lemma is complete. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. By Lemma 5.3 and the stationary increments property of X, it only remains to show

$$\lim_{\varepsilon\to 0+}\sup_{s\in B_E(\varepsilon)} \frac{|X(s)|}{\tau_E(s)\sqrt{\log\log(1+\tau_E(s)^{-1})}} \ge C \tag{5.7}$$

for some constant $C > 0$.

For any $0 < \mu < 1$ and $n \ge 1$, we define $s_n = (0, \cdots, 0, e^{-a_p n^{1+\mu}}) \in \mathrm{R}^N$. By (2.7)

$$C_3^{-1} e^{-n^{1+\mu}} |a_p n^{1+\mu}|^{-\frac{l_p-1}{a_p}} \le \tau_{_E}(s_n) \le C_3 e^{-n^{1+\mu}} |a_p n^{1+\mu}|^{\frac{l_p-1}{a_p}}. \tag{5.8}$$

For every integer $n \ge 1$, let $d_n = \exp(n^{1+\mu} + n^\mu)$. Denote $U = \exp(\mu(n-1)^\mu)$. Notice that as $n \to \infty$,

$$\tau_E(Us_n)d_{n-1} \le C\, U^{1/a_p}\Big|a_p n^{1+\mu} - \mu(n-1)^{\mu}\Big|^{\frac{l_p-1}{a_p}} \exp(-n^{1+\mu} + (n-1)^{1+\mu} + (n-1)^{\mu})$$
$$\le C\,\Big|a_p n^{1+\mu} - \mu(n-1)^{\mu}\Big|^{\frac{l_p-1}{a_p}} \exp\Big(-\mu(1-\frac{1}{a_p})(n-1)^{\mu}\Big) \to 0.$$

It follows from Lemma 5.4, Lemma 3.1 and (2.7) that

$$\int_{\{\tau_{E'}(\xi)\le d_{n-1}\}} \langle s_n, \xi\rangle^2 \frac{d\xi}{\psi(\xi)^{2+Q}} = U^{-2}\int_{\{\tau_{E'}(\xi)\le d_{n-1}\}} \langle s_n U, \xi\rangle^2 \frac{d\xi}{\psi(\xi)^{2+Q}}$$
$$\le CU^{-2} d_X^2(Us_n, 0) \le CU^{-2}\tau_E^2(Us_n)$$
$$\le CU^{-2}U^{\frac{2}{a_p}}\big|\ln\|s_n\|\big|^{2\frac{l_p-1}{a_p}}\big|\ln\|Us_n\|\big|^{2\frac{l_p-1}{a_p}}\tau_E^2(s_n)$$
$$\le C\exp\Big(-(1-\frac{1}{a_p})\mu(n-1)^{\mu}\Big)\tau_E^2(s_n) \tag{5.9}$$

for n large enough. On the other hand, noting that ψ is E'-homogeneous, by using [8, Proposition 2.3], we obtain that

$$\int_{\{\tau_{E'}(\xi)>d_n\}} \frac{d\xi}{\psi(\xi)^{2+Q}} = \int_{d_n}^{\infty} dr \int_{S_{E'}} \frac{1}{r^{2+Q}\psi(\theta)} r^{Q-1}\sigma(d\theta) \le C d_n^{-2},$$

since $\sigma(d\theta)$ is a finite measure on $S_{E'}$. Furthermore,

$$d_n^{-2} = \mathrm{e}^{-2n^{1+\mu}-2n^{\mu}} = \mathrm{e}^{-2n^{1+\mu}}|\ln \mathrm{e}^{-a_p n^{1+\mu}}|^{-2\frac{l_p-1}{a_p}}|a_p n^{1+\mu}|^{2\frac{l_p-1}{a_p}}\mathrm{e}^{-2n^{\mu}}$$
$$\le C\tau_E^2(s_n)|n^{1+\mu}|^{2\frac{l_p-1}{a_p}}\mathrm{e}^{-2n^{\mu}},$$

when n is large enough. Therefore, for sufficiently large n,

$$\int_{\{\tau_{E'}(\xi)>d_n\}} \frac{d\xi}{\psi(\xi)^{2+Q}} \le C\tau_E^2(s_n)\mathrm{e}^{-n^{\mu}}. \tag{5.10}$$

