On intersections of independent anisotropic Gaussian random fields

CHEN ZhenLong[1,†] & XIAO Yimin[2]

[1] School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310035, China;

[2] Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824, USA

(email: zlchen@zjsu.edu.cn, xiao@stt.msu.edu)

Abstract Let $X^H = \{X^H(s), s \in \mathbb{R}^{N_1}\}$ and $X^K = \{X^K(t), t \in \mathbb{R}^{N_2}\}$ be two independent anisotropic Gaussian random fields with values in $\mathbb{R}^d$ with indices $H = (H_1, \ldots, H_{N_1}) \in (0,1)^{N_1}$, $K = (K_1, \ldots, K_{N_2}) \in (0,1)^{N_2}$, respectively. Existence of intersections of the sample paths of X^H and X^K is studied. More generally, let $E_1 \subseteq \mathbb{R}^{N_1}$, $E_2 \subseteq \mathbb{R}^{N_2}$ and $F \subset \mathbb{R}^d$ be Borel sets. A necessary condition and a sufficient condition for $\mathbb{P}\{(X^H(E_1) \cap X^K(E_2)) \cap F \neq \varnothing\} > 0$ in terms of the Bessel-Riesz type capacity and Hausdorff measure of $E_1 \times E_2 \times F$ in the metric space $(\mathbb{R}^{N_1+N_2+d}, \widetilde{\rho})$ are proved, where $\widetilde{\rho}$ is a metric defined in terms of H and K. These results are applicable to solutions of stochastic heat equations driven by space-time Gaussian noise and fractional Brownian sheets.

Keywords: intersection, anisotropic Gaussian fields, hitting probability, Hausdorff dimension, stochastic heat equation, fractional Brownian sheet

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1 Introduction

Many authors have investigated intersections of the trajectories of stochastic processes. For Brownian motion, the questions have been studied by A. Dvoretzky, P. Erdös, S. Kakutani, S. J. Taylor, and J.-F. LeGall. See Khoshnevisan [19] for historical accounts and a very nice proof for the existence theorem using an elementary argument based on the self-similarity and Markov property of Brownian motion. The results on intersections of Brownian motion have been extended to Lévy processes, Gaussian processes and other processes. We refer to the survey papers of Taylor [30] and Xiao [39] for further information.

This paper is concerned with existence of intersections of Gaussian random fields. Our approach is based on the results on hitting probabilities of Gaussian random fields in Biermé, Lacaux and Xiao [3]. This is different from the approaches based on intersection local times in Rosen [28, 29], Hu and Nualart [13], Wu and Xiao [36], where fractional Brownian motions are considered. The approach in this paper can be applied to a wide class of Gaussian random fields and can be extended for studying intersections of non-Gaussian random fields.

Let $X^H = \{X^H(s), s \in \mathbb{R}^{N_1}\}$ and $X^K = \{X^K(t), t \in \mathbb{R}^{N_2}\}$ be two independent Gaussian random fields taking values in $\mathbb{R}^d$. More specifically, we assume that X^H is defined as

$$X^H(s) = (X_1^H(s), \ldots, X_d^H(s)), \qquad s \in \mathbb{R}^{N_1}, \tag{1}$$

where $X_1^H, \ldots, X_d^H$ are independent copies of a real-valued, centered Gaussian random field X_0^H. The Gaussian random field X^K is defined in the same way. Here $H \in (0,1)^N$ and

† Corresponding author

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$K \in (0,1)^N$ are constant vectors whose meanings will be specified later.

We say the two Gaussian fields X^H and X^K intersect if there exist $s \in \mathbb{R}^{N_1}$ and $t \in \mathbb{R}^{N_2}$ such that $X^H(s) = X^K(t)$. In this paper we study the existence of intersections and, particularly, the following problems.

(i) When do X^H and X^K intersect (with positive probability)?

(ii) Let $E_1 \subseteq \mathbb{R}^{N_1}$ and $E_2 \subseteq \mathbb{R}^{N_2}$ be arbitrary Borel sets. When do X^H and X^K intersect if we restrict the "time" $s \in E_1$ and $t \in E_2$? More precisely, when is

$$\mathbb{P}\Big\{X^H(E_1) \cap X^K(E_2) \neq \varnothing\Big\} > 0?$$

(iii) Given a Borel set $F \subset \mathbb{R}^d$, when does F contain intersection points of $X^H(s)$, $(s \in E_1)$ and $X^K(t)$, $(t \in E_2)$? That is, when is

$$\mathbb{P}\Big\{X^H(E_1) \cap X^K(E_2) \cap F \neq \varnothing\Big\} > 0?$$

Clearly, Question (i) is a special case of Question (ii), which is a special case of Question (iii). The order reflects the historic development on this topic. While Question (i) has been studied extensively since the pioneering work of A. Dvoretzky, P. Erdös and S. Kakutani, Questions (ii) and (iii) have only attracted attention of a few researchers. Kahane [14, 15] was the first to consider intersections of a symmetric stable Lévy process when its time is restricted to disjoint compact sets, which is related to Question (ii). By applying potential theory for additive Lévy processes, Khoshnevisan and Xiao [20] provided a necessary and sufficient condition in terms of a class of natural capacity for the self-intersections of a Lévy process when the time t is restricted to two disjoint sets and thus resolved a long standing conjecture of Kahane [14]. According to Peres [26], Question (iii) for Brownian motion was raised by S. Kakutani (extending a question of P. Lévy). Evans [10] and Tongring [32] provided sufficient conditions for a Bore set $F \subseteq \mathbb{R}^d$ $(d = 2, 3)$ to contain multiple points of a Brownian motion in $\mathbb{R}^d$. Later Fitzsimmons and Salisbury [12] completed the work by showing that the sufficient condition of Evans [10] and Tongring [32] for $\mathbb{R}^2$ is also necessary. See Peres [26] for the case of Brownian motion in $\mathbb{R}^3$ and further information. Question (iii) was recently studied by Dalang, et al. [7] for independent Brownian sheets in the special case when E_1 and E_2 are intervals. The special dependence structures of the Brownian sheet such as independence of increments over disjoint intervals play crucial roles in Dalang, et al. [7].

In this paper we answer the above questions (i)–(iii) by providing sufficient conditions and necessary conditions for the existence of intersections in terms of H, K and the geometry of E_1, E_2 and F. Our method is based on results on hitting probabilities of Gaussian random fields in Biermé, Lacaux and Xiao [3] and Xiao [40].

