Generalized dimensions of images of measures under Gaussian processes

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Abstract

We show that for certain Gaussian random processes and fields $X : \mathbb{R}^N \to \mathbb{R}^d$,

$$D_q(\mu_X) = \min\Big\{d, \frac{1}{\alpha} D_q(\mu)\Big\} \quad \text{a.s.},$$

for an index α which depends on Hölder properties and strong local nondeterminism of X and for $q > 1$, where D_q denotes generalized q-dimension and where μ_X is the image of the measure μ under X. In particular this holds for index-α fractional Brownian motion, for fractional Riesz-Bessel motions and for certain infinity scale fractional Brownian motions.

1 Introduction

Dimensions of images of sets under stochastic processes have been studied for many years. The Hausdorff dimension of the image or sample path of Brownian motion $X : \mathbb{R}_+ \to \mathbb{R}^d$ is almost surely equal to

$$\dim_{\mathrm{H}} X(\mathbb{R}_+) = \min\big\{d, 2\big\},$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension, see Lévy [17], with the exact gauge function for the dimension established by Ciesielski and Taylor [6] for $d \geq 3$ and by Ray [27] and Taylor [32] for $d = 2$. Similar questions were subsequently studied for other processes, notably for sample paths of stable Lévy processes, see [33], and for fractional Brownian motion, see [1, 2, 18, 23, 30, 31]. There are several comprehensive surveys of this work [2, 18, 34, 37] which contain many further references.

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A more general, but very natural, question is to find the almost sure dimensions of the image $X(E)$ of a Borel set $E \subseteq \mathbb{R}^N$ under a process $X : \mathbb{R}^N \to \mathbb{R}^d$, in terms of the dimension of E. In particular, Kahane [18] showed that

$$\dim_{\mathrm{H}} X(E) = \min\left\{d, \frac{\dim_{\mathrm{H}} E}{\alpha}\right\} \quad \text{a.s.} \tag{1.1}$$

if X is index-α fractional Brownian motion (which reduces to standard Brownian motion when $\alpha = \frac{1}{2}$ and $N = 1$).

The corresponding question for packing dimension $\dim_{\mathrm{P}}$, where dimensions of images of sets can behave in a more subtle manner, was not answered until rather later, when Xiao [36] showed that for index-α fractional Brownian motion,

$$\dim_{\mathrm{P}} X(E) = \frac{\dim_P^{\alpha d} E}{\alpha} \quad \text{a.s.},$$

where $\dim_P^s E$ is the 'packing dimension profile' of E, a notion introduced in connection with linear projections of sets by Falconer and Howroyd [14], and which is defined in terms of a certain s-dimensional kernel.

In recent years, many other dimensional properties of the range, graph, level sets and images of given sets have been studied for a wide range of random processes, see [2, 19, 35, 39, 41] for surveys of this work.

It is natural to study dimensional properties of images of measures under random processes or fields in an analogous way to images of sets. For μ a Borel measure on $\mathbb{R}^N$ and $X : \mathbb{R}^N \to \mathbb{R}^d$, the random image measure μ_X on $\mathbb{R}^d$ is defined by

$$\mu_X(A) = \mu\{x : X(x) \in A\}, \quad A \subseteq \mathbb{R}^d.$$

When μ is the Lebesgue measure on $\mathbb{R}^N$ and X is a Gaussian process, the properties of the corresponding image measure μ_X have played important roles in studying the exact Hausdorff measure functions for the range, graph and level sets of X [30, 35]. For more general Borel measures μ, one can look at the almost sure Hausdorff and packing dimensions of the measures (given by the minimal dimension of any set of full measure); indeed, by supporting suitable measures on sets, this approach is often implicit in the set dimension results mentioned above. Explicit results for Hausdorff and packing dimensions of image measures under a wide range of processes are given in [29], with dimension profiles again key in the packing dimension cases.

However, the singularity structure of a measure may be very rich, and multifractal analysis in various forms has evolved to exhibit this structure as a function or spectrum; for general discussions see, for example, [12, 16, 21, 25]. In this paper we consider generalized q-dimensions which reflect the asymptotic behaviour as $r \searrow 0$ of the qth-moment sums $M_r(q) = \sum_C \mu(C)^q$ over the mesh cubes C of side r in $\mathbb{R}^N$. It will be convenient for us to work with the equivalent qth-moment integrals $\int \mu(B(x,r))^{q-1} d\mu(x)$, where $B(x,r)$ is the ball centre x and radius r, see Section 2 for further details of q-dimensions.

Our main results are natural measure analogues of (1.1) for the generalized q-dimension. Thus, for example, for Gaussian processes $X : \mathbb{R}^N \to \mathbb{R}^d$ which are strongly locally α-nondeterministic and which satisfy an α-Hölder condition, and for a compactly supported probability measure μ on $\mathbb{R}^N$, we show that

$$\underline{D}_q(\mu_X) = \min\left\{d, \frac{1}{\alpha}\underline{D}_q(\mu)\right\} \quad \text{a.s.},$$

where $\underline{D}_q$ denotes (lower) generalized q-dimension. Such processes include index-α fractional Brownian motion, fractional Riesz-Bessel motions and infinity scale fractional Brownian motion. We restrict attention to 'larger' moments, that is where $q > 1$.

Typically, upper bounds for the generalized dimensions follow easily from almost sure Hölder continuity of the sample paths. Lower bounds are more elusive because $X : \mathbb{R}^N \to \mathbb{R}^d$ may map many disparate points into a small ball. Simplifying things somewhat, we need to estimate expectations of $\mu_f(B(X(y), r))^{q-1}$. Taking n to be the integer with $n \leq q < n+1$, this involves estimating the probability that the images under X of points $\{x_1, \ldots, x_n, y\}$ in $\mathbb{R}^N$ are within distance r of each other, that is, essentially,

$$\mathsf{P}\big\{|X(y) - X(x_1)| \leq r, |X(y) - X(x_2)| \leq r, \ldots, |X(y) - X(x_n)| \leq r\big\}.$$

We use a tower of conditional expectations together with strong local non-determinism of X (i.e. that conditional variances are dominated by those points that are closest to each other in a uniform sense) to obtain a bound for this probability in terms of an expression $\phi(x_1, \ldots, x_n, y)$ which reflects the mutual distance distribution of the points $\{x_1, \ldots, x_n, y\}$. To estimate this, we integrate ϕ with respect to μ for each $x_1, \ldots, x_n$, then a further integration over y bounds the generalized q-dimension in terms of integrals of ϕ.

This reduces the problem to estimating 'multipotential' integrals of the form

$$\int \left[\int \cdots \int \phi(x_1, \ldots, x_n, y) d\mu(x_1) \ldots d\mu(x_n) \right]^{(q-1)/n} d\mu(y).$$

Two stages are then needed to estimate such integrals. In Section 3 we introduce a device that allows us to replace integrals over Euclidean space by more tractable ones over an ultrametric space, so that the integral becomes an infinite sum over the vertices of an $(n+1)$-ary tree. We then estimate this in Section 4 using an induction process over 'join' vertices of tree, at each step using Hölder's inequality to estimate the relevant sums over increasingly large sets of vertices on the tree. Obtaining these estimates turns out to be significantly more awkward when q is non-integral where qth powers of sums cannot conveniently be decomposed into sums of individual products.

2 Main definitions and results

This section presents the basic notions of generalized q-dimensions of measures and of the random processes that we will be concerned with, to enable us to state our main results.

We first review generalized q-dimensions of measures, which are the basis for the 'coarse' approach to multifractal analysis, that is the approach based on 'box sums', as opposed to the 'fine' approach based on Hausdorff or packing dimensions, see [12, 15, 16, 21, 25] for various treatments. The *r-mesh cubes* in $\mathbb{R}^N$ are the cubes in the family

$$\mathcal{M}_r = \big\{[i_1 r, (i_1+1)r) \times \cdots \times [i_N r, (i_N+1)r) \subseteq \mathbb{R}^N : i_1, \ldots, i_N \in \mathbb{Z}\big\}.$$

Let μ be a finite Borel measure of bounded support on $\mathbb{R}^N$. For $q > 0$ and $r > 0$ the qth-*moment sums* of μ are given by

$$M_r(q) = \sum_{C \in \mathcal{M}_r} \mu(C)^q,$$

For $q > 0, q \neq 1$, we define the *lower* and *upper generalized q-dimensions* or *Rényi dimensions* of μ to be

$$\underline{D}_q(\mu) = \liminf_{r \to 0} \frac{\log M_r(q)}{(q-1)\log r} \quad \text{and} \quad \overline{D}_q(\mu) = \limsup_{r \to 0} \frac{\log M_r(q)}{(q-1)\log r}. \tag{2.1}$$

If, as happens for many measures, $\underline{D}_q(\mu) = \overline{D}_q(\mu)$, we write $D_q(\mu)$ for the common value which we refer to as the *generalized q-dimension.* Note that when we write $D_q(\mu)$, etc. it is implicit that the generalized q-dimension exists.

Note that the definitions of q-dimensions are independent of the origin and coordinate orientation chosen for the mesh cubes.

There are useful integral forms of $\underline{D}_q$ and $\overline{D}_q$. For $q > 0, q \neq 1$,

$$\underline{D}_q(\mu) = \liminf_{r \to 0} \frac{\log \int \mu(B(x,r))^{q-1} d\mu(x)}{(q-1)\log r} \tag{2.2}$$

$$\text{and} \quad \overline{D}_q(\mu) = \limsup_{r \to 0} \frac{\log \int \mu(B(x,r))^{q-1} d\mu(x)}{(q-1)\log r}; \tag{2.3}$$

see [21] for $q > 1$ and [24] for $0 < q < 1$.

It is easily verified that $\underline{D}_q(\mu)$ and $\overline{D}_q(\mu)$ are each nonincreasing in q and continuous (for $q \neq 1$), and that $0 \leq \underline{D}_q(\mu) \leq \overline{D}_q(\mu) \leq N$ for all q.

Let $f : \mathbb{R}^N \to \mathbb{R}^d$ be Borel measurable and bounded on bounded subsets of $\mathbb{R}^N$. Let μ be a Borel probability measure on $\mathbb{R}^N$ with bounded support. The *image measure* μ_f of μ under f is the Borel measure of bounded support defined by

$$\mu_f(A) = \mu\{x : f(x) \in A\}, \quad A \subseteq \mathbb{R}^d.$$

In particular, for any measurable $g : \mathbb{R}^d \to \mathbb{R}_+$

$$\int_{\mathbb{R}^d} g(y) d\mu_f(y) = \int_{\mathbb{R}^N} g(f(x)) d\mu(x).$$

It follows easily from (2.2) and (2.3) that if $f : \mathbb{R}^N \to \mathbb{R}^d$ is α-Hölder on compact intervals in $\mathbb{R}^N$, where $0 < \alpha \leq 1$, and $q > 0, q \neq 1$, then

$$\underline{D}_q(\mu_f) \leq \min\Big\{d, \frac{1}{\alpha}\underline{D}_q(\mu)\Big\} \quad \text{and} \quad \overline{D}_q(\mu_f) \leq \min\Big\{d, \frac{1}{\alpha}\overline{D}_q(\mu)\Big\}. \tag{2.4}$$

These inqualities may be regarded as measure analogues of the relationships between the Hausdorff, packing and box dimensions of sets and their images under Hölder mappings, see [12].