Now we decompose X into two independent parts as follows.

$$\widetilde{X}_n(t) = \int_{\{\tau_{E'}(\xi)\notin(d_{n-1},d_n]\}} \big(e^{i\langle t,\xi\rangle}-1\big)\frac{\widetilde{\mathcal{M}}(d\xi)}{\psi(\xi)^{1+Q/2}} \tag{5.11}$$

and

$$X_n(t) = \int_{\{\tau_{E'}(\xi)\in(d_{n-1},d_n]\}} \big(e^{i\langle t,\xi\rangle}-1\big)\frac{\widetilde{\mathcal{M}}(d\xi)}{\psi(\xi)^{1+Q/2}}. \tag{5.12}$$

Notice that the random fields $\{X_n(t), t\in \mathrm{R}^N\}$, $n = 1, 2, \ldots$ are independent.

Let

$$I_1(n) = \frac{|X_n(s_n)|}{\tau_{_E}(s_n)\sqrt{\log\log(1+\tau_{_E}(s_n)^{-1})}}$$

and

$$I_2(n) = \frac{|\widetilde{X}_n(s_n)|}{\tau_{_E}(s_n)\sqrt{\log\log(1+\tau_{_E}(s_n)^{-1})}}.$$

Then

$$\begin{aligned}\lim_{\varepsilon\to 0+}\sup_{s\in B_E(\varepsilon)} \frac{|X(s)|}{\tau_E(s)\sqrt{\log\log(1+\tau_E(s)^{-1})}} &\geq \limsup_{n\to\infty} \frac{|X(s_n)|}{\tau_{_E}(s_n)\sqrt{\log\log(1+\tau_{_E}(s_n)^{-1})}} \\ &\geq \limsup_{n\to\infty} I_1(n) - \limsup_{n\to\infty} I_2(n). \qquad (5.13)\end{aligned}$$

By using (5.9), (5.10) and the same argument in the proof of Theorem 5.5 in [22], we can readily get that

$$\limsup_{n\to\infty} I_2(n) = 0, \quad \text{a.s.} \qquad (5.14)$$

In order to estimate $\limsup_{n\to\infty} I_1(n)$, using Lemma 3.1 again, we have that

$$\mathrm{E}\big(X_n(s_n)\big)^2 \leq d_X^2(s_n, 0) \leq C_{13}\,\tau_{_E}^2(s_n).$$

Again, by the corresponding argument in the proof of Theorem 5.5 in [22], it is easy to get that

$$\limsup_{n\to\infty} I_1(n) \geq \sqrt{2C_{13}} \qquad \text{a.s.} \qquad (5.15)$$

Hence (5.7) follows from (5.13), (5.14) and (5.15). □

6. Examples

Finally we provide two examples of operator scaling Gaussian random fields with stationary increments to illustrate our results and compare them with those in Meerschaert, Wang and Xiao [22]. In particular, Example 6.2 shows that the regularity properties of X depend subtly on its scaling exponent E.

Example 6.1 If E has a Jordan canonical form (2.1) such that, for all $k = 1, 2, \cdots, p$, $\tilde{l}_k = 1$ if J_k is a Jordan cell matrix and $\tilde{l}_k = 2$ if J_k is not a Jordan cell matrix. Then for any $t = (t_1, \cdots, t_N) \in \mathrm{R}^N$, by (2.7) and Lemma 2.2, we have

$$\tau_{_E}(t) \asymp \sum_{i=1}^{N} |t_i|^{1/a_i},$$

where a_i is the real part of eigenvalue(s) corresponding to J_k such that $\sum_{j=1}^{k-1}\tilde{l}_j + 1 \le i \le \sum_{j=1}^{k}\tilde{l}_j$. Therefore, in this case, Theorems 4.2 and 5.1 are of the same form as the corresponding results in Meerschaert, Wang and Xiao [22].