The rest of this paper is organized as follows. In Section 2 we prove a theorem on hitting probabilities of a Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$. This result extends and refines those in Biermé, Lacaux and Xiao [3] and Xiao [40], and is suitable for studying Questions (i) and (ii).

In Section 3, we apply the results in Section 2 to provide answers to Questions (i)–(iii). We will see that, in order to answer Question (iii), it is useful to study the hitting probability problem for the $\mathbb{R}^{2d}$-valued Gaussian random field $Y(s,t) = (X^H(s), X^K(t))$, where X^H and X^K may have different distributions.

Finally in Section 4, we show that our main results in Section 3 can be applied to solutions to stochastic partial differential equations and to fractional Brownian sheets.

Throughout this paper, we will use c to denote unspecified positive finite constants which may be different in each appearance. More specific constants are numbered as $c_1,\ c_2,\cdots$.

2 Hitting probabilities

In this section, we consider the problem on hitting probabilities of Gaussian random fields under some general conditions. Our main result is Theorem 2.1 which refines Theorem 2.1 of Biermé, Lacaux and Xiao [3] and Theorem 7.6 in Xiao [40].

We will continue to use the same setting as in Biermé, Lacaux and Xiao [3]. Let $H = (H_1, \dots, H_N) \in (0,1)^N$ be a fixed vector and, for $a, b \in \mathbb{R}^N$ with $a_j < b_j$ $(j = 1, \dots, N)$, let $I = [a,b] := \prod_{j=1}^N [a_j, b_j] \subseteq \mathbb{R}^N$ denote a compact interval (or a rectangle). For example, we may take $I = [\varepsilon_0, 1]^N$, where $\varepsilon_0 \in (0,1)$ is a fixed constant.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$X(t) = \big(X_1(t), \dots, X_d(t)\big), \quad \forall\, t \in \mathbb{R}^N. \tag{2}$$

We will call X an (N,d)-Gaussian random field. We assume that the coordinate processes $X_1, \dots, X_d$ are independent copies of a real-valued, centered Gaussian random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$, which satisfies the following conditions:

(C1) There exists a positive and finite constant c_1 such that

$$\mathbb{E}\Big[\big(X_0(s) - X_0(t)\big)^2\Big] \leqslant c_1 \sum_{j=1}^N |s_j - t_j|^{2H_j}, \quad \forall\, s,\, t \in I. \tag{3}$$

(C2) There exists a constant $c_2 > 0$ such that for all $s, t \in I$,

$$\mathrm{Var}\big(X_0(t) \big| X_0(s)\big) \geqslant c_2 \sum_{j=1}^N |s_j - t_j|^{2H_j}. \tag{4}$$

Here $\mathrm{Var}(X_0(t)|X_0(s))$ denotes the conditional variance of $X_0(t)$ given $X_0(s)$.

Notice that (C1) implies that the function $\sigma^2(t) = \mathbb{E}\big[(X_0(t))^2\big]$ is continuous on I and (C2) implies that $\sigma^2(t) > 0$ for every $t \in I$. Hence there is a positive constant c such that $\mathbb{E}[X_0(t)^2] \geqslant c$ for all $t \in I$. This fact will be used in the proofs of Theorems 2.1 and 3.5. The class of Gaussian random fields that satisfy Conditions (C1) and (C2) is large. It includes not only the well-known fractional Brownian motion and the Brownian sheet, but also fractional Brownian sheets (cf. Ayache and Xiao [2]), solutions to stochastic heat equation driven by space-time white noise (Dalang, Khoshnevisan and Nualart [5, 6], Dalang and Nualart [8], Mueller and Tribe [25]) and many more. See Xiao [40] for more examples and further information.

In studying sample path properties of Gaussian random fields which satisfy (C1) and (C2), the following metric ρ on $\mathbb{R}^N$ defined by

$$\rho(s,t)=\sum_{j=1}^{N}|s_j-t_j|^{H_j}, \qquad \forall s,\, t\in\mathbb{R}^N \tag{5}$$

has played important roles (cf. Xiao [40]). Before stating the main result of this section, we define the metric $\widetilde{\rho}$ on $\mathbb{R}^N\times\mathbb{R}^d$ by

$$\widetilde{\rho}\big((s,x),(t,y)\big)=\max\big\{\rho(s,t),\ \ \|x-y\|\big\}, \qquad \forall (s,x),(t,y)\in\mathbb{R}^N\times\mathbb{R}^d, \tag{6}$$

where $\|\cdot\|$ denotes the Euclidean metric on $\mathbb{R}^d$. For any $\beta>0$ and $G\subseteq\mathbb{R}^{N+d}$, define the β-dimensional Hausdorff measure in the metric $\widetilde{\rho}$ of G by

$$\mathcal{H}^{\beta}_{\widetilde{\rho}}(G)=\lim_{\delta\to 0}\inf\bigg\{\sum_{n=1}^{\infty}(2r_n)^{\beta}:\ G\subseteq\bigcup_{n=1}^{\infty}B_{\widetilde{\rho}}(r_n),\ r_n\leqslant\delta\bigg\}, \tag{7}$$

where $B_{\widetilde{\rho}}(r)$ denotes an open ball of radius r in the metric space $(\mathbb{R}^{N+d},\widetilde{\rho})$. It can be proved that $\mathcal{H}^{\beta}_{\widetilde{\rho}}$ is a metric outer measure and all Borel sets in $\mathbb{R}^{N+d}$ are $\mathcal{H}^{\beta}_{\widetilde{\rho}}$-measurable (cf. Falconer [11] or Rogers [27]). The corresponding Hausdorff dimension of G is defined by

$$\dim^{\widetilde{\rho}}_{\rm H} G=\inf\big\{\beta>0:\ \mathcal{H}^{\beta}_{\widetilde{\rho}}(G)=0\big\}. \tag{8}$$

It is known that Frostman's lemma still holds. See, for example, Xiao [40, Lemma 6.10].

The Bessel-Riesz type capacity of order α on the metric space $(\mathbb{R}^{N+d},\widetilde{\rho})$ is defined by

$$\mathcal{C}_{\widetilde{\rho},\alpha}(G)=\bigg[\inf_{\mu\in\mathcal{P}(G)}\int_{\mathbb{R}^{N+d}}\int_{\mathbb{R}^{N+d}}f_{\alpha}\big(\widetilde{\rho}(u,v)\big)\mu(du)\mu(dv)\bigg]^{-1}, \tag{9}$$

where $\mathcal{P}(G)$ is the family of probability measures carried by G and the function $f_\alpha:(0,\infty)\to(0,\infty)$ is defined by

$$f_\alpha(r)=\begin{cases} r^{-\alpha} & \text{if } \alpha>0,\\ \log\big(\frac{e}{r\wedge 1}\big) & \text{if } \alpha=0,\\ 1 & \text{if } \alpha<0.\end{cases} \tag{10}$$

As in Xiao [40] and Biermé, Lacaux and Xiao [3], the usual β-dimensional Hausdorff measure and Bessel-Riesz capacity of order α in Euclidean metric $\|\cdot\|$ are denoted by $\mathcal{H}^\beta$ and $\mathcal{C}_\alpha$, respectively.