Now let $X : \mathbb{R}^N \to \mathbb{R}^d$ be a continuous random process or random field on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and let E denote expectation. Then for μ a Borel probability measure on $\mathbb{R}^N$ with compact support, the random image measure μ_X on $\mathbb{R}^d$ is defined by

$$\mu_X(A) = \mu\{x : X(x) \in A\}, \quad A \text{ a Borel subset of } \mathbb{R}^d.$$

This section completes the main aim of the paper by relating the q-dimensions of μ_X and μ for certain processes.

Immediately from (2.4), if $X : \mathbb{R}^N \to \mathbb{R}^d$ almost surely has a Hölder exponent α on a compact interval $K \subseteq \mathbb{R}^N$ which contains the support of μ, then for $q > 0, q \neq 1$

$$\underline{D}_q(\mu_X) \le \min\Big\{d,\ \frac{1}{\alpha}\,\underline{D}_q(\mu)\Big\} \quad \text{and} \quad \overline{D}_q(\mu_X) \le \min\Big\{d,\ \frac{1}{\alpha}\,\overline{D}_q(\mu)\Big\} \text{ a.s.} \tag{2.5}$$

For many processes, Kolmogorov's continuity theorem, see for example [28], provides a suitable Hölder exponent.

Henceforth we assume that $X : \mathbb{R}^N \to \mathbb{R}^d$ is a Gaussian random field defined by

$$X(x) = \big(X_1(x), \ldots, X_d(x)\big), \qquad x \in \mathbb{R}^N, \tag{2.6}$$

where $X_1, \ldots, X_d$ are independent copies of a mean zero Gaussian process $X_0 : \mathbb{R}^N \to \mathbb{R}$ with $X_0(0) = 0$ a.s. We also assume that X_0 satisfies the following Condition **(C)**:

(C) For some $\delta_0 > 0$, let $\psi : [0, \delta_0) \to [0, \infty)$ be a non-decreasing, right continuous function with $\psi(0) = 0$ such that, for some constant $C_1 > 0$,

$$\frac{\psi(2r)}{\psi(r)} \le C_1 \quad \text{for all } \ r \in (0, \delta_0/2).$$

(C1) There is a constant $C_2 > 0$ such that for all $x, h \in \mathbb{R}^N$ with $|h| \le \delta_0$,

$$\mathsf{E}\Big[\big(X_0(x+h) - X_0(x)\big)^2\Big] \le C_2 \psi(|h|).$$

(C2) For all $T > 0$, the process X_0 is *strongly locally ψ-nondeterministic* on $[-T, T]^N$, that is, there exist positive constants C_3 and r_0 such that for all $x \in [-T, T]^N$ and all $0 < r \le \min\{|x|,\ r_0\}$,

$$\mathrm{Var}\big(X_0(x)\big|X_0(y)\ :\ y \in [-T, T]^N,\ r \le |x - y| \le r_0\big) \ge C_3\, \psi(r). \tag{2.7}$$

The concept of local nondeterminism was first introduced by Berman [4] for Gaussian processes and was subsequently extended by Pitt [26] to random fields. The above definition of strong local ψ-nondeterminism is essentially due to Cuzick and DuPreez [8] (who considered the case $N = 1$). For brief historical details and various applications of strong local nondeterminism see [38, 39].

Dimensional properties of such Gaussian fields have been studied in [2, 19, 35, 39] in increasing generality. The almost sure Hausdorff dimensions of certain random sets associated with a random process X given by (2.6), such as the range $X([0,1]^N)$, graph $\mathrm{Gr}X([0,1]^N) = \{(x, X(x)) : x \in [0,1]^N\}$ and level sets $X^{-1}(z) = \{x \in [0,1]^N : X(x) = z\}$, are determined by the *upper index of ψ at* 0 defined by

$$\alpha^* = \inf\Big\{\beta \ge 0 : \lim_{r \to 0} \frac{\psi(r)}{r^{2\beta}} = \infty\Big\} \tag{2.8}$$

(with the convention $\inf \emptyset = \infty$). Analogously, we can define the *lower index of ψ at* 0 by

$$\alpha_* = \sup\Big\{\beta \ge 0 : \lim_{r \to 0} \frac{\psi(r)}{r^{2\beta}} = 0\Big\}. \tag{2.9}$$

Clearly, $0 \le \alpha_* \le \alpha^* \le \infty$. Xiao [41] showed that the packing dimension of the range $X([0,1]^N)$ is determined by the lower index α_*. Moreover, if $\psi(r)$ is regularly varying at $r = 0$ of index $2\alpha \in (0, 2]$, then $\alpha_* = \alpha^* = \alpha$.

We may now state the main result of the paper.

Theorem 2.1 *Let $X : \mathbb{R}^N \to \mathbb{R}^d$ be the Gaussian random field defined by* (2.6) *and assume that the associated random field X_0 satisfies Condition* **(C)**. *Let $q > 1$ and let μ be a Borel probability measure on $\mathbb{R}^N$ with compact support.*

(i) If $0 < \alpha_* = \alpha^* = \alpha < 1$, then

$$\underline{D}_q(\mu_X) = \min\left\{d, \frac{1}{\alpha}\underline{D}_q(\mu)\right\} \quad a.s. \tag{2.10}$$

(ii) If the generalized q-dimension $D_q(\mu)$ of μ exists and $0 < \alpha_* \le \alpha^* < 1$, then

$$\min\left\{d, \frac{1}{\alpha^*}D_q(\mu)\right\} = \underline{D}_q(\mu_X) \le \overline{D}_q(\mu_X) \le \min\left\{d, \frac{1}{\alpha_*}D_q(\mu)\right\} \quad a.s. \tag{2.11}$$

(iii) If the generalized q-dimension $D_q(\mu)$ of μ exists and $0 < \alpha_* = \alpha^* = \alpha < 1$, then the generalized q-dimension $D_q(\mu_X)$ of μ_X exists almost surely, and

$$D_q(\mu_X) = \min\left\{d, \frac{1}{\alpha}D_q(\mu)\right\} \quad a.s.$$

We give several examples to which Theorem 2.1 is readily applicable. The most important Gaussian random fields which satisfy Condition **(C)** are the fractional Brownian motions. Recall that, for $0 < \alpha < 1$, a real-valued *index-α fractional Brownian motion* $B^\alpha : \mathbb{R}^N \to \mathbb{R}$ is the centered Gaussian random field with covariance function

$$\mathsf{E}\big(B^\alpha(x)B^\alpha(y)\big) = \frac{1}{2}\big(|x|^{2\alpha} + |y|^{2\alpha} - |x-y|^{2\alpha}\big), \quad x, y \in \mathbb{R}^N, \tag{2.12}$$

introduced by Mandelbrot and Van Ness [23] for $N = 1$. When $N > 1$ and $\alpha = 1/2$, then B^α is Lévy's N-parameter Brownian motion, see [18, Chapter 18]. It follows that $\mathsf{E}\big[\big(B^\alpha(x) - B^\alpha(y)\big)^2\big] = |x-y|^{2\alpha}$ so B^α has stationary, isotropic increments and is α-self-similar. Strong local ψ-nondeterminism of B^α, with $\psi(r) = r^{2\alpha}$, follows from Lemma 7.1 of Pitt [26], whose proof relies on the self-similarity of B^α. A different proof using Fourier analysis can be found in [39].

Corollary 2.2 (Fractional Brownian motion) *Let $X : \mathbb{R}^N \to \mathbb{R}^d$ be index-α fractional Brownian motion* (2.12). *Let μ be a Borel probability measure on $\mathbb{R}^N$ with compact support. Then for all $q > 1$,*

$$\underline{D}_q(\mu_X) = \min\left\{d, \frac{1}{\alpha}\underline{D}_q(\mu)\right\} \quad \text{and} \quad D_q(\mu_X) = \min\left\{d, \frac{1}{\alpha}D_q(\mu)\right\} \quad a.s.$$

Another example is *fractional Riesz-Bessel motion* $Y^{\gamma,\beta} : \mathbb{R}^N \to \mathbb{R}$, with indices γ and β, introduced by Anh, Angulo and Ruiz-Medina [3]. This is a centered Gaussian random field with stationary increments and spectral density

$$f_{\gamma,\beta}(\lambda) = \frac{c(\gamma,\beta,N)}{|\lambda|^{2\gamma}(1+|\lambda|^2)^\beta}, \tag{2.13}$$

where γ and β are constants satisfying

$$\beta + \gamma - \tfrac{N}{2} > 0, \quad 0 < \gamma < 1 + \tfrac{N}{2}$$

and $c(\gamma, \beta, N) > 0$ is a normalizing constant. These random fields are important for modelling as they exhibit long range dependence and intermittency simultaneously, see [3].

It may be shown that if $\gamma+\beta-\frac{N}{2} > 1$ then the sample function $Y^{\gamma,\beta}(x)$ has continuous (first order) partial derivatives almost surely, so the generalized dimensions of μ_X are the same as those of μ. For $0 < \gamma+\beta-\frac{N}{2} < 1$, it is proved in [39] that $Y^{\gamma,\beta}$ satisfies Condition **(C)** with $\psi(r) = r^{2(\gamma+\beta-\frac{N}{2})}$, so applying Theorem 2.1 leads to the following statement.

Corollary 2.3 (Fractional Riesz-Bessel motion) *Let $X : \mathbb{R}^N \to \mathbb{R}^d$ be index-(γ, β) fractional Riesz-Bessel motion (2.13). Let μ be a Borel probability measure on $\mathbb{R}^N$ with compact support. If $\gamma + \beta - \frac{1}{2}N > 1$ then $\underline{D}_q(\mu_X) = \underline{D}_q(\mu)$ and $D_q(\mu_X) = D_q(\mu)$ a.s. for all $q > 0, q \neq 1$. If $0 < \gamma + \beta - \frac{1}{2}N < 1$ then for all $q > 1$,*

$$\underline{D}_q(\mu_X) = \min\left\{d, \frac{\underline{D}_q(\mu)}{(\gamma+\beta-\frac{1}{2}N)}\right\} \text{ and } D_q(\mu_X) = \min\left\{d, \frac{D_q(\mu)}{(\gamma+\beta-\frac{1}{2}N)}\right\} \quad a.s.$$

When X_0 has stationary and isotropic increments, α^* and α_* coincide with the upper and lower indices of $\sigma^2(h)$ defined in an analogous manner to (2.8)-(2.9), where

$$\sigma^2(h) = \mathsf{E}\big[\big(X_0(x+h) - X_0(x)\big)^2\big], \qquad x, h \in \mathbb{R}^N$$

(by isotropy $\sigma^2(h)$ is a function of $\|h\|$). Many interesting examples of Gaussian random fields with stationary increments which satisfy condition **(C)** can be constructed, see [10, 22, 35, 39, 40]. Their approach is similar to the method for constructing Lévy processes with different upper and lower Blumenthal-Getoor indices [5]. We remark that, while Blumenthal and Getoor's indices are concerned with the asymptotic behavior of $\sigma^2(h)$ as $\|h\| \to \infty$, we are interested in the behavior of $\sigma^2(h)$ near $h = 0$.