Example 6.2 We consider the Gaussian random field $\{X(t), t \in \mathrm{R}^2\}$ defined by (2.12) with scaling exponent E, a Jordan matrix, as follows

$$E = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix},$$

where $a > 1$ is a constant. Then $p = 1$ and $\tilde{l}_1 = 2$. For any $t > 0$, by straightforward computations, we have

$$t^E = t^a \begin{pmatrix} 1 & 0 \\ \ln t & 1 \end{pmatrix}.$$

According to Lemma 6.1.5 in [21], the norm $\|\cdot\|_E$ induced by E is defined as that for any $x \in \mathrm{R}^2$

$$\|x\|_E = \int_0^1 \frac{\|t^E x\|}{t} dt.$$

Note that we can uniquely represent $x \in \mathrm{R}^2$ as $(0, s)$ or $(s, \theta s)$ for some $s \in \mathrm{R}$, $\theta \in \mathrm{R}$. When $x = (0, s)$,

$$\|x\|_E = \int_0^1 |s| t^{a-1} dt = \frac{|s|}{a}, \tag{6.1}$$

and when $x = (s, \theta s)$,

$$\|x\|_E = \int_0^1 |s| t^{a-1} \sqrt{1 + (\theta + \ln t)^2} dt =: |s|\alpha(\theta). \tag{6.2}$$

It is easy to see that $\alpha(\theta)$ is continuous on $\theta \in \mathrm{R}$ with $\alpha(\theta) > 1/a$ and that $|\theta|/\alpha(\theta)$ is bounded since $|\theta|/\alpha(\theta)$ is continuous and

$$\lim_{\theta\to\infty} \frac{|\theta|}{\alpha(\theta)} = a. \tag{6.3}$$

We have $\alpha := \min_\theta \alpha(\theta) > 1/a$. From (6.1) and (6.2) we have

$$S_E = \{x : \|x\|_E = 1\} = \left\{ \pm \begin{pmatrix} 0 \\ a \end{pmatrix}, \pm \frac{1}{\alpha(\theta)} \begin{pmatrix} 1 \\ \theta \end{pmatrix} : \theta \in \mathrm{R} \right\},$$

and $\mathrm{R}^2 = \{s^E y : s \ge 0, y \in S_E\}$.

To unify the notation, we set

$$\frac{\theta}{\alpha(\theta)} = \pm a \quad \text{and} \quad \frac{1}{\alpha(\theta)} = 0$$

when $\theta = \pm\infty$. Then for any $x \in \mathrm{R}^2$ with $\tau_{_E}(x) = s$, there exists $\theta \in [-\infty, +\infty]$ such that

$$x = \pm s^E \frac{1}{\alpha(\theta)} \begin{pmatrix} 1 \\ \theta \end{pmatrix} = \pm \frac{s^a}{\alpha(\theta)} \begin{pmatrix} 1 \\ \theta + \ln s \end{pmatrix}, \tag{6.4}$$

where $s^a \ln s|_{s=0} := 0$ and the sign $+$ or $-$ depends on x.

Now we reformulate Theorem 4.2 and Theorem 5.1 for the present case. For convenience, we express the vector $y \in \mathrm{R}^2$ in terms of $s = \tau_E(y)$ and θ by

$$y = y(s, \theta, w) = (-1)^w \Big(\frac{s^a}{\alpha(\theta)}, \frac{s^a}{\alpha(\theta)} (\theta + \ln s) \Big),$$

where $w \in \{0, 1\}$.

Conclusion A. Let $I = [0,1]^2$. Then

$$\lim_{r \to 0+} \sup_{\substack{s \le r,\ \theta \in [-\infty, +\infty] \\ w \in \{0,1\}, x, x+y \in I}} \frac{|X(x + y(s,\theta,w)) - X(x)|}{s\sqrt{\log(1 + s^{-1})}} = C_{17} \quad \text{a.s.}, \tag{6.5}$$

and that for any $x_0 \in I$,

$$\lim_{r \to 0+} \sup_{s \le r,\ \theta \in [-\infty, +\infty],\ w \in \{0,1\}} \frac{|X(x_0 + y(s,\theta,w)) - X(x_0)|}{s\sqrt{\log\log(1 + s^{-1})}} = C_{18} \quad \text{a.s.}, \tag{6.6}$$

where C_{17} and C_{18} are positive and finite constants. □

Next we describe the asymptotic behavior of $\tau_E(y)$ as $\|y\| \to 0$ along three types of curves in R^2:

(i) If $\theta = -\ln s + c$ for a constant $c \in \mathrm{R}$, then $y = y(s, \theta, w) = (-1)^w (s^a/\alpha(\theta), cs^a/\alpha(\theta))$ satisfies

$$\|y\| = \frac{\sqrt{1 + c^2} s^a}{\alpha(\theta)} = \frac{\sqrt{1 + c^2} s^a}{\alpha(c - \ln s)}.$$

This, together with (6.3), implies that as $\|y\| \to 0$,

$$s = \tau_E(y) \sim \|y\|^{1/a} \Big| \ln \|y\| \Big|^{1/a}, \tag{6.7}$$

where the notation "$\sim$" means that as $\|y\| \to 0$ the quotient of the two sides of $\sim$ goes to a positive constant.

(ii) If $\theta = \pm\infty$, then $y(s, \theta, w) = (-1)^w (0, as^a)$ and

$$s = \tau_E(y) = \frac{1}{a^{1/a}} \|y\|^{1/a}. \tag{6.8}$$

(iii) If θ is fixed in $(-\infty, +\infty)$, then for $y = y(s, \theta, w)$,

$$\|y\| = \frac{s^a}{\alpha(\theta)} \sqrt{1 + (\theta + \ln s)^2},$$

which implies that as $\|y\| \to 0$,

$$s = \tau_E(y) \sim \|y\|^{1/a} \Big| \ln \|y\| \Big|^{-1/a}. \tag{6.9}$$

In the following, we derive the exact uniform moduli of continuity of $X(x)$ by using the norm $\|\cdot\|$ in three different cases which are intuitively corresponding to the three types mentioned above. These results illustrate the subtle changes of the regularity properties of X. For the exact local moduli of continuity, similar results are true as well. In order not to make the paper too lengthy, we leave it to interested readers.

Conclusion B (1) If $I_1 = \{(t, t) : \ t \in [0, 1]\}$, then

$$\lim_{\|y\| \to 0} \sup_{x, x+y \in I_1} \frac{|X(x + y) - X(x)|}{(\|y\| \big| \ln \|y\| \big|)^{1/a} \sqrt{\log(1 + \|y\|^{-1})}} = C_{19} \in (0, \infty) \quad \text{a.s.} \tag{6.10}$$

(2) If $I_2 = \{(0, t) : \ t \in [0, 1]\}$, then

$$\lim_{\|y\| \to 0} \sup_{x, x+y \in I_2} \frac{|X(x + y) - X(x)|}{\|y\|^{1/a} \sqrt{\log(1 + \|y\|^{-1})}} = C_{20} \in (0, \infty) \quad \text{a.s.} \tag{6.11}$$

(3) Let $\theta_0 \in \arg\min_\theta \alpha(\theta) = \{\vartheta, \alpha(\vartheta) \le \alpha(\theta), \theta \in [-\infty, +\infty]\}$. Then

$$\lim_{\|y\| \to 0} \sup_{\substack{y = y(r, \theta_0, 0) \\ x, x+y \in I}} \frac{\big| \ln \|y\| \big|^{1/a} |X(x + y) - X(x)|}{\|y\|^{1/a} \sqrt{\log(1 + \|y\|^{-1})}} = C_{21} \in (0, \infty) \quad \text{a.s.} \tag{6.12}$$

Proof. (1) Observe that, in the proof of (4.2) in the case of $N = 2$, one can choose the sequences of $\{x_i^{(k)}\}$ such that all the points $x_i^{(k)}$ and the differences $x_{i+1}^{(k)} - x_i^{(k)}$ lie in $I_1 = \{(t, t) : \ t \in [0, 1]\}$. Therefore, the proof of (4.2) essentially shows that

$$\lim_{y \to 0} \sup_{x, x+y \in I_1} \frac{|X(x + y) - X(x)|}{\tau_E(y) \sqrt{\log(1 + \tau_E(y)^{-1})}} \ge C > 0 \quad \text{a.s.} \tag{6.13}$$

Thanks to the formula (6.7), we see that (6.10) follows from the proof of Theorem 4.2.

(2) To prove (6.11), choose $x_i^{(n)} = (0, i2^{-n})$ for $i = 0, 1, \cdots, 2^n$. Then by some obvious modifications, one can easily check that (6.13) is also true with I_2 instead of I_1. Therefore, by (6.8), (6.11) also follows from in the proof of Theorem 4.2.