The following is the main result of this section.

Theorem 2.1 *Let $X=\{X(t),t\in\mathbb{R}^N\}$ be an (N,d)-Gaussian random field defined by* (2) *and assume that X_0 satisfies Conditions (C1) and (C2). If $E\subseteq I$ and $F\subseteq\mathbb{R}^d$ are Borel sets, then*

$$c_3^{-1}\,\mathcal{C}_{\widetilde{\rho},d}(E\times F)\leqslant\mathbb{P}\Big\{X(E)\cap F\neq\varnothing\Big\}\leqslant c_3\,\mathcal{H}^{d}_{\widetilde{\rho}}(E\times F), \tag{11}$$

where $c_3\geqslant 1$ is a finite constant which depends on I, F and H only.

As a corollary we obtain Theorem 7.6 in Xiao [40] and Theorem 2.1 of Biermé, Lacaux and Xiao [3].

Corollary 2.2 *Assume that the conditions of Theorem 2.1 hold. Then for every Borel set $F \subseteq \mathbb{R}^d$, we have*

$$c_4^{-1}\,\mathcal{C}_{d-Q}(F) \leqslant \mathbb{P}\Big\{X(I) \cap F \neq \varnothing\Big\} \leqslant c_4\,\mathcal{H}^{d-Q}(F), \tag{12}$$

where $c_4 \geqslant 1$ is a finite constant depending on I, F and H only. In the above, $Q := \sum_{j=1}^N 1/H_j$, $\mathcal{C}_\alpha$ is the Bessel-Riesz capacity of order α in the Euclidean metric, and $\mathcal{H}^q(F)$ is defined as the q-dimensional Hausdorff measure of F when $q > 0$, and $\mathcal{H}^q(F) = 1$ whenever $q \leqslant 0$.

Proof It is sufficient to verify that for any interval $I \subseteq \mathbb{R}^N$, there exist positive and finite constants c_5 and c_6 such that for all Borel sets F in $\mathbb{R}^d$ we have

(i) $\mathcal{H}^d_{\widetilde{\rho}}(I \times F) \leqslant c_5\,\mathcal{H}^{d-Q}(F)$ and

(ii) $\mathcal{C}_{d-Q}(F) \leqslant c_6\,\mathcal{C}_{\widetilde{\rho},d}(I \times F)$.

To prove (i), let $\gamma > \mathcal{H}^{d-Q}(F)$ be arbitrary. Then there is a covering of F by balls $B(r_n)$ of radius r_n (in the Euclidean metric) such that

$$F \subset \bigcup_{n=1}^{\infty} B(r_n) \quad \text{and} \quad \sum_{n=1}^{\infty} r_n^{d-Q} \leqslant \gamma. \tag{13}$$

Since I can be covered by $c r_n^{-Q}$ cubes $C_{n,j}$ in $\mathbb{R}^N$ of sides r_n^{1/H_i} $(i = 1, \ldots, N)$ (or balls of radius r_n in the metric ρ), we see that

$$I \times F \subset \bigcup_{n=1}^{\infty} \bigcup_j C_{n,j} \times B(r_n).$$

This gives a covering of $I \times F$ by balls of radius r_n in the metric $\widetilde{\rho}$. Moreover, the inequality in (13) implies

$$\sum_{n=1}^{\infty} \sum_j (2r_n)^d \leqslant c \sum_{n=1}^{\infty} r_n^{d-Q} \leqslant c\gamma.$$

This implies (i).

Next we prove (ii). We assume $\mathcal{C}_{d-Q}(F) > 0$, otherwise there is nothing to prove. For any $0 < \gamma < \mathcal{C}_{d-Q}(F)$, there is a probability measure σ on F such that

$$\int\!\!\int \frac{\sigma(dx)\sigma(dy)}{\|x-y\|^{d-Q}} \leqslant \gamma^{-1} \tag{14}$$

provided $d > Q$. When $d = Q$, the kernel in (14) is the logarithmic function. Let λ be the restriction of the normalized Lebesgue measure on I and let $\mu = \lambda \times \sigma$. Then μ is a probability measure on $I \times F$. When $d > Q$, it follows from Equation (173) in Xiao [40] that

$$\int_{I\times F}\int_{I\times F} \frac{ds dt \sigma(dx)\sigma(dy)}{\widetilde{\rho}\big((s,x),(t,y)\big)^d} \leqslant c \int\!\!\int \frac{\sigma(dx)\sigma(dy)}{\|x-y\|^{d-Q}} \leqslant c\,\gamma^{-1}. \tag{15}$$

When $d = Q$, Equation (3.32) in Biermé, Lacaux and Xiao [3] shows that the corresponding inequality with the logarithmic kernel still holds. By (15) we have $\mathcal{C}_{\widetilde{\rho},d}(I \times F) \geqslant c^{-1}\gamma$. Since $\gamma < \mathcal{C}_{d-Q}(F)$ is arbitrary, this proves (ii) and the Corollary. □

In the remaining part of this section, we prove Theorem 2.1. We will make use of the following two lemmas proved in Biermé, Lacaux and Xiao [3]. Lemma 2.3 will be applied to prove the upper bound in (11), and Lemma 2.4 will be applied to prove the lower bound in (11).

Lemma 2.3 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field defined by* (2) *with X_0 satisfying Conditions (C1) and (C2). For any constant $M > 0$, there exist positive constants c and δ_0 such that for all $r \in (0, \delta_0)$, $t \in I$ and all $x \in [-M, M]^d$,*

$$\mathbb{P}\left\{ \inf_{s \in B_\rho(t,r) \cap I} \big\| X(s) - x \big\| \leqslant r \right\} \leqslant c\, r^d. \tag{16}$$

In the above $B_\rho(t,r) = \{s \in \mathbb{R}^N : \rho(s,t) \leqslant r\}$ denotes the closed ball of radius r in the metric ρ in $\mathbb{R}^N$ defined by (5).