We recall a class of Gaussian random fields with $\alpha_* < \alpha^*$, due to Clausel [7]. Let $\mathbf{H} = \{H_j, j \geq 0\}$ be a sequence of real numbers such that

$$0 < \liminf_{j\to\infty} H_j \leq \limsup_{j\to\infty} H_j < 1.$$

A real-valued Gaussian random field $B_{\mathbf{H}} : \mathbb{R}^N \to \mathbb{R}$ with stationary increments may be defined by the harmonizable representation:

$$B_{\mathbf{H}}(t) = \sum_{j=0}^{\infty} \int_{D_j} \frac{e^{i\langle t,\lambda\rangle} - 1}{\|\lambda\|^{H_j+\frac{N}{2}}}\, dW(\lambda),$$

where $D_0 = \{\lambda \in \mathbb{R}^N : \|\lambda\| < 1\}$ and $D_j = \{\lambda \in \mathbb{R}^N : 2^{j-1} \leq \|\lambda\| < 2^j\}$ for $j \geq 1$. Then $B_{\mathbf{H}}$ is called the *infinity scale fractional Brownian motion* with indices $\mathbf{H} = \{H_j, j \geq 0\}$. It is proved in [10] that $\alpha_* = \liminf_{j\to\infty} H_j$ and, under an extra condition on $\{H_j, j \geq 0\}$, we have $\alpha^* = \limsup_{j\to\infty} H_j$. To be more precise, let $\underline{H} = \liminf_{j\to\infty} H_j$ and $\overline{H} = \limsup_{j\to\infty} H_j$. For each $\varepsilon \in (0, \underline{H})$, we define a sequence $T_n = T_n(\varepsilon)$ as follows

$$T_1 = \inf\{j : H_j \geq \overline{H} - \varepsilon\}, \qquad T_2 = \inf\{j > T_1 : H_j < \overline{H} - \varepsilon\},$$

and for all $k \geq 1$ define inductively

$$T_{2k+1} = \inf\{j > T_{2k} : H_j \geq \overline{H} - \varepsilon\}$$

and

$$T_{2k+2} = \inf\{j > T_{2k+1} : H_j < \overline{H} - \varepsilon\}.$$

If we assume that for every $\varepsilon \in (0, \underline{H})$,

$$T_{2k+2} > \frac{(\overline{H} - \varepsilon)(1 - \underline{H} + \varepsilon)}{(\underline{H} - \varepsilon)(1 - \overline{H} + \varepsilon)} T_{2k+1} \tag{2.14}$$

for all k large enough, then it can be verified that $\alpha^* = \limsup_{j\to\infty} H_j$, see [10].

Corollary 2.4 (Infinity scale fractional Brownian motion) *Let $X : \mathbb{R}^N \to \mathbb{R}^d$ be an infinity scale fractional Brownian motion with indices $\mathbf{H} = \{H_j,\, j \geq 0\}$, which satisfies* (2.14) *for all $\varepsilon > 0$ small enough. Let $q > 1$ and let μ be a Borel probability measure on $\mathbb{R}^N$ with compact support.*

(i) If $\lim_{n\to\infty} H_j = \alpha \in (0, 1)$, then

$$\underline{D}_q(\mu_X) = \min\left\{d,\ \frac{1}{\alpha}\underline{D}_q(\mu)\right\} \quad a.s.$$

(ii) If $0 < \liminf_{j\to\infty} H_j < \limsup_{j\to\infty} H_j < 1$, then if μ has generaliz ed q-dimension $D_q(\mu)$, we have

$$\underline{D}_q(\mu_X) = \min\left\{d,\ \frac{1}{\overline{H}}D_q(\mu)\right\} \quad a.s.,$$

where $\overline{H} = \limsup_{j\to\infty} H_j$.

The proof of Theorem {main] will be completed in Section 5. Before we can make the necessary probabiiity estimates we need some technical results, which are developed in Sections 3 and 4.

3 Equivalent ultrametrics

In this section we construct, for each $n \in \mathbb{N}$, an ultrametric d on the unit cube in $\mathbb{R}^N$ such that, given any set of n points in $[0, \frac{1}{2})^N$, we can choose one of a finite number of translates of d whose restriction to these points is equivalent to the Euclidean metric in a uniform manner. We will need this so we can replace the Euclidean metric by an ultrametric when estimating the expectations that arise in Section 5.

For $m \geq 2$ we construct a hierarchy of m-ary subcubes of the unit cube $[0, 1)^N$ in the usual way. For $k = 0, 1, 2, \ldots$ define the set of *k-th level cubes*

$$\mathcal{C}_k = \Big\{[i_1 m^{-k}, (i_1+1)m^{-k}) \times \cdots \times [i_N m^{-k}, (i_N+1)m^{-k}) : 0 \leq i_1, \ldots, i_N \leq m^k - 1\Big\}. \tag{3.1}$$

These cubes define an ultrametric d on $[0, 1)^N$ given by

$$d(x, y) = m^{-k}, \qquad x, y \in [0, 1)^N,$$

where k is the greatest integer such that x and y are in the same cube of $\mathcal{C}_k$, with $d(x, x) = 0$.

Whilst it is easy to see that $|x-y| \leq \text{const}\, d(x,y)$, where $|x-y|$ is the Euclidean distance between x and y, the opposite inequality is not uniformly valid. To address this, we consider translates of d to get a family of ultrametrics on $[0,\frac{1}{2})^N$, from which we can always select one that will suit our needs.

Assume (to avoid the need for rounding fractions) that m is even. Let $\mathcal{A}_m$ denote the family of translation vectors:

$$\mathcal{A}_m = \Big\{ \Big(\frac{j_1}{m-1}, \ldots, \frac{j_N}{m-1} \Big) : 0 \leq j_1, \ldots, j_N \leq \frac{m}{2} - 1 \Big\}. \tag{3.2}$$

For each $a \in \mathcal{A}_m$ define

$$d_a(x,y) = d(x+a, y+a), \qquad x, y \in [0, \tfrac{1}{2})^N; \tag{3.3}$$

then d_a is an ultrametric on $[0,\frac{1}{2})^N$ which we may think of as a translate of d by the vector a. (Note that the restriction on the indices $0 \leq j_l \leq \frac{m}{2} - 1$ in (3.2) ensures that the d_a are defined throughout $[0,\frac{1}{2})^N$.)

For $a \in \mathcal{A}_m$ we write $\mathcal{C}_k^a$ for the cubes obtained by translating the family $\mathcal{C}_k$ by a vector $-a$, that is

$$\mathcal{C}_k^a = \{ C - a : C \in \mathcal{C}_k \}; \tag{3.4}$$

thus $d_a(x,y)$ is also given by the greatest integer such k such that x and y are in the same cube of $\mathcal{C}_k^a$.

Proposition 3.1 *(i) For all $a \in \mathcal{A}_m$ we have*

$$|x-y| \leq N^{1/2} d_a(x,y) \qquad x, y \in [0, \tfrac{1}{2})^N.$$

(ii) Given $x, y \in [0,\frac{1}{2})^N$, we have

$$d_a(x,y) \leq 8m(m-1)|x-y| \tag{3.5}$$

for all except at most $N(\frac{m}{2})^{N-1}$ vectors $a \in \mathcal{A}_m$.

Proof. (i) With k the greatest integer such that $x+a$ and $y+a$ are in the same k-th level cube of $\mathcal{C}_k$,

$$d_a(x,y) = d(x+a, y+a) = m^{-k} = N^{-1/2} N^{1/2} m^{-k} \geq N^{-1/2}|x-y|.$$

(ii) We first prove (ii) in the case $N=1$.

Let $x \in [0,\frac{1}{2})$ and let $k \geq 1$. We claim that, for all j such that $0 \leq j \leq \frac{m}{2} - 1$ with at most one exception,

$$\Big| x + \frac{j}{m-1} - i m^{-k} \Big| \geq \frac{1}{4m^k(m-1)} \qquad \text{for all } i \in \mathbb{Z}. \tag{3.6}$$

Suppose, for a contradiction, that there are $0 \leq j \neq j' \leq \frac{m}{2} - 1$ and $i, i' \in \mathbb{Z}$ such that both

$$\Big| x + \frac{j}{m-1} - i m^{-k} \Big| < \frac{1}{4m^k(m-1)} \qquad \text{and} \qquad \Big| x + \frac{j'}{m-1} - i' m^{-k} \Big| < \frac{1}{4m^k(m-1)}.$$

Then

$$\Big|\frac{j-j'}{m-1}-(i-i')m^{-k}\Big| < \frac{1}{2m^k(m-1)}$$

so

$$\big|(j-j')m^{k-1}(1+m^{-1}+m^{-2}+\cdots)-(i-i')\big| = \Big|\frac{(j-j')m^k}{m-1}-(i-i')\Big| < \frac{1}{2(m-1)}.$$

Thus the integer $z=(j-j')m^{k-1}(1+m^{-1}\cdots+m^{-k+1})-(i-i')$ satisfies

$$\Big|z+\frac{j-j'}{m-1}\Big| = \big|z+(j-j')(m^{-1}+m^{-2}+\cdots)\big| < \frac{1}{2(m-1)},$$

which cannot hold for any $0\leq j\neq j'\leq \frac{m}{2}-1$, proving the claim (3.6).

Now suppose $x,y\in[0,\frac{1}{2})$ satisfy $|x-y|\leq 1/(8m(m-1))$ and let $k\geq 1$ be the integer such that

$$\frac{1}{8m^{k+1}(m-1)} < |x-y| \leq \frac{1}{8m^k(m-1)}. \tag{3.7}$$

For all j such that (3.6) holds for this x and k, we have for all $i\in\mathbb{Z}$

$$\begin{aligned}\Big|y+\frac{j}{m-1}-im^{-k}\Big| &\geq \Big|x+\frac{j}{m-1}-im^{-k}\Big|-|x-y|\\ &\geq \frac{1}{4m^k(m-1)}-|x-y|\\ &\geq |x-y| \qquad\qquad \text{(using (3.7))}\\ &= \Big|\Big(y+\frac{j}{m-1}\Big)-\Big(x+\frac{j}{m-1}\Big)\Big|.\end{aligned}$$

Hence, $y+j/(m-1)$ is in the same interval of $\mathcal{C}_k$ as $x+j/(m-1)$, so for all j such that (3.6) holds, that is for all except at most one value of j,

$$d_{j/(m-1)}(x,y)=d\big(x+j/(m-1),y+j/(m-1)\big)\leq m^{-k}\leq 8m(m-1)|x-y|.$$

If $1/(8m(m-1)) < |x-y| \leq \frac{1}{2}$ then $d_{j/(m-1)}(x,y)\leq 1 < 8m(m-1)|x-y|$ for all j, completing the proof of (ii) when $N=1$.