(3) Note that $\alpha(\theta)$ is continuous and as $\theta \to \infty$, $\alpha(\theta)/|\theta| \to a$. The set $\arg\min_\theta \alpha(\theta)$ is not empty. Let $\alpha_0 = \alpha(\theta_0)$ and

$$x_i^{(n)} = iy(2^{-n}, \theta_0, 0) + (0, 1) = \Big(\frac{i2^{-an}}{\alpha_0}, \frac{i2^{-an}}{\alpha_0} (\theta_0 - n \ln 2) + 1 \Big)$$

for $i = 0, 1, 2, \cdots, K_n$, where

$$K_n = \max\{i, x_i^{(n)} \in [0,1]^2\}.$$

Manifestly, for sufficiently large n, $K_n > 2^n$. Let $r_n := \tau_E(y(2^{-n}, \theta_0, 0))$. Then

$$\lim_{r\to 0} \sup_{\substack{y=y(r,\theta_0,0)\\ x,x+y\in I}} \frac{|X(x+y)-X(x)|}{r\sqrt{\log(1+r^{-1})}}$$
$$\geq \liminf_{n\to\infty} \max_{0\leq i\leq K_n-1} \frac{|X(x_{i+1}^{(n)})-X(x_i^{(n)})|}{r_n\sqrt{\log(1+r_n^{-1})}} =: \liminf_{k\to\infty} \mathcal{J}_n.$$

Note that for $k \geq 1$,

$$ky(2^{-n}, \theta_0, 0) = \Big(\frac{k2^{-an}}{\alpha_0}, \frac{k2^{-an}}{\alpha_0}(\theta_0 - n\ln 2)\Big).$$

There exist some $\theta \in (-\infty, \infty)$, $w \in \{0,1\}$ and $s = \tau_E(ky(2^{-n}, \theta_0, 0))$ such that

$$ky(2^{-n}, \theta_0, 0) = y(s, \theta, w),$$

which implies that $w = 0$ and

$$\frac{s^a}{\alpha(\theta)} = \frac{k2^{-an}}{\alpha_0}.$$

Because $\alpha_0 = \min_\theta \alpha(\theta)$

$$s = \tau_E(ky(2^{-n}, \theta_0, 0)) \geq 2^{-n} := r_n.$$

Therefore, from Theorem 3.2 and Lemma 2.4 we obtain that

$$\mathrm{Var}\Big(X(x_{i+1}^{(n)}) - X(x_i^{(n)})\Big|X(x_k^{(n)}), 0\leq k\leq i\Big) \geq C_5 \min_{1\leq k\leq i+1} \tau_E^2(ky(2^{-n}, \theta_0, 0)) \geq C_5 r_n^2.$$

By the same proof of (4.2) with some obvious modifications, we have that

$$\lim_{r\to 0} \sup_{\substack{y=y(r,\theta_0,0)\\ x,x+y\in I}} \frac{|X(x+y)-X(x)|}{r\sqrt{\log(1+r^{-1})}} \geq C > 0. \tag{6.14}$$

Reviewing the proof of Lemma 7.1.1 in Marcus and Rosen [19], one can easily get that

$$\lim_{r\to 0} \sup_{\substack{y=y(r,\theta_0,0)\\ x,x+y\in I}} \frac{|X(x+y)-X(x)|}{r\sqrt{\log(1+r^{-1})}} \leq C, \quad \text{a.s. for some constant } C < \infty$$

implies that

$$\lim_{r\to 0} \sup_{\substack{y=y(r,\theta_0,0)\\ x,x+y\in I}} \frac{|X(x+y)-X(x)|}{r\sqrt{\log(1+r^{-1})}} = C', \quad \text{a.s. for some constant } C' < \infty.$$

Therefore, from Theorem 4.2 and (6.14) it follows that

$$\lim_{r\to 0} \sup_{\substack{y=y(r,\theta_0,0)\\ x,x+y\in I}} \frac{|X(x+y)-X(x)|}{r\sqrt{\log(1+r^{-1})}} = C \in (0,\infty).$$

This and (6.9) imply (6.12). □

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