Lemma 2.4 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field defined by* (2) *with X_0 satisfying Conditions (C1) and (C2). Then there exists a positive and finite constant c such that for all $\varepsilon \in (0,1)$, $s,t \in I$ and $x,y \in \mathbb{R}^d$, we have*

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} e^{-i(\langle \xi, x\rangle + \langle \eta, y\rangle)} \exp\left(-\frac{1}{2}(\xi,\, \eta)\left(\varepsilon I_{2d} + \mathrm{Cov}\big(X(s), X(t)\big)\right)(\xi,\, \eta)^T \right) d\xi d\eta \\ &\quad \leqslant \frac{c}{\widetilde{\rho}\big((s,x),(t,y)\big)^d}. \end{aligned} \tag{17}$$

In the above, I_{2d} denotes the identity matrix of order $2d$, $\mathrm{Cov}\big(X(s), X(t)\big)$ denotes the covariance matrix of the random vector $(X(s), X(t))$, and $(\xi, \eta)^T$ is the transpose of the row vector (ξ, η).

Proof of Theorem 2.1 The proof is a modification of the proof of Theorem 7.6 in Xiao [40]. The upper bound in (11) is proved by using a simple covering argument. Choose and fix an arbitrary constant $\gamma > \mathcal{H}^d_{\widetilde{\rho}}(E \times F)$. Then there is a sequence of balls $\{B_{\widetilde{\rho}}\big((t_j, y_j),\, r_j\big), j \geqslant 1\}$ in $\mathbb{R}^{N+d}$ such that r_j's are small,

$$E \times F \subseteq \bigcup_{j=1}^{\infty} B_{\widetilde{\rho}}\big((t_j, y_j),\, r_j\big) \quad \text{and} \quad \sum_{j=1}^{\infty} (2r_j)^d \leqslant \gamma. \tag{18}$$

Note that $B_{\widetilde{\rho}}\big((t_j, y_j),\, r_j\big)$ can be covered by $B_\rho(t_j,\, r_j) \times B(y_j,\, r_j)$, we have

$$\big\{X(E) \cap F \neq \varnothing\big\} \subseteq \bigcup_{j=1}^{\infty} \Big\{ X\big(B_\rho(t_j,\, r_j)\big) \cap B(y_j,\, r_j) \neq \varnothing \Big\}. \tag{19}$$

It follows from Lemma 2.3 that

$$\mathbb{P}\Big\{ X\big(B_\rho(t_j,\, r_j)\big) \cap B(y_j,\, r_j) \neq \varnothing \Big\} \leqslant c\, r_j^d. \tag{20}$$

Combining (19), (20) and (18), we obtain that $\mathbb{P}\big\{X(E) \cap F \neq \varnothing\big\} \leqslant c\gamma$. Since $\gamma > \mathcal{H}^d_{\widetilde{\rho}}(E \times F)$ is arbitrary, the upper bound in (11) follows.

The lower bound in (11) can be proved by using a second moment argument. Without loss of generality, we assume $\mathcal{C}_{\widetilde{\rho},d}(E \times F) > 0$ otherwise there is nothing to prove. By Choquet's capacity theorem (cf. Khoshnevisan [18]), we may and will assume that F is compact and let $M > 0$ be a constant such that $F \subseteq [-M, M]^d$.

For any Borel probability measure μ on $E \times F$ such that

$$\mathcal{E}_{\widetilde{\rho},\, d}(\mu) = \int\int \frac{1}{\widetilde{\rho}(u, v)^d}\, \mu(du)\mu(dv) \leqslant \frac{2}{\mathcal{C}_{\widetilde{\rho},d}(E \times F)}. \tag{21}$$

For all integers $n \geqslant 1$, we consider a family of random measures ν_n on $E \times F$ defined by

$$\begin{aligned} \int_{E\times F} g(s,x)\, \nu_n(ds, dx) &= \int_{E\times F} (2\pi n)^{d/2} \exp\Big(-\frac{n}{2}\, \|X(s) - x\|^2\Big)\, g(s,x)\, \mu(ds,\, dx) \\ &= \int_{E\times F} \int_{\mathbb{R}^d} \exp\bigg(-\frac{\|\xi\|^2}{2n} + i\langle \xi, X(s) - x\rangle \bigg)\, g(s,x)\, d\xi\, \mu(ds,\, dx), \end{aligned} \tag{22}$$

where g is an arbitrary measurable function on $\mathbb{R}^N \times \mathbb{R}^d$. Denote the total mass of ν_n by $\|\nu_n\|$. We claim that the following two inequalities hold:

$$\mathbb{E}\big(\|\nu_n\|\big) \geqslant c_7, \qquad \mathbb{E}\big(\|\nu_n\|^2\big) \leqslant c_8\, \mathcal{E}_{\widetilde{\rho},\, d}(\mu), \tag{23}$$

where the constants c_7 and c_8 are independent of μ and n.

By (22) and Fubini's theorem we have

$$\begin{aligned} \mathbb{E}\big(\|\nu_n\|\big) &= \int_{E\times F} \int_{\mathbb{R}^d} \exp\bigg(-\frac{1}{2}\Big(\frac{1}{n} + \sigma^2(s)\Big) \|\xi\|^2 - i\langle \xi, x\rangle \bigg)\, d\xi\, \mu(ds, dx) \\ &\geqslant \int_{E\times F} \frac{(2\pi)^{d/2}}{(1 + \sigma^2(s))^{d/2}} \exp\bigg(-\frac{\|x\|^2}{2\sigma^2(s)}\bigg)\, \mu(ds, dx) \\ &\geqslant c_7 > 0, \end{aligned}$$

where $\sigma^2(s) = \mathbb{E}\big(X_0(s)^2\big)$ and c_7 does not depend on μ and n because F is bounded and μ is a probability measure. This gives the first inequality in (23).

Next we prove the second inequality in (23). By using (22) and Fubini's theorem again, we obtain

$$\begin{aligned} \mathbb{E}\big(\|\nu_n\|^2\big) &= \int_{E\times F} \int_{E\times F} \mu(ds, dx)\mu(dt, dy) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i(\langle \xi,\, x\rangle + \langle \eta,\, y\rangle)} \\ &\qquad\qquad \times \exp\bigg(-\frac{1}{2}(\xi,\, \eta)\big(n^{-1} I_{2d} + \mathrm{Cov}\big(X(s), X(t)\big)\big)\, (\xi,\, \eta)^T \bigg)\, d\xi d\eta \\ &\leqslant c \int_{E\times F} \int_{E\times F} \frac{1}{\widetilde{\rho}\big((s,x),\, (t,y)\big)^d}\, \mu(ds, dx)\mu(dt, dy), \end{aligned}$$

where the last inequality follows from Lemma 2.4. This verifies the second inequality in (23).