For $N\geq 2$, write, in coordinate form, $x=(x_1,\ldots,x_N), y=(y_1,\ldots,y_N)\in\mathbb{R}^N$ and $a=(a_1,\ldots,a_N)\in\mathcal{A}_m$. Applying the result for $N=1$ to each coordinate,

$$d_a(x,y)=\max_{1\leq l\leq N} d_{a_l}(x_l,y_l)\leq 8m(m-1)\max_{1\leq l\leq N}|x_l-y_l|\leq 8m(m-1)|x-y|,$$

provided that a_l is not an exceptional value for the 1-dimensional case for any coordinate l. There are at most $N(\frac{m}{2})^{N-1}$ such exceptional vectors in $a\in\mathcal{A}_m$, otherwise (3.5) holds. □

Corollary 3.2 *Let $x_1,\ldots,x_n\in[0,\frac{1}{2})^N$. If $m>2n^2N$ then there exists $a\in\mathcal{A}_m$ such that*

$$N^{-1/2}|x_i-x_j|\leq d_a(x_i,x_j)\leq 8m(m-1)|x_i-x_j| \tag{3.8}$$

for all $1\leq i,j\leq n$.

Proof. By Proposition 3.1, for each pair x_i,x_j there are at most $N(\frac{m}{2})^{N-1}$ vectors $a\in\mathcal{A}_m$ for which (3.8) fails. Since there are a total of $(\frac{m}{2})^N$ vectors in $\mathcal{A}_m$, this leaves at least $(\frac{m}{2})^N-n^2N(\frac{m}{2})^{N-1}$ vectors in $\mathcal{A}_m$ such that (3.8) holds for all $1\leq i,j\leq n$, and this number is positive if $m>2n^2N$. □

4 Integral estimates

This section is devoted to bounding the integral (4.11) which arises estimating the q-dimensions of the image measures. A related procedure was used in [11] in connection with self-affine measures. We work with a *code space* or *word space* on M symbols, which we may identify with the vertices of the M-ary rooted tree in the usual way. In Section 5 this will in turn be identified with the hierarchy of m-ary cubes (3.1) in the ultrametric construction of Section 3, with $M = m^N$.

For $k = 0, 1, 2, \ldots$ let I_k be the set of all k-term sequences or words formed from the integers $1, 2, \ldots, M$, that is $I_k = \{1, 2, \ldots, M\}^k$, with I_0 comprising the empty word $\emptyset$. We write $|v| = k$ for the length of a word $v \in I_k$. Let $I = \cup_{k=0}^{\infty} I_k$ denote the set of all finite words, and let $I_\infty = \{1, 2, \ldots, M\}^{\mathbb{N}}$ denote the corresponding set of infinite words. We write $x|_k$ for the *curtailment* of $x \in I \cup I_\infty$ after k terms, that is the word comprising the initial k terms of x. For $v \in I$ and $x \in I \cup I_\infty$ we write $v \preceq x$ to mean that v is an initial subword of x. If $x, y \in I_\infty$ then $x \wedge y$ is the maximal word such that both $x \wedge y \preceq x$ and $x \wedge y \preceq y$.

We may topologise I_∞ in the natural way by the metric $d(x, y) = 2^{-|x \wedge y|}$ for distinct $x, y \in I_\infty$ to make I_∞ into a compact metric space, with the *cylinders* $C_v = \{y \in I_\infty : v \preceq y\}$ for $v \in I$ forming a base of open and closed neighborhoods of I_∞.

It is convenient to identify I with the vertices of an M-ary rooted tree with root $\emptyset$. The edges of this tree join each vertex $v \in I$ to its M 'children' $v1, \ldots, vM$.

The *join set* $\wedge(x_1, \ldots, x_n)$ of $x_1, \ldots, x_n \in I_\infty$ is the set of vertices $\{x_k \wedge x_{k'} : 1 \leq k \neq k' \leq n\}$. We say that $u \in \wedge(x_1, \ldots, x_n)$ has *multiplicity* $r - 1$ if r is the greatest integer for which there are distinct indices $k_1, \ldots, k_r$ with $x_{k_p} \wedge x_{k_q} = u$ for all $1 \leq p < q \leq r$. (In the case of $M = 2$ where I is a binary tree, every vertex of a join set has multiplicity 1.) Counting according to multiplicity, the join set $\wedge(x_1, \ldots, x_n)$ always comprises $n - 1$ vertices of I. The *top vertex* $\wedge^T(x_1, \ldots, x_n)$ of a join set is the vertex $v \in \wedge(x_1, \ldots, x_n)$ such that $v \preceq x_l$ for all $1 \leq l \leq n$.

To establish (4.11) below, we will split the domain of integration into subdomains consisting of n-tuples $(x_1, \ldots, x_n)$ lying in different orbits of automorphisms of the tree I. We will use induction over certain classes of orbit to estimate the integrals over each such domain, with Hölder's inequality playing a natural role at each step. It is convenient to phrase the argument using a little terminology from group actions.

Let Aut be the group of automorphisms of the rooted tree I that fix the root $\emptyset$; these automorphisms act on the infinite tree I_∞ and thus on the n-tuples $(I_\infty)^n$ in the obvious way. For each $n \in \mathbb{N}$ let

$$S(n) = \{(x_1, \ldots, x_n) : x_k \in I_\infty\},$$

for each $n \in \mathbb{N}$ and $l = 0, 1, 2, \ldots$ let

$$S_l(n) = \{(x_1, \ldots, x_n) : x_k \in I_\infty, |\wedge^T(x_1, \ldots, x_n)| \geq l\},$$

and for each $n \in \mathbb{N}$ and $v \in I$ let

$$\begin{aligned} S(v, n) &= \{(x_1, \ldots, x_n) : x_k \in I_\infty, x_k \succeq v \text{ for } k = 1, \ldots, n\} \\ &= \{(x_1, \ldots, x_n) : x_k \in I_\infty, \wedge^T(x_1, \ldots, x_n) \succeq v \text{ for } k = 1, \ldots, n\}. \end{aligned}$$

The group Aut acts on $S(n)$ by $g(x_1, \ldots, x_n) = (g(x_1), \ldots, g(x_n))$ for $g \in$ Aut, and in particular acts on each $S_l(n)$, since if $g \in$ Aut maps $(x_1, \ldots, x_n)$ to $(x'_1, \ldots, x'_n)$ then g

maps the join set $\wedge(x_1,\dots,x_n)$ to the join set $\wedge(x'_1,\dots,x'_n)$. Similarly, for each v, the subgroup Aut_v of Aut that fixes the vertex $v\in I$ acts on $S(v,n)$. We write $\mathrm{Orb}_l(n)$ for the set of orbits of $S_l(n)$ under Aut, and $\mathrm{Orb}(v,n)$ for the set of orbits of $S(v,n)$ under Aut_v. In other words, defining an equivalence relation $\sim$ on $S(n)$ by

$$(x_1,\dots,x_n)\sim(x'_1,\dots,x'_n) \text{ if there exists } g\in \mathrm{Aut} \text{ such that } g(x_k)=x'_k \text{ for all } 1\le k\le n,$$

then

$$\mathrm{Orb}_l(n)=S_l(n)/\sim \qquad \text{and} \qquad \mathrm{Orb}(v,n)=S(v,n)/\sim .$$

For each l and n we have $S_l(n)=\bigcup_{|v|=l}S(v,n)$ with this union disjoint. Thus if $|v|=l$ then the orbit $\mathcal{O}\in\mathrm{Orb}_l(n)$ restricts to an orbit $\mathcal{O}(v)\in\mathrm{Orb}(v,n)$ in the obvious way, that is

$$\mathcal{O}(v)=\{(x_1,\dots,x_n)\in\mathcal{O}:\wedge^T(x_1,\dots,x_n)\succeq v\}.$$

The *level* of a vertex $v\in I$ is just the length of the word $|v|$, and the *set of levels of a join set* is the set of $n-1$ levels of the vertices in $\wedge(x_1,\dots,x_n)$, counting by multiplicity. Since the set of levels of a join set is constant across each orbit, we may define the *set of levels of an orbit* $\mathcal{O}\in\mathrm{Orb}_l(n)$ or $\mathcal{O}\in\mathrm{Orb}(v,n)$, written $L(\mathcal{O})$, to be the set of levels of the join set $\wedge(x_1,\dots,x_n)$ of any $(x_1,\dots,x_n)\in\mathcal{O}$.

Throughout this section we will be working with products over sets of levels of certain orbits. To aid keeping track of terms, particularly when Hölder's inequality is invoked, we use a number in square brackets above the product sign to indicate the number of terms in this product (for example the product in (4.1) is over $n-1$ levels).

The following proposition provides our basic estimate for the μ^n-measure of n-tuples of points in I_∞ lying in a given orbit, in terms of the measures of cylinders at the levels of the join classes of the orbit.

Proposition 4.1 *Let $q>1$ and $n\ge 1$ be such that $q\ge n$. Let $v\in I$ and let $\mathcal{O}\in\mathrm{Orb}(v,n)$. Then*

$$\mu^n\big\{(x_1,\dots,x_n)\in\mathcal{O}\big\}\le\mu(C_v)^{(q-n)/(q-1)}\prod_{l\in L(\mathcal{O})}^{[n-1]}\Big(\sum_{|u|=l,u\succeq v}\mu(C_u)^q\Big)^{1/(q-1)}. \tag{4.1}$$

Proof. We prove (4.1) by induction on n. When $n=1$, the only orbit $\mathcal{O}\in\mathrm{Orb}(v,n)$ comprises the set of x_1 such that $x_1\succeq v$, so

$$\mu\big\{(x_1)\in\mathcal{O}\big\}=\mu\big\{x_1:x_1\succeq v\big\}=\mu(C_v),$$

which is (4.1) in this case.

Now assume inductively that (4.1) holds for all $\mathcal{O}\in\mathrm{Orb}(v,n)$ for all $v\in I$ and all $1\le n\le n_0$. We show that (4.1) holds with $n=n_0+1$.

Let $\mathcal{O}\in\mathrm{Orb}(v,n)$. We first consider the case where $v=\wedge^T(x_1,\dots,x_n)$ for some, and therefore for all, $(x_1,\dots,x_n)\in\mathcal{O}$ (thus v is the top vertex of the $\wedge(x_1,\dots,x_n)$ in the orbit). Then each $(x_1,\dots,x_n)\in\mathcal{O}$ decomposes into $2\le r\le n$ subsets

$$\{(x_1^1,\dots,x_{n_1}^1),\dots,(x_1^r,\dots,x_{n_r}^r)\}$$

such that the top vertices $v_k = \wedge^T(x_1^k, \ldots, x_{n_k}^k)$ are distinct with $v_k \succ v, v_k \neq v$ and with the paths in the tree I that join the v_k to v meeting only at v. Then $1 \leq n_k < n$ for each k, and

$$n_1 + \cdots + n_r = n. \tag{4.2}$$

The orbit $\mathcal{O}$ induces orbits $\mathcal{O}_k \in \mathrm{Orb}(v, n_k)$ of $(x_1^k, \ldots, x_{n_k}^k)$ for each k, and thus $\mathcal{O}$ may be decomposed as a subset of the product of the $\{\mathcal{O}_k : 1 \leq k \leq r\}$. Thus applying the inductive hypothesis (4.1) to each $\mathcal{O}_k$,

$$\begin{aligned}
&\mu^n\big\{(x_1, \ldots, x_n) \in \mathcal{O}\big\} \\
&\quad \leq \mu^{n_1}\big\{(x_1^1, \ldots, x_{n_1}^1) \in \mathcal{O}_1\big\} \times \cdots \times \mu^{n_r}\big\{(x_1^r, \ldots, x_{n_r}^r) \in \mathcal{O}_r\big\} \\
&\quad \leq \mu(C_v)^{(q-n_1)/(q-1)} \prod_{l \in L(\mathcal{O}_1)}^{[n_1-1]} \Big(\sum_{|u|=l, u \succeq v} \mu(C_u)^q \Big)^{1/(q-1)} \\
&\qquad\qquad \times \cdots \times \mu(C_v)^{(q-n_r)/(q-1)} \prod_{l \in L(\mathcal{O}_r)}^{[n_r-1]} \Big(\sum_{|u|=l, u \succeq v} \mu(C_u)^q \Big)^{1/(q-1)}, \\
&\quad = \mu(C_v)^{(q-n_1-\cdots-n_r)/(q-1)} \big(\mu(C_v)^q\big)^{(r-1)/(q-1)} \prod_{l \in L(\mathcal{O}_1) \cup \cdots \cup L(\mathcal{O}_r)}^{[n_1+\cdots+n_r-r]} \Big(\sum_{|u|=l, u \succeq v} \mu(C_u)^q \Big)^{1/(q-1)} \\
&\quad \leq \mu(C_v)^{(q-n)/(q-1)} \prod_{l \in L(\mathcal{O})}^{[n-1]} \Big(\sum_{|u|=l, u \succeq v} \mu(C_u)^q \Big)^{1/(q-1)},
\end{aligned}$$

where $L(\mathcal{O})$ is the complete set of levels of $\mathcal{O}$ (including level $|v|$ with multiplicity $r-1$) and where we have used (4.2). This is (4.1) in the case where v is the top vertex of the join sets of the n-tuples in $\mathcal{O}$.

Finally, let $\mathcal{O} \in \mathrm{Orb}(v, n)$ be such that $|\wedge^T(x_1, \ldots, x_n)| = l'$ for each $(x_1, \ldots, x_n) \in \mathcal{O}$ with $l' > |v|$ (so the top vertex of $\wedge(x_1, \ldots, x_n)$ is strictly below v). The orbit $\mathcal{O}$ may be decomposed into orbits $\mathcal{O}_w \in \mathrm{Orb}(w, n)$ where $w \succeq v$ and $|w| = l'$. We have shown that (4.1) holds for such w, so summing and using Hölder's inequality,

$$\begin{aligned}
&\mu^n\big\{(x_1, \ldots, x_n) \in \mathcal{O}\big\} \\
&\quad \leq \sum_{|w|=l', w \succeq v} \Big(\mu(C_w)^{(q-n)/(q-1)} \prod_{l \in L(\mathcal{O}_w)}^{[n-1]} \Big(\sum_{|u|=l, u \succeq w} \mu(C_u)^q \Big)^{1/(q-1)} \Big) \\
&\quad \leq \Big(\sum_{|w|=l', w \succeq v} \mu(C_w) \Big)^{(q-n)/(q-1)} \prod_{l \in L(\mathcal{O})}^{[n-1]} \Big(\sum_{|w|=l', w \succeq v} \Big(\sum_{|u|=l, u \succeq w} \mu(C_u)^q \Big) \Big)^{1/(q-1)} \\
&\quad = \mu(C_v)^{(q-n)/(q-1)} \prod_{l \in L(\mathcal{O})}^{[n-1]} \Big(\sum_{|u|=l, u \succeq v} \mu(C_u)^q \Big)^{1/(q-1)}.
\end{aligned}$$

This completes the induction and the proof. □

Proposition 4.1 would be enough for our purposes when q is an integer. However, when estimating (4.11) for non-integral $q > 1$ we need a generalization where one of the points $y \in I_\infty$ is distinguished. The proof of Proposition 4.2 again uses induction on join sets and Hölder's inequality, but the argument is more intricate than that of Proposition 4.1 on which it depends.

Let $1 \leq p \leq n$, $0 \leq l_1 < l_2 < \ldots < l_p$ be levels and $m_1, \ldots, m_p \in \mathbb{N}$ be such that $m_1 + \cdots + m_p = n$. For each $y \in I_\infty$ and $1 \leq r \leq p$, write $y_r = y|_{l_r}$ and, given an orbit $\mathcal{O}_r \in \mathrm{Orb}_{l_r}(m_r)$, write $\mathcal{O}_r(y_r) \in \mathrm{Orb}(y_r, m_r)$ for the suborbit under automorphisms fixing y_r.

Proposition 4.2 *Let $q > 1$ and let n be an integer with $n \geq q - 1$. As above, let $1 \leq p \leq n$, let $0 \leq l_1 < l_2 < \ldots < l_p$ be levels and let $m_1 + \cdots + m_p = n$. For each $r = 1, \ldots, p$ let $\mathcal{O}_r \in \mathrm{Orb}_{l_r}(m_r)$ be given. Then*

$$\int_{y \in I_\infty} \mu^n \big\{(x_1 \ldots, x_n) : (x_1, \ldots, x_{m_1}) \in \mathcal{O}_1(y_1), \ldots, (x_{n-m_p+1}, \ldots, x_n) \in \mathcal{O}_p(y_p)\big\}^{(q-1)/n} d\mu(y)$$
$$\leq \prod_{l \in L}^{[n]} \Big(\sum_{|u|=l} \mu(C_u)^q \Big)^{1/n}, \tag{4.3}$$

where L denotes the aggregate set of levels of $\{l_1, \ldots, l_p, L(\mathcal{O}_1), \ldots, L(\mathcal{O}_p)\}$.

Proof. We proceed by induction on $r \leq p$, starting with $r = p$ and working backwards to $r = 1$, taking as the inductive hypothesis:
For all $y_r \in I_{l_r}$,

$$\int_{y \succeq y_r} \mu^{m_r + \cdots + m_p} \big\{(x_1^r, \ldots, x_{m_r}^r, \ldots, x_1^p, \ldots, x_{m_p}^p)$$
$$: (x_1^r, \ldots, x_{m_r}^r) \in \mathcal{O}_r(y_r), \ldots, (x_1^p, \ldots, x_{m_p}^p) \in \mathcal{O}_p(y_p)\big\}^{(q-1)/n} d\mu(y)$$
$$\leq \mu(C_{y_r})^{(n-n_r)/n} \prod_{l \in L_r}^{[n_r]} \Big(\sum_{|u|=l, u \succeq y_r} \mu(C_u)^q \Big)^{1/n}, \tag{4.4}$$

where $n_r = m_r + \cdots + m_p$ and L_r denotes the set of levels of $\{l_r, \ldots, l_p, L(\mathcal{O}_r), \ldots, L(\mathcal{O}_p)\}$ counted by multiplicity (so that L_r consists of $m_r + \cdots + m_p = n_r$ levels).

To start the induction, we apply Proposition 4.1 to get, for each $y_p \in I_{l_p}$,

$$\int_{y \succeq y_p} \mu^{m_p} \big\{(x_1^p, \ldots, x_{m_p}^p) \in \mathcal{O}_p(y_p)\big\}^{(q-1)/n} d\mu(y) \tag{4.5}$$
$$\leq \int_{y \succeq y_p} \Bigg[\mu(C_{y_p})^{(q-m_p)/(q-1)} \prod_{l \in L(\mathcal{O}_p(y_p))}^{[m_p - 1]} \Big(\sum_{|u|=l, u \succeq y_p} \mu(C_u)^q \Big)^{1/(q-1)} \Bigg]^{(q-1)/n} d\mu(y)$$
$$= \mu(C_{y_p})^{(n-m_p)/n} \big(\mu(C_{y_p})^q\big)^{1/n} \prod_{l \in L(\mathcal{O}_p)}^{[m_p - 1]} \Big(\sum_{|u|=l, u \succeq y_p} \mu(C_u)^q \Big)^{1/n}$$
$$= \mu(C_{y_p})^{(n-n_p)/n} \prod_{l \in L_r}^{[n_p]} \Big(\sum_{|u|=l, u \succeq y_p} \mu(C_u)^q \Big)^{1/n},$$

on incorporating $(\mu(C_{y_p})^q)^{1/n}$ in the main product and noting that $m_p = n_p$. (Observe that this remains valid if $m_p = 1$, in which case the measure in (4.5) is at most $\mu(C_{y_p})$.) This establishes the inductive hypothesis (4.4) when $r = p$.

Now assume that (4.4) is valid for $r = k, \dots, p$ for some $2 \le k \le p$. Then for each $y_{k-1} \in I_{l_{k-1}}$,

$$\begin{aligned}
I := \int_{y \succeq y_{k-1}} & \mu^{m_{k-1}+m_k+\cdots+m_p}\big\{(x_1^{k-1}, \dots, x_{m_{k-1}}^{k-1}, x_1^k, \dots, x_{m_k}^k, \dots, x_1^p, \dots, x_{m_p}^p) \\
&\qquad : (x_1^{k-1}, \dots, x_{m_{k-1}}^{k-1}) \in \mathcal{O}_{k-1}(y_{k-1}), (x_1^k, \dots, x_{m_k}^k) \in \mathcal{O}_k(y_k), \\
&\qquad\qquad \dots, (x_1^p, \dots, x_{m_p}^p) \in \mathcal{O}_p(y_p)\big\}^{(q-1)/n} d\mu(y) \\
\le \mu^{m_{k-1}} & \big\{(x_1^{k-1}, \dots, x_{m_{k-1}}^{k-1}) \in \mathcal{O}_{k-1}(y_{k-1})\big\}^{(q-1)/n} \\
& \times \sum_{y_k \succeq y_{k-1}, |y_k| = l_k} \int_{y \succeq y_k} \mu^{m_k+\cdots+m_p}\big\{(x_1^k, \dots, x_{m_k}^k, \dots, x_1^p, \dots, x_{m_p}^p) \\
&\qquad : (x_1^k, \dots, x_{m_k}^k) \in \mathcal{O}_k(y_k), \dots, (x_1^p, \dots, x_{m_p}^p) \in \mathcal{O}_p(y_p)\big\}^{(q-1)/n} d\mu(y) \\
\le & \Bigg[\mu(C_{y_{k-1}})^{(q-m_{k-1})/n} \prod_{l \in L(\mathcal{O}_{k-1}(y_{k-1}))}^{[m_{k-1}-1]} \Big(\sum_{|u|=l, u \succeq y_{k-1}} \mu(C_u)^q\Big)^{1/n}\Bigg] \\
& \times \sum_{y_k \succeq y_{k-1}, |y_k| = l_k} \Bigg(\mu(C_{y_k})^{(n-n_k)/n} \prod_{l \in L_k}^{[n_k]} \Big(\sum_{|u|=l, u \succeq y_k} \mu(C_u)^q\Big)^{1/n}\Bigg),
\end{aligned} \tag{4.6}$$

where in obtaining the last inequality we have used Proposition 4.1 to estimate the first part and the inductive hypothesis (4.4) for the second part. Using Hölder's inequality for each y_{k-1}:

$$\begin{aligned}
\sum_{y_k \succeq y_{k-1}, |y_k| = l_k} & \Bigg(\mu(C_{y_k})^{(n-n_k)/n} \prod_{l \in L_k}^{[n_k]} \Big(\sum_{|u|=l, u \succeq y_k} \mu(C_u)^q\Big)^{1/n}\Bigg) \\
& \le \Big(\sum_{y_k \succeq y_{k-1}, |y_k| = l_k} \mu(C_{y_k})\Big)^{(n-n_k)/n} \prod_{l \in L_k}^{[n_k]} \Big(\sum_{y_k \succeq y_{k-1}, |y_k| = l_k} \Big(\sum_{|u|=l, u \succeq y_k} \mu(C_u)^q\Big)\Big)^{1/n} \\
& = \mu(C_{y_{k-1}})^{(n-n_k)/n} \prod_{l \in L_k}^{[n_k]} \Big(\sum_{|u|=l, u \succeq y_{k-1}} \mu(C_u)^q\Big)^{1/n}.
\end{aligned}$$