By using (23) and the Paley-Zygmund inequality (cf. Kahane [15], p.8), one can verify that there is a subsequence of $\{\nu_n, n \geqslant 1\}$ that converges weakly to a finite measure ν which is positive with positive probability [depending on c_7 and c_8 only] and ν also satisfies (23) (see Testart [31]). Since ν is supported on the set $\{(s,x) \in E \times F : X(s) = x\}$, we use the Paley-Zygmund inequality again to derive

$$\mathbb{P}\Big\{X(E) \cap F \neq \varnothing\Big\} \geqslant \mathbb{P}\big\{\|\nu\| > 0\big\} \geqslant \frac{\big[\mathbb{E}(\|\nu\|)\big]^2}{\mathbb{E}\big[\|\nu\|^2\big]} \geqslant c_9\, \mathcal{C}_{\widetilde{\rho},d}(E \times F),$$

where $c_9 = c_7^2/c_8$. This implies the lower bound in (11). □

Remark 2.5 Theorem 2.1 assumes that the coordinate processes of X are i.i.d. It would be interesting to consider (N, d)-Gaussian random fields whose coordinate processes are dependent or/and not identically distributed. An interesting example of such Gaussian random field is operator-fractional Brownian motion studied by Mason and Xiao [24], Didier and Pipiras [9], among others. More general examples can be found in Li and Xiao [21]. In Section 3.2, in order to answer Question (iii) in the Introduction, we will consider the hitting probability problem for the Gaussian random field $Y(s,t) = (X^H(s), X^K(t))$, where X^H and X^K have different distributions when $H \neq K$. □

3 Intersections of Gaussian random fields

In this section we apply Theorem 2.1 and the method in its proof to answer Questions (i)–(iii) in the Introduction.

3.1 Results for Questions (i) and (ii)

Let $H = (H_1, \ldots, H_{N_1}) \in (0,1)^{N_1}$ and $K = (K_1, \ldots, K_{N_2}) \in (0,1)^{N_2}$ be two constant vectors. Let $X^H = \{X^H(s), s \in \mathbb{R}^{N_1}\}$ and $X^K = \{X^K(t), t \in \mathbb{R}^{N_2}\}$ be two independent Gaussian random fields with values in $\mathbb{R}^d$ as defined in (1). We assume that the associate real-valued random fields X_0^H and X_0^K satisfy Conditions (C1) and (C2) respectively on interval $I_1 \subseteq \mathbb{R}^{N_1}$ with indices $H = (H_1, \ldots, H_{N_1})$ and on $I_2 \subseteq \mathbb{R}^{N_2}$ with indices $K = (K_1, \ldots, K_{N_2})$. In the rest of this paper, we let $N = N_1 + N_2$ and $I = I_1 \times I_2$.

Let $Z = \{Z(s,t), (s,t) \in \mathbb{R}^N\}$ be the (N, d)-Gaussian random field defined by

$$Z(s,t) \equiv X^H(s) - X^K(t), \qquad s \in \mathbb{R}^{N_1}, t \in \mathbb{R}^{N_2}. \tag{24}$$

Let $E_1 \subseteq I_1$ and $E_2 \subseteq I_2$ be Borel sets. We study conditions on E_1 and E_2 such that $\mathbb{P}\big\{X^H(E_1) \cap X^K(E_2) \neq \varnothing\big\} > 0$, which is equivalent to $\mathbb{P}\big\{Z(E_1 \times E_2) \cap \{0\} \neq \varnothing\big\} > 0$.

The random fields X_0^H and X_0^K induce the metrics ρ^H and ρ^K on $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$, respectively, where

$$\rho^H(s, s') = \sum_{i=1}^{N_1} |s_i - s'_i|^{H_i} \tag{25}$$

and the metric ρ^K on $\mathbb{R}^{N_2}$ is defined in a similar way by using $K = (K_1, \ldots, K_{N_2})$. So a natural metric on $\mathbb{R}^N$ is

$$\rho\big((s,t), (s',t')\big) = \rho^H(s, s') + \rho^K(t, t'), \quad \forall\ (s,t),\ (s',t') \in \mathbb{R}^N. \tag{26}$$

Theorem 3.1 *Under the above conditions, there exists a finite constant $c_{10} \geqslant 1$ such that*

$$c_{10}^{-1}\, \mathcal{C}_{\rho,d}(E_1 \times E_2) \leqslant \mathbb{P}\Big\{X^H(E_1) \cap X^K(E_2) \neq \varnothing\Big\} \leqslant c_{10}\, \mathcal{H}_\rho^d(E_1 \times E_2). \tag{27}$$

In the above, $\mathcal{H}_\rho^d$ and $\mathcal{C}_{\rho,d}$ denote the d-dimensional Hausdorff measure and the Bessel-Riesz capacity of order d in the metric space $(\mathbb{R}^N, \rho)$.

Proof First we verify that the Gaussian field $Z_0(s,t) = X_0^H(s) - X_0^K(t)$ satisfies Conditions (C1) and (C2) on the interval $I = I_1 \times I_2$ with indices $(H_1, \ldots, H_{N_1}, K_1, \ldots, K_{N_2}) \in (0,1)^N$.

Because X_0^H and X_0^K are independent, it is straightforward to verify (C1). Hence we only need to show that (C2) holds. For any (s,t), $(s',t') \in I$, we use independence again to get

$$\mathrm{Var}\big(Z_0(s,t)|Z_0(s',t')\big) \geqslant \mathrm{Var}\big(X_0^H(s)|X^H(s')\big) \geqslant c \sum_{j=1}^{N_1} |s_j - s_j'|^{2H_j}.$$

Similarly we have

$$\mathrm{Var}\big(Z_0(s,t)|Z_0(s',t')\big) \geqslant \mathrm{Var}\big(X_0^K(t)|X^K(t')\big) \geqslant c \sum_{j=1}^{N_2} |t_j - t_j'|^{2K_j}.$$

Adding up these two inequality shows that Z_0 satisfies Condition (C2) as desired.