Thus from (4.6)

$$\begin{aligned}
I & \le \mu(C_{y_{k-1}})^{(n-n_k-m_{k-1})/n} \big(\mu(C_{y_{k-1}})^q\big)^{1/n} \\
& \quad \times \prod_{l \in L(\mathcal{O}_{k-1}(y_{k-1}))}^{[m_{k-1}-1]} \Big(\sum_{|u|=l, u \succeq y_{k-1}} \mu(C_u)^q\Big)^{1/n} \prod_{l \in L_k}^{[n_k]} \Big(\sum_{|u|=l, u \succeq y_{k-1}} \mu(C_u)^q\Big)^{1/n} \\
& = \mu(C_{y_{k-1}})^{(n-n_k-m_{k-1})/n} \prod_{l \in L_{k-1}}^{[m_{k-1}+n_k]} \Big(\sum_{|u|=l, u \succeq y_{k-1}} \mu(C_u)^q\Big)^{1/n},
\end{aligned}$$

which is (4.4) with $r = k-1$, noting that $m_{k-1} + n_k = n_{k-1}$.

Finally, taking $r = 1$ in (4.4) and noting that $n_1 = n$,

$$\int_{y \succeq y_1} \mu^n \big\{(x_1, \ldots, x_n) : (x_1, \ldots, x_{m_1}) \in \mathcal{O}_1(y_1), \ldots, (x_{n-n_p+1}, \ldots, x_n) \in \mathcal{O}_p(y_p)\big\}^{(q-1)/n} d\mu(y)$$
$$\leq \prod_{l \in L_1}^{[n]} \Big(\sum_{|u|=l, u \succeq y_1} \mu(C_u)^q \Big)^{1/n},$$

and summing over all y_1 at level l_1 and using Hölder's inequality again, gives (4.3). □

To use Proposition 4.2 to determine when the integral in (4.11) converges we need to bound the number of orbits that have prescribed sets of levels. Let $0 \leq k_1 \leq \cdots \leq k_n$ be (not necessarily distinct) levels. Write

$$\begin{aligned} N(k_1, \ldots, k_n) = \#\Big\{ &(l_1, \ldots, l_p, \mathcal{O}_1, \ldots, \mathcal{O}_p) : 1 \leq p \leq n, 0 \leq l_1 < \cdots < l_p, \\ &\mathcal{O}_r \in \mathrm{Orb}_{l_r}(m_r) \text{ for some } m_r \text{ where } m_1 + \cdots + m_r = n, \\ &\text{such that } \{l_1, \ldots, l_p, L(\mathcal{O}_1), \ldots, L(\mathcal{O}_p)\} = \{k_1, \ldots, k_n\}\Big\}. \end{aligned} \tag{4.7}$$

Lemma 4.3 *Let $n \in \mathbb{N}$ and $0 < \lambda < 1$. Then*

$$\sum_{0 \leq k_1 \leq \cdots \leq k_n} N(k_1, \ldots, k_n) \lambda^{(k_1 + \cdots + k_n)/n} < \infty.$$

Proof. The crucial observation here is that we may find an upper bound for $N(k_1, \ldots, k_n)$ that depends on n but not on the particular levels $(k_1, \ldots, k_n)$.

Let $N_0(k_1, \ldots, k_n)$ be the total number of orbits in $\mathrm{Orb}(\emptyset, n+1)$ (where $\emptyset$ is the root of the tree I) with levels $0 \leq k_1 \leq \cdots \leq k_n$. Every join set with levels $0 \leq k_1 \leq \cdots \leq k_n$ may be obtained by adding a vertex at level k_n of the form $x_i|_{k_n}$ to a join set $\wedge(x_1, \ldots, x_n)$ with levels $0 \leq k_1 \leq \cdots \leq k_{n-1}$ for some $1 \leq i \leq n$, and this may be done in at most n ways. It follows that $N_0(k_1, \ldots, k_n) \leq n N_0(k_1, \ldots, k_{n-1})$, so since $N_0(k_1) = 1$, we obtain $N_0(k_1, \ldots, k_n) \leq n!$.

The number $N(k_1, \ldots, k_n)$ given by (4.7) is no more than the number of orbits in $\mathrm{Orb}(\emptyset, n+1)$ having levels $0 \leq k_1 \leq \cdots \leq k_n$ with a subset of the join vertices (to within equivalence) of each member of the orbit distinguished to correspond to levels $l_1, \ldots, l_p$. But given $(x_1, \ldots, x_{n+1}) \in \mathcal{O}$ where $\mathcal{O}$ has join levels $k_1, \ldots, k_n$, there are at most 2^n ways of choosing a distinguished subset $(y_1, \ldots, y_p)$ of the join set $\wedge(x_1, \ldots, x_{n+1})$; these vertices then determine m_r as well as $\mathcal{O}_r(y_r) = \mathcal{O}_r \in \mathrm{Orb}_{l_r}(m_r)$ for $r = 1, \ldots, p$. (Note that we are considerably over-counting since the same contributions to (4.7) may come from different orbits of $\mathrm{Orb}(\emptyset, n+1)$.) Hence

$$N(k_1, \ldots, k_n) \leq 2^n N_0(k_1, \ldots, k_n) \leq 2^n n!.$$

Thus

$$\begin{aligned} \sum_{0 \leq k_1 \leq \cdots \leq k_n} N(k_1, \ldots, k_n) \lambda^{(k_1 + \cdots + k_n)/n} &\leq 2^n n! \sum_{0 \leq k_1 \leq \cdots \leq k_n} \lambda^{(k_1 + \cdots + k_n)/n} \\ &\leq 2^n n! \sum_{k=0}^{\infty} P(k) \lambda^{k/n}, \end{aligned} \tag{4.8}$$

where $P(k)$ is the number of distinct ways of partitioning the integer k into a sum of n integers $k = k_1 + \cdots + k_n$ with $0 \le k_1 \le \cdots \le k_n$. Since $P(k)$ is polynomially bounded (trivially $P(k) \le (k+1)^{n-1}$), (4.8) converges for $0 < \lambda < 1$. $\square$

To obtain the main estimate, we use Lemma 4.3 to count the domains of integration to which we apply Proposition 4.2.

Let $f : \mathbb{N}_0 \to \mathbb{R}_+$ be a function. Define the *multipotential kernel* $\phi : I^{n+1} \to \mathbb{R}_+$ to be the product of f evaluated at the levels of the vertices of each join set, that is

$$\phi(x_1, \ldots, x_n, y) = f(l_1)f(l_2)\cdots f(l_n) \text{ where } L(\wedge(x_1, \ldots, x_n, y)) = \{l_1, l_2, \ldots l_n\}. \tag{4.9}$$

Theorem 4.4 *Let $n \in \mathbb{N}$ and $q > 1$ with $n \le q < n+1$. Suppose that*

$$\limsup_{l \to \infty} \frac{\log \left(f(l)^{q-1} \sum_{|u|=l} \mu(C_u)^q \right)}{l} < 0. \tag{4.10}$$

Then, with ϕ as in (4.9),

$$I := \int \left[\int \cdots \int \phi(x_1, \ldots, x_n, y) d\mu(x_1) \ldots d\mu(x_n) \right]^{(q-1)/n} d\mu(y) < \infty. \tag{4.11}$$

Proof. For each $y \in I_\infty$ we decompose the integral inside the square brackets as a sum of integrals taken over all p, all $0 \le l_1 < \cdots < l_p$, all $m_1, \ldots, m_p \ge 1$ such that $m_1 + \cdots + m_p = n$, and all orbits $\mathcal{O}_1 \in \mathrm{Orb}_{l_1}(m_1), \ldots, \mathcal{O}_p \in \mathrm{Orb}_{l_p}(m_p)$. As before, for each r we write $y_r = y|_{l_r}$, and $\mathcal{O}_r(y_r)$ for the suborbit of $\mathcal{O}_r$ in $\mathrm{Orb}(y_r, m_p)$. Thus, using the power-sum inequality, (4.9) and (4.3), and noting that ϕ depends only on the levels of the join sets,

$$\begin{aligned}
I &= \\
&\int \Bigg[\sum_{\substack{0 \le l_1 < \cdots < l_p \\ m_1 + \cdots + m_p = n \\ \mathcal{O}_1, \ldots, \mathcal{O}_p}} \int_{(x_1, \ldots, x_{m_1}) \in \mathcal{O}_1(y_1)} \cdots \int_{(x_{n-m_p+1}, \ldots, x_n) \in \mathcal{O}_p(y_p)} \phi(x_1, \ldots, x_n, y) d\mu(x_1) \ldots d\mu(x_n) \Bigg]^{(q-1)/n} d\mu(y) \\
&\le \sum_{\substack{0 \le l_1 < \cdots < l_p \\ m_1 + \cdots + m_p = n \\ \mathcal{O}_1, \ldots, \mathcal{O}_p}} \int \Bigg[\mu^n \big\{ (x_1, \ldots, x_n) : (x_1, \ldots, x_{m_1}) \in \mathcal{O}_1(y_1), \\
&\qquad \ldots, (x_{n-m_p+1}, \ldots, x_n) \in \mathcal{O}_p(y_p) \big\} \phi(x_1, \ldots, x_n, y) \Bigg]^{(q-1)/n} d\mu(y) \\
&\le \sum_{\substack{0 \le l_1 < \cdots < l_p \\ m_1 + \cdots + m_p = n \\ \mathcal{O}_1, \ldots, \mathcal{O}_p}} \prod_{l \in L}^{[n]} \Big(f(l)^{q-1} \sum_{|u|=l} \mu(C_u)^q \Big)^{1/n},
\end{aligned}$$

where the products are over the set of levels $L = \{l_1, \ldots, l_p, L(\mathcal{O}_1), \ldots, L(\mathcal{O}_p)\}$ counted with repetitions Condition (4.10) implies that $f(l)^{q-1} \sum_{|u|=l} \mu(C_u)^q \le c\lambda^l$ for all l, for some $c > 0$ and some $\lambda < 1$. Thus, with $N(k_1, \ldots, k_n)$ as in (4.7),

$$\begin{aligned}
I &\leq \sum_{0 \leq k_1 \leq \ldots \leq k_n} N(k_1, \ldots, k_n) \prod_{i=1}^{n} \Big(f(k_i)^{q-1} \sum_{|u|=k_i} \mu(C_u)^q \Big)^{1/n} \\
&\leq \sum_{0 \leq k_1 \leq \ldots \leq k_n} N(k_1, \ldots, k_n) \prod_{i=1}^{n} \Big(c\lambda^{k_i} \big)^{1/n} \\
&\leq c \sum_{0 \leq k_1 \leq \ldots \leq k_n} N(k_1, \ldots, k_n) \lambda^{(k_1 + \cdots + k_n)/n} < \infty
\end{aligned}$$

using Lemma 4.3. □

5 Proofs of main results

We can now complete the proof of our main results.

Firstly, when working with upper and lower indices, it is convenient to have a variant of of (2.4) under a weakened Hölder condition

Lemma 5.1 *Suppose that $f : \mathbb{R}^N \to \mathbb{R}^d$ and that for some compact interval $K \subseteq \mathbb{R}^N$ and $0 < \alpha \leq 1$, there exist a sequence $r_n \searrow 0$ and a constant c such that*

$$\sup_{x,y \in K : |x-y| \leq r_n} \big| f(x) - f(y) \big| \leq c\, r_n^{\alpha}. \tag{5.1}$$

Then for all $q > 0, q \neq 1$ and every finite Borel measure μ with support contained in K, we have

$$\underline{D}_q\,(\mu_f) \leq \min\Big\{ d,\, \frac{1}{\alpha}\, \overline{D}_q\,(\mu) \Big\}.$$

In particular, if the generalized q-dimension $D_q(\mu)$ exists, then

$$\underline{D}_q\,(\mu_f) \leq \min\Big\{ d,\, \frac{1}{\alpha}\, D_q(\mu) \Big\}.$$

Proof. Let $\rho_n = c\, r_n^{\alpha}$ for all $n \geq 1$. Then (5.1) implies that

$$\begin{aligned}
\int_{\mathbb{R}^d} \mu_f\big(B(z, \rho_n)\big)^{q-1} d\mu_f(z) &= \int_K \mu_f\big(B(f(x), \rho_n)\big)^{q-1} d\mu(x) \\
&\geq \int_K \mu\big(B(x, r_n)\big)^{q-1} d\mu(x)
\end{aligned}$$

for $q > 1$, with the reverse inequality for $0 < q < 1$. Hence, in both cases,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\int_{\mathbb{R}^d} \mu_f\big(B(z, \rho_n)\big)^{q-1} d\mu_f(z)}{(q-1)\log \rho_n} &\leq \limsup_{n\to\infty} \frac{\int \mu\big(B(x, r_n)\big)^{q-1} d\mu(x)}{(q-1)\log \rho_n} \\
&= \frac{1}{\alpha}\, \overline{D}_q\,(\mu).
\end{aligned}$$

□

The following well-known lemma on the modulus of continuity of a Gaussian process follows from [9, Corollary 2.3].

Lemma 5.2 *Let $X_0 : \mathbb{R}^N \to \mathbb{R}$ be a centered Gaussian random field which satisfies Condition* **(C1)** *for some ψ with $0 < \alpha_* \leq \alpha^* < 1$. Given a compact interval $K \subseteq \mathbb{R}^N$, let*

$$\omega_{X_0}(\delta) = \sup_{\substack{x,\ x+y \in K \\ |y| \leq \delta}} |X_0(x+y) - X_0(x)|$$

be the uniform modulus of continuity of $X_0(x)$ on K. Then there exists a finite constant $C_4 > 0$ such that

$$\limsup_{\delta \to 0} \frac{\omega_{X_0}(\delta)}{\sqrt{\psi(\delta) \log \frac{1}{\delta}}} \leq C_4, \quad a.s.$$

Combining (2.4), Lemmas 5.1 and 5.2 yields an upper bound for the generalized q-dimension of the image measure μ_X of μ.

Proposition 5.3 *Let $X : \mathbb{R}^N \to \mathbb{R}^d$ be the Gaussian random field defined by (2.6) such that the associated random field X_0 satisfies Condition* **(C1)***, and let $q > 0, q \neq 1$. Let μ be a Borel probability measure μ on $\mathbb{R}^N$ with compact support.*

(i) If $0 < \alpha_* = \alpha^* = \alpha < 1$, then

$$\underline{D}_q(\mu_X) \leq \min\Big\{d, \frac{1}{\alpha}\underline{D}_q(\mu)\Big\} \ a.s. \quad \text{and} \quad \overline{D}_q(\mu_X) \leq \min\Big\{d, \frac{1}{\alpha}\overline{D}_q(\mu)\Big\} \ a.s.$$

(ii) If $0 < \alpha_* \leq \alpha^* < 1$, then

$$\underline{D}_q(\mu_X) \leq \min\Big\{d, \frac{1}{\alpha^*}\overline{D}_q(\mu)\Big\} \ a.s. \quad \text{and} \quad \overline{D}_q(\mu_X) \leq \min\Big\{d, \frac{1}{\alpha_*}\overline{D}_q(\mu)\Big\} \ a.s.$$

As is often the case when finding dimensions or generalized dimensions, lower bounds are more elusive than upper bounds. For this, we use the strong local ψ-nondeterminism of X along with Corollary 3.2 to bound the probability that points $X(x_1), \ldots, X(x_n)$ all lie in the ball $B(X(y), r)$ in terms of a multipotential kernel defined using an ultrametric d_a. We then apply Theorem 4.4 to bound an integral involving this kernel.

Recall that an isotropic multivariate Gaussian random variable Z in $\mathbb{R}^d$ with variance σ^2 satisfies

$$\mathsf{P}\big\{|Z - u| \leq r\big\} \leq c\Big(\frac{r}{\sigma}\Big)^s, \qquad u \in \mathbb{R}^d, r > 0$$

for all $0 < s \leq d$, for some constant $c \equiv c_{d,s}$. Since the conditional distributions in a Gaussian process are still Gaussian, it follows from the strong local nondeterminism of X_0 given by **(C2)** that, for $t_0 > 0$, there exists $c > 0$ such that

$$\mathsf{P}\big\{|X(x) - u| \leq r \ \big|\ X(y) : t \leq |x - y| \leq t_0\big\} \leq c r^s \psi(t)^{-s/2} \tag{5.2}$$

for all $u \in \mathbb{R}^d$, $x \in \mathbb{R}^N$, $t > 0$ and $r > 0$.

We now introduce the multipotential kernel that will be used when applying Theorem 4.4. Let $m \geq 2$ and $a \in \mathcal{A}_m$, see (3.2). Recall that $\mathcal{C}_k^a$ denotes the set of kth level cubes in the hierarchy of m-ary cubes that define the metric d_a, i.e. the cubes of $\mathcal{C}_k$ translated by the vector $-a$, see (3.4). Let $\mathcal{C}^a = \cup_{k=0}^{\infty} \mathcal{C}_k^a$.

For a given $a \in \mathcal{A}_m$ and $w, z \in [0, \frac{1}{2})^N$, let $w \wedge z$ denote the smallest cube $C \in \mathcal{C}^a$ such that $w, z \in C$. Then if $w_1, \ldots, w_{n+1}$ are distinct points of $[0, \frac{1}{2})^N$ there is a uniquely defined set of n *join cubes*, $C_1, \ldots, C_n \in \mathcal{C}^a$, with the property that $w_i \wedge w_j$ is one of the cubes C_l for all $i \neq j$. To ensure that a set of $n+1$ points has exactly n join cubes we regard a join cube C as having *multiplicity* $r \geq 1$ if r is the greatest integer such that there are distinct $w_{j_1}, \ldots, w_{j_{r+1}}$ with $w_{j_p} \wedge w_{j_q}$ as the cube C for all $1 \leq p < q \leq r+1$. Note that the cubes of $\mathcal{C}^a$ may be identified naturally with the vertices of an m^N-ary tree I, which in turn identifies the points of $[0,1)^N - a$ with the infinite tree I_∞; this identification will be used to relate the ultrametrics to the trees of Section 4.

Write $k(C)$ for the *level* of the cube $C \in \mathcal{C}^a$, so that $C \in \mathcal{C}^a_{k(C)}$. We define the *multipotential kernel* ϕ_a by

$$\phi_a(w_1, \ldots, w_{n+1}) = m^{k(C_1)} m^{k(C_2)} \cdots m^{k(C_n)}, \tag{5.3}$$

where $C_1, \ldots, C_n$ are the join cubes of $w_1, \ldots, w_{n+1}$ in the hierarchy $\mathcal{C}^a$.

The following proposition uses strong local nondeterminism, in the form of (5.2), inductively to estimate the probability that the images of a set of points all lie inside a ball.

Proposition 5.4 *Let $X : \mathbb{R}^N \to \mathbb{R}^d$ be the Gaussian random field defined by* (2.6), *and assume the associated random field $X_0 : \mathbb{R}^N \to \mathbb{R}$ satisfies Condition* **(C2)**. *Given N, d, n, s and α, where $0 < s \leq d$ and $\alpha > \alpha^*$, there are positive constants c_2 and r_0 and an integer $m \geq 2$ such that, for all $x_1, x_2, \ldots, x_n, y \in [0, \frac{1}{2})^N$, we may choose a vector $a \in \mathcal{A}_m$, see* (3.2), *such that for all $0 < r \leq r_0$*

$$\begin{aligned} \mathsf{P}\Big\{|X(y) - X(x_1)| \leq r, |X(y) - X(x_2)| \leq r, \ldots, |X(y) - X(x_n)| \leq r\Big\} \\ \leq c_2 r^{sn} \phi_a(x_1, x_2, \ldots, x_n, y)^{\alpha s}. \end{aligned} \tag{5.4}$$

In particular, for all $x_1, x_2, \ldots, x_n, y \in [0, \frac{1}{2})^N$ and $0 < r \leq r_0$,

$$\begin{aligned} \mathsf{P}\Big\{|X(y) - X(x_1)| \leq r, |X(y) - X(x_2)| \leq r, \ldots, |X(y) - X(x_n)| \leq r\Big\} \\ \leq c_2 r^{sn} \sum_{a \in \mathcal{A}_m} \phi_a(x_1, x_2, \ldots, x_n, y)^{\alpha s}. \end{aligned} \tag{5.5}$$

Proof. Let $m = 2n^2 N + 2$ and $c_0 = \max\{8m(m-1), N^{1/2}\}$. By Corollary 3.2, given $x_1, x_2, \ldots, x_n, y \in [0, \frac{1}{2})^N$, there exists $a \in \mathcal{A}_m$ such that

$$c_0^{-1}|z - w| \leq d_a(z, w) \leq c_0 |z - w|, \qquad z, w \in \{x_1, x_2, \ldots, x_n, y\}; \tag{5.6}$$

thus d_a restricted to the set of points $\{x_1, x_2, \ldots, x_n, y\}$ is equivalent to the Euclidean metric with constant c_0.