Now we apply Theorem 2.1 to the Gaussian random field Z with $E = E_1 \times E_2$ and $F = \{0\}$ to get

$$c_{_4}^{-1}\, \mathcal{C}_{\widetilde{\rho},d}(E_1 \times E_2 \times \{0\}) \leqslant \mathbb{P}\Big\{X^H(E_1) \cap X^K(E_2) \neq \varnothing\Big\} \leqslant c_{_4}\, \mathcal{H}^d_{\widetilde{\rho}}(E_1 \times E_2 \times \{0\}).$$

This gives (27). □

By taking $E_1 = I_1$ and $E_2 = I_2$ we obtain the following corollary, which provides an answer to Question (i).

Corollary 3.2 *Under the conditions of Theorem 3.1, we have*

(i) If $d > Q$, then $\mathbb{P}\big\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\big\} = 0$.

(ii If $d < Q$, then $\mathbb{P}\big\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\big\} > 0$.

In the above, $Q := \sum_{j=1}^{N_1} 1/H_j + \sum_{j=1}^{N_2} 1/K_j$.

Proof Applying Corollary 2.2 to Z with $F = \{0\}$ yields the corollary. □

Remark 3.3 The critical case of $Q = d$ is more subtle and is an open problem except in a few special cases. If X^H and X^K are two independent Brownian sheets (with $N_1 = N_2$ and $H = K = (1/2, \ldots, 1/2)$), the problem has been recently solved by Dalang, et al. [7] who proved that there is almost surely no intersection. When X^H and X^K are two independent fractional Brownian motions, Xiao [38, Theorem 3.2] proved that if $N_1/H + N_2/K \leqslant d$ then $X^H(I_1) \cap X^K(I_2) = \varnothing$ almost surely. By using a recent result of Luan and Xiao [22], one can extend this result to certain anisotropic Gaussian random fields with stationary increments which satisfy strong local nondeterminism. We leave the details to an interested reader. □

When $\sum_{j=1}^{N_1} 1/H_j + \sum_{j=1}^{N_2} 1/K_j > d$, Corollary 3.2 implies the sample paths $\{X^H(s), s \in I_1\}$ and $\{X^K(t), t \in I_2\}$ intersect with positive probability. It is of interest to study the fractal properties such as the Hausdorff and packing dimensions of the set L_2 of intersection times

$$L_2 = \{(s,t) \in I_1 \times I_2 : X^H(s) = X^K(t)\}$$

and the set of intersection points $X^H(I_1) \cap X^K(I_2)$. While we can apply Theorem 7.1 in Xiao [4] to determine the Hausdorff dimension of the intersection times, we have not been able to determine the Hausdorff dimension of $X^H(I_1) \cap X^K(I_2)$, except in the special case when X^H and X^K are two independent fractional Brownian motions (see Wu and Xiao [36]).

Corollary 3.4 *Assume the conditions of Theorem 3.1 hold and* $\sum_{j=1}^{N_1} 1/H_j + \sum_{j=1}^{N_2} 1/K_j > d$. *If the indices* $H_1, \ldots, H_{N_1}, K_1, \ldots K_{N_2}$ *are reordered as* $0 < H_1' \leqslant H_2' \leqslant \ldots \leqslant H_N' < 1$, *then*

$$\begin{aligned}\dim_{\rm H} L_2 &= \min_{1\leqslant k\leqslant N}\Big\{\sum_{j=1}^{k}\frac{H_k'}{H_j'} + N - k - H_k' d\Big\}\\ &= \sum_{j=1}^{k}\frac{H_k'}{H_j'} + N - k - H_k' d, \qquad \text{if } \sum_{j=1}^{k-1}\frac{1}{H_j'} \leqslant d < \sum_{j=1}^{k}\frac{1}{H_j'}.\end{aligned}$$

Proof Since the set of intersection times is the same as the level set $\{(s,t) \in I : Z(s,t) = 0\}$, and the Gaussian field $Z_0(s,t) = X_0^H(s) - X_0^K(t)$ satisfies Conditions (C1) and (C2) on the interval $I = I_1 \times I_2$ with indices $(H_1, \ldots, H_{N_1}, K_1, \ldots, K_{N_2}) \in (0,1)^N$. The corollary follows from Theorem 7.1 in Xiao [40] immediately. □

If, in addition to conditions (C1) and (C2), the Gaussian fields X^H and X^K satisfy the sectorial local nondeterminism, then the results in Wu and Xiao [37] can be applied to show that X^H and X^K have a jointly continuous intersection local times provided $\sum_{j=1}^{N_1} 1/H_j + \sum_{j=1}^{N_2} 1/K_j > d$. Moreover, it is possible to establish sharp Hölder conditions for the intersection local times which lead to a lower bound for the exact Hausdorff measure of L_2. Nevertheless, the problem for determining the exact Hausdorff measure function for L_2 has not been completely solved.

3.2 Results for Question (iii)

Let $X^H = \{X^H(s), s \in \mathbb{R}^{N_1}\}$ and $X^K = \{X^K(t), t \in \mathbb{R}^{N_2}\}$ be two independent Gaussian random fields with values in $\mathbb{R}^d$. As in Section 3.1, we assume that the associate real-valued random fields X_0^H and X_0^K satisfy Conditions (C1) and (C2) respectively on interval $I_1 \subset \mathbb{R}^{N_1}$ with indices $H = (H_1, \ldots, H_{N_1}) \in (0,1)^{N_1}$ and on $I_2 \subset \mathbb{R}^{N_2}$ with indices $K = (K_1, \ldots, K_{N_2}) \in (0,1)^{N_2}$.

Given Borel sets $E_1 \subseteq I_1$, $E_2 \subseteq I_2$ and a Borel set $F \subseteq \mathbb{R}^d$, we now study the question when F contains intersection points of the sample paths $\{X^H(s), s \in E_1\}$ and $\{X^K(t), t \in E_2\}$. That is, when can we have

$$\mathbb{P}\Big\{X^H(E_1) \cap X^K(E_2) \cap F \neq \varnothing\Big\} > 0? \tag{28}$$

For this purpose, we consider the Gaussian random field $Y = \{Y(s,t), (s,t) \in \mathbb{R}^N\}$ with values in $\mathbb{R}^{2d}$ defined by

$$Y(s,t) = \big(X^H(s), X^K(t)\big), \qquad \forall (s,t) \in \mathbb{R}^N.$$

Then (28) holds if and only if

$$\mathbb{P}\Big\{Y(E_1 \times E_2) \cap \widetilde{F} \neq \varnothing\Big\} > 0, \tag{29}$$

where $\widetilde{F} = \{(x,x) : x \in F\} \subseteq \mathbb{R}^{2d}$.

Note that the coordinate processes of Y are not i.i.d. if $H \neq K$. Hence we need to modify the methods in Section 2 in order to prove hitting probability results for Y.

We define a metric on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^d$ by

$$\overline{\rho}\big((s,t,x),(s',t',x')\big) = \max\Big\{\sum_{\ell=1}^{N_1}|s_\ell - s'_\ell|^{H_\ell} + \sum_{\ell=1}^{N_2}|t_\ell - t'_\ell|^{K_\ell}, \|x-x'\|\Big\}, \tag{30}$$

where $(s,t),(s',t') \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $x, x' \in \mathbb{R}^d$. For any $r > 0$ and $(s,t,x) \in \mathbb{R}^N \times \mathbb{R}^d$, let $B_{\overline{\rho}}((s,t,x),r)$ denote the open ball in the metric space $(\mathbb{R}^N \times \mathbb{R}^d, \overline{\rho})$ centered at (s,t,x) with radius r. Note that $B_{\overline{\rho}}((s,t,x),r)$ is contained in $B_{\rho^H}(s,r) \times B_{\rho^K}(t,r) \times B(x,r)$, where ρ^H is the metric on $\mathbb{R}^{N_1}$ by (25).

For $\beta > 0$ and $G \subseteq \mathbb{R}^N \times \mathbb{R}^d$, the β-dimensional Hausdorff measure of G under the metric $\overline{\rho}$, now denoted by $\mathcal{H}^\beta_{\overline{\rho}}(G)$, is defined in the same way as (7), and the corresponding Hausdorff dimension is denoted by $\dim^{\overline{\rho}}_{\rm H} G$.

We will make use of the following Bessel-Riesz type capacity of order (α_1, α_2) on $\mathbb{R}^N \times \mathbb{R}^d$. Let $\widetilde{\rho}^H$ be the metric on $\mathbb{R}^{N_1+d}$ defined by

$$\widetilde{\rho}^H((s,x),(s',x')) = \max\big\{\rho^H(s,s'),\ \|x-x'\|\big\} \tag{31}$$

and let $\widetilde{\rho}^K((t,x),(t',x'))$ be the metric on $\mathbb{R}^{N_2+d}$ defined similarly.

For any real numbers α_1 and α_2, consider the kernel $\psi_{\alpha_1,\alpha_2} : \mathbb{R}^{N+d} \to \mathbb{R}_+$ defined by

$$\psi_{\alpha_1,\alpha_2}\big((s,t,x),\,(s',t',x')\big) = f_{\alpha_1}\big(\widetilde{\rho}^H((s,x),(s',x'))\big) f_{\alpha_2}\big(\widetilde{\rho}^K((t,x),(t',x'))\big), \tag{32}$$

where the function f_α is defined in (10).

For any Borel set $G \subseteq \mathbb{R}^{N+d}$, the capacity of G of order (α_1,α_2) is defined by

$$\mathcal{C}_{\alpha_1,\alpha_2}(G) = \left[\inf_{\mu\in\mathcal{P}(G)} \int_{\mathbb{R}^{N+d}}\int_{\mathbb{R}^{N+d}} \psi_{\alpha_1,\alpha_2}\big((s,t,x),\,(s',t',x')\big)\mu(ds,dt,dx)\mu(ds',dt',dx')\right]^{-1}, \tag{33}$$

where $\mathcal{P}(G)$ is the family of probability measures carried by G.

The following is the main theorem of this section, which extends the results of Evans [10] and Tongring [32] for Brownian motion in two directions. It allows the time variables s and t to be restricted to any Borel sets E_1 and E_2 (which can be fractal sets) and it is for anisotropic Gaussian random fields.

Theorem 3.5 *Let $X^H = \{X^H(s), s \in \mathbb{R}^{N_1}\}$ and $X^K = \{X^K(t), t \in \mathbb{R}^{N_2}\}$ be two independent Gaussian random fields with values in $\mathbb{R}^d$ as above. Then for any Borel sets $E_1 \subseteq I_1$, $E_2 \subseteq I_2$ and $F \subseteq \mathbb{R}^d$,*

$$c_{11}^{-1}\, \mathcal{C}_{d,d}(E_1 \times E_2 \times F) \leqslant \mathbb{P}\Big\{X^H(E_1) \cap X^K(E_2) \cap F \neq \varnothing\Big\} \leqslant c_{11}\, \mathcal{H}^{2d}_{\overline{\rho}}(E_1 \times E_2 \times F), \tag{34}$$

where $c_{11} \geqslant 1$ is a finite constant which depends only on I_1, I_2, F, H and K.

Proof By (28) and (29), it suffices to show that

$$c_{11}^{-1}\, \mathcal{C}_{d,d}(E_1 \times E_2 \times F) \leqslant \mathbb{P}\Big\{Y(E_1 \times E_2) \cap \widetilde{F} \neq \varnothing\Big\} \leqslant c_{11}\, \mathcal{H}^{2d}_{\overline{\rho}}(E_1 \times E_2 \times F) \tag{35}$$

for some finite constant $c_{11} \geqslant 1$. The proof of (35) is similar to that of Theorem 2.1 and relies on Lemmas 2.3 and 2.4.

For any constant $\gamma > \mathcal{H}^{2d}_{\overline{\rho}}(E_1 \times E_2 \times F)$, there is a sequence of balls $\{B_{\overline{\rho}}\big((s_j, t_j, y_j), r_j\big), j \geqslant 1\}$ in $\mathbb{R}^{N+d}$ with radius r_j small such that

$$E_1 \times E_2 \times F \subseteq \bigcup_{j=1}^{\infty} B_{\overline{\rho}}\big((s_j, t_j, y_j), r_j\big) \quad \text{and} \quad \sum_{j=1}^{\infty} (2r_j)^{2d} \leqslant \gamma. \tag{36}$$

Since $B_{\overline{\rho}}\big((s_j, t_j, y_j), r_j\big)$ can be covered by $B_{\rho^H}(s_j, r_j) \times B_{\rho^K}(t_j, r_j) \times B(y_j, r_j)$, where ρ^H is the metric on $\mathbb{R}^{N_1}$ defined in (25), we have

$$\begin{aligned}\Big\{Y(E_1 \times E_2) \cap \widetilde{F} \neq \varnothing\Big\} \subseteq \bigcup_{j=1}^{\infty} \Big\{ & X^H\big(B_{\rho^H}(s_j, r_j)\big) \cap B(y_j, r_j) \neq \varnothing, \\ & X^K\big(B_{\rho^K}(t_j, r_j)\big) \cap B(y_j, r_j) \neq \varnothing\Big\}.\end{aligned} \tag{37}$$

It follows from the independence of X^H and X^K and Lemma 2.3 that for every $j \geqslant 1$,

$$\mathbb{P}\Big\{X^H\big(B_{\rho^H}(s_j, r_j)\big) \cap B(y_j, r_j) \neq \varnothing,\ X^K\big(B_{\rho^K}(t_j, r_j)\big) \cap B(y_j, r_j) \neq \varnothing\Big\} \leqslant c\, r_j^{2d}. \tag{38}$$

Combining (37), (38) and (36), we obtain that $\mathbb{P}\big\{Y(E_1 \times E_2) \cap \widetilde{F} \neq \varnothing\big\} \leqslant c\gamma$. Since $\gamma > \mathcal{H}^{2d}_{\overline{\rho}}(E_1 \times E_2 \times F)$ is arbitrary, the upper bound in (35) follows.

Next we prove the lower bound in (35). As in the proof of (11), we assume, without loss of generality, that $\mathcal{C}_{d,d}(E_1 \times E_2 \times F) > 0$ and F is compact. Thus $F \subseteq [-M, M]^d$ for some constant $M > 0$.

For any Borel probability measure μ on $E_1 \times E_2 \times F$ such that

$$\begin{aligned}\mathcal{E}_{d,\,d}(\mu) &= \int\!\!\int \frac{\mu(ds, dt, dx)\mu(ds', dt', dx')}{\widetilde{\rho}^H((s, x), (s', x'))^d \widetilde{\rho}^K((t, x), (t', x'))^d} \\ &\leqslant \frac{2}{\mathcal{C}_{d,d}(E_1 \times E_2 \times F)}.\end{aligned} \tag{39}$$

Recall that $\widetilde{\rho}^H((s, x), (s', x')) = \max\{\rho^H(s, s'), \|x - x'\|\}$ and $\widetilde{\rho}^K((t, x), (t', x'))$ is defined similarly.

For all integers $n \geqslant 1$, we consider a family of random measures ν_n on $E_1 \times E_2 \times F$ defined by

$$\begin{aligned}&\int_{E_1 \times E_2 \times F} g(s, t, x)\, \nu_n(ds, dt, dx) \\ &= \int (2\pi n)^d \exp\bigg(-\frac{n}{2}\big(\|X^H(s) - x\|^2 + \|X^K(t) - x\|^2\big)\bigg)\, g(s, t, x)\, \mu(ds, dt, dx) \\ &= \int_{E_1 \times E_2 \times F} \int_{\mathbb{R}^{2d}} \exp\bigg(-\frac{\|\xi\|^2 + \|\eta\|^2}{2n} + i\langle \xi, X^H(s) - x\rangle + i\langle \eta, X^K(t) - x\rangle\bigg) \\ &\qquad\qquad \times g(s, t, x)\, d\xi d\eta\, \mu(ds, dt, dx),\end{aligned} \tag{40}$$

where g is an arbitrary measurable function on $\mathbb{R}^N \times \mathbb{R}^d$. Again we denote the total mass of ν_n by $\|\nu_n\|$ and claim that the following two inequalities hold:

$$\mathbb{E}\big(\|\nu_n\|\big) \geqslant c_{12}, \qquad \mathbb{E}\big(\|\nu_n\|^2\big) \leqslant c_{13}\, \mathcal{E}_{d,\,d}(\mu), \tag{41}$$

where the positive and finite constants c_{12} and c_{13} are independent of μ and n.

By (40) and Fubini's theorem we have

$$\begin{aligned}\mathbb{E}\big(\|\nu_n\|\big) &= \int_{E_1\times E_2\times F} \frac{(2\pi)^d}{(1+\sigma_H^2(s))^{d/2}(1+\sigma_K^2(t))^{d/2}} \\ &\qquad\qquad \times \exp\bigg(-\frac{\|x\|^2}{2\sigma_H^2(s)} - \frac{\|x\|^2}{2\sigma_K^2(t)}\bigg)\,\mu(ds,dt,dx) \\ &\geqslant c_{12} > 0, \end{aligned} \tag{42}$$

where $\sigma_H^2(s) = \mathbb{E}\big(X_0^H(s)^2\big)$ and the constant c_{12} does not depend on μ and n. This gives the first inequality in (41).

Next we prove the second inequality in (41). Let $\Gamma_{H,n} = n^{-1} I_{2d} + \mathrm{Cov}\big(X^H(s), X^H(s')\big)$, $\Gamma_{K,n} = n^{-1} I_{2d} + \mathrm{Cov}\big(X^K(s), X^K(s')\big)$ and $(\xi,\xi')^T$ be the transpose of the row vector (ξ,ξ').

It follows from (40), Fubini's theorem and the independence of X^H and X^K that

$$\begin{aligned}\mathbb{E}\big(\|\nu_n\|^2\big) &= \int_{E_1\times E_2\times F}\int_{E_1\times E_2\times F} \mu(ds,dt,dx)\mu(ds',dt',dx') \\ &\qquad \times \bigg\{\int_{\mathbb{R}^{2d}} e^{-i(\langle\xi,x\rangle - \langle\xi',x'\rangle)} \exp\Big(-\frac{1}{2}(\xi,\,\xi')\,\Gamma_{H,n}\,(\xi,\,\xi')^T\Big)\,d\xi d\xi' \\ &\qquad \times \int_{\mathbb{R}^{2d}} e^{-i(\langle\eta,x\rangle - \langle\eta',x'\rangle)} \exp\Big(-\frac{1}{2}(\eta,\,\eta')\,\Gamma_{K,n}\,(\eta,\,\eta)^T\Big)\,d\eta\,d\eta'\bigg\} \\ &\leqslant c\int_{E_1\times E_2\times F}\int_{E_1\times E_2\times F} \frac{\mu(ds,dt,dx)\mu(ds',dt',dx')}{\widetilde{\rho}^H\big((s,x),(s',x')\big)^d\,\widetilde{\rho}^K\big((t,x),(t',x')\big)^d}. \end{aligned} \tag{43}$$

In the above, the last inequality follows from Lemma 2.4. This verifies the second inequality in (41).

By using (41) and the Paley-Zygmund inequality again, we see that there is a subsequence of $\{\nu_n, n\geqslant 1\}$ that converges weakly to a finite measure ν which is positive with positive probability. Moreover ν is supported on the set $\{(s,t,x)\in E_1\times E_2\times F : Y(s,t) = (x,x)\}$ and also satisfies