We now appeal to the strong local ψ-nondeterminism of X. For $i = 2, 3, \ldots, n$ let w_i be the point (or one of the points) from $\{x_1, \ldots, x_{i-1}, y\}$ such that $d_a(w_i, x_i)$ is least. By local nondeterminism (5.2), noting the equivalence of the metrics (5.6), there are constants c_1 and r_0 such that

$$\begin{aligned} \mathsf{P}\big\{|X(w_i) - X(x_i)| \leq 2r \;\big|\; X(x_1), \ldots, X(x_{i-1}), X(y)\big\} &\leq c_1\, r^s \psi(|w_i - x_i|)^{-s/2} \\ &\leq c_1 r^s |w_i - x_i|^{-\alpha s} \end{aligned} \tag{5.7}$$

for each $i = 2, \ldots, n$ and $0 < r \le r_0$, where the last inequality follows from the fact that $\psi(r) \ge r^{2\alpha}$.

Starting with $\mathsf{P}\big\{|X(y) - X(x_1)| \le 2r\big\} \le c_1 r^s |y - x_1|^{-\alpha s}$, and applying the conditional probabilities in (5.7) inductively, we obtain

$$\begin{aligned}
&\mathsf{P}\big\{|X(y) - X(x_1)| \le r, |X(y) - X(x_2)| \le r, \ldots, |X(y) - X(x_n)| \le r\big\} \\
&\quad \le \mathsf{P}\big\{|X(y) - X(x_1)| \le 2r, |X(w_2) - X(x_2)| \le 2r, \ldots, |X(w_n) - X(x_n)| \le 2r\big\} \\
&\quad \le (c_1)^n r^{ns} |y - x_1|^{-\alpha s} |w_2 - x_2|^{-\alpha s} \ldots |w_n - x_n|^{-\alpha s} \\
&\quad \le (c_1)^n (c_0)^{n\alpha s} r^{ns} d_a(y, x_1)^{-\alpha s} d_a(w_2, x_2)^{-\alpha s} \ldots d_a(w_n, x_n)^{-\alpha s} \\
&\quad = c_2\, r^{ns} \phi_a(x_1, x_2, \ldots, x_n, y)^{\alpha s},
\end{aligned}$$

using the definitions (3.3) and (5.3) of d_a and ϕ_a and the choice of the w_i.

Inequality (5.5) is immediate from (5.4). $\square$

Proposition 5.5 *Let $n \ge 1$ and $1 < q \le n + 1$. Then for all $0 < s \le d$, there exist numbers $c_3 > 0$ and $r_0 > 0$ such that for all $0 < r \le r_0$,*

$$\begin{aligned}
&\mathsf{E} \int \mu_X(B(z, r))^{q-1} d\mu_X(z) \\
&\quad \le c_3 r^{s(q-1)} \sum_{a \in \mathcal{A}_m} \int \left[\int \cdots \int \phi_a(x_1, \ldots, x_n, y)^{\alpha s} d\mu(x_1) \ldots d\mu(x_n) \right]^{(q-1)/n} d\mu(y).
\end{aligned} \tag{5.8}$$

Proof. First note that, for every $y \in \mathbb{R}^N$, using Fubini's theorem and (5.5), we obtain

$$\begin{aligned}
&\mathsf{E}\big(\mu_X(B(X(y), r))^n\big) \\
&\quad = \mathsf{E}\big(\mu\{x : |X(y) - X(x)| \le r\}^n\big) \\
&\quad = \int \cdots \int \mathsf{P}\Big\{|X(y) - X(x_1)| \le r, |X(y) - X(x_2)| \le r, \\
&\qquad\qquad \ldots, |X(y) - X(x_n)| \le r\Big\} d\mu(x_1) \ldots d\mu(x_n) \\
&\quad \le c_2 r^{ns} \int \cdots \int \sum_{a \in \mathcal{A}_m} \phi_a(x_1, x_2, \ldots, x_n, y)^{\alpha s} d\mu(x_1) \ldots d\mu(x_n).
\end{aligned} \tag{5.9}$$

Since $n/(q-1) \ge 1$, Jensen's inequality, (5.9) and the power-sum inequality give

$$\begin{aligned}
&\mathsf{E} \int \mu_X(B(z, r))^{q-1} d\mu_X(z) \\
&\quad = \mathsf{E} \int \mu_X(B(X(y), r))^{q-1} d\mu(y) \\
&\quad \le \int \Big[\mathsf{E}\big(\mu_X(B(X(y), r))^n\big)\Big]^{(q-1)/n} d\mu(y) \\
&\quad \le c_3 r^{s(q-1)} \int \left[\sum_{a \in \mathcal{A}_m} \int \cdots \int \phi_a(x_1, x_2, \ldots, x_n, y)^{\alpha s} d\mu(x_1) \ldots d\mu(x_n) \right]^{(q-1)/n} d\mu(y) \\
&\quad \le c_3 r^{s(q-1)} \int \sum_{a \in \mathcal{A}_m} \left[\int \cdots \int \phi_a(x_1, x_2, \ldots, x_n, y)^{\alpha s} d\mu(x_1) \ldots d\mu(x_n) \right]^{(q-1)/n} d\mu(y),
\end{aligned}$$

to give (5.8). $\square$

We now derive the almost sure lower bound for $\underline{D}_q(\mu_X)$.

Proposition 5.6 *Let $X : \mathbb{R}^N \to \mathbb{R}^d$ be the Gaussian random field defined by* (2.6) *and assume that the associated random field X_0 satisfies Condition* **(C2)**. *Let μ be a Borel probability measure on $\mathbb{R}^N$ with compact support. Then for all $q > 1$,*

$$\underline{D}_q(\mu_X) \geq \min\left\{d, \frac{1}{\alpha^*}\underline{D}_q(\mu)\right\} \quad a.s. \tag{5.10}$$

Proof. Without loss of generality we may assume that the support of μ lies in the cube $[0, \frac{1}{2})^N$. As before, for each $a \in \mathcal{A}_m$ we write $\mathcal{C}_k^a$ for the kth level cubes in the hierarchy of m-ary cubes that define the metric d_a.

Let $\alpha > \alpha^*$ and $0 < s < \min\{d, \underline{D}_q(\mu)/\alpha\}$. From (2.1)

$$\liminf_{k\to\infty} \frac{\log \sum_{C\in\mathcal{C}_k^a} \mu(C)^q}{(q-1)\log m^{-k}} = \underline{D}_q(\mu) > s\alpha$$

so

$$\limsup_{k\to\infty} \frac{\log \left[m^{s(q-1)\alpha k} \sum_{C\in\mathcal{C}_k^a} \mu(C)^q\right]}{k} < 0.$$

Note that this estimate holds for all $a \in \mathcal{A}_m$ since the definition of the lower generalized dimension (2.1) is independent of the origin selected for the mesh cubes used for the moment sums.

Let n be the integer such that $n \leq q < n+1$. For each $a \in \mathcal{A}_m$ in turn, identify the cubes of $\mathcal{C}_k^a$ with the kth level vertices of the M-ary tree of Section 3 in the natural way, where $M = m^N$. Thus, with $x_j \in I_\infty$ identified with $x_j \in \mathbb{R}^N$ and $y \in I_\infty$ with $y \in \mathbb{R}^N$, we have that $l_i = k(C_i)$, $i = 1, \ldots, n$, are the levels both of the cubes and the equivalent vertices in the tree I_∞ in the join set of $x_1, \ldots, x_n, y$. Setting $f(l) = m^{\alpha s l}$ in (4.9) and using (5.3), we get

$$\phi(x_1, \ldots, x_n, y) = m^{\alpha s l_1} \cdots m^{\alpha s l_n} = m^{\alpha s k(C_1)} \cdots m^{\alpha s k(C_n)} = \phi_a(x_1, \ldots, x_n, y)^{\alpha s}.$$

Thus, Proposition 5.5 together with Theorem 4.4 gives that

$$\mathsf{E} \int \mu_X(B(z,r))^{q-1} d\mu_X(z) \leq c_4 r^{s(q-1)} \tag{5.11}$$

for all $r \leq 1$, for some constant $0 < c_4 < \infty$.

For all $0 < t < s < \min\{d, \underline{D}_q(\mu)/\alpha\}$, summing (5.11) over $r = 2^{-k}, k = 0, 1, 2, \ldots,$ gives

$$\mathsf{E}\left(\sum_{k=0}^{\infty} 2^{kt(q-1)} \int \mu_X(B(z, 2^{-k}))^{q-1} d\mu_X(z)\right) \leq c_4 \sum_{k=0}^{\infty} 2^{-k(s-t)(q-1)} < \infty.$$

Thus the bracketed series on the left converges almost surely, so as the generalized dimensions (2.2) are determined by the sequence of scales $r = 2^{-k}, k = 0, 1, 2, \ldots,$ we conclude that $\underline{D}_q(\mu_X) \geq t$ almost surely for all $0 < t < \min\{d, \underline{D}_q(\mu)/\alpha\}$. Since $\alpha > \alpha^*$ is arbitrary, (5.10) follows. □

Proof of Theorem 2.1. The upper bound in (2.10) follows from Proposition 5.3, and the lower bound in (2.10) follows from Proposition 5.6. This proves (i). Part (ii) also follows from Propositions 5.3 and 5.6. Finally, (iii) follows from (ii). □

6 Further remarks

Here are some open problems and remarks about generalized dimensions of random fields which are not covered by this paper.

1. Note that (2.11) in Theorem 2.1 only provides an upper bound for $\overline{D}_q(\mu_X)$. While we believe that if the generalized q-dimension of $D_q(\mu)$ exists then

$$\overline{D}_q(\mu_X) = \min\left\{d,\, \frac{1}{\alpha_*} D_q(\mu)\right\} \quad \text{a.s.},$$

we have not been able to prove it, because the last inequality in (5.7) fails when $\alpha_* < \alpha < \alpha^*$.

2. Besides fractional Brownian motion, another important Gaussian random field is the Brownian sheet $W : \mathbb{R}^N_+ \to \mathbb{R}^d$, $W(x) = (W_1(x), \ldots, W_d(x))$, which is a centered Gaussian random field with covariance function given by

$$\mathsf{E}\big[W_i(x)W_j(y)\big] = \delta_{ij} \prod_{k=1}^{N} x_k \wedge y_k, \qquad x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in \mathbb{R}^N_+,$$

where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, see [19]. The Brownian sheet W is not strongly locally nondeterministic, but satisfies a weaker form of local nondeterminism, namely, sectorial local nondeterminism as it is called in [20]. We expect that the conclusion of Corollary 2.2 still holds for the Brownian sheet, but since strong local nondeterminism played an important rôle in Section 5, a different method may be needed to study the effect of the Brownian sheet on generalized dimensions.

3. In recent years several authors have constructed and investigated anisotropic random fields, see [40] and references therein. Random fractal images under anisotropic random fields have a richer geometry than isotropic random fields such as fractional Brownian motion. It is not clear to what extent our arguments can be modified for non-isotropic Gaussian fields. Furthermore, it is not clear to what extent our approach can be used for non-Gaussian random fields, such as linear and harmonizable fractional stable random fields, see [42]. For example, difficulties arise from possible discontinuities of the sample paths.

4. The case of small moments, that is for $0 < q < 1$, is interesting but is likely to need rather different methods. Extrapolating from the case $q = 0$ (when the generalized dimension reduces to box counting dimension of the support of the measure), it is possible that some kind of generalized dimension profile is needed, see [14, 36].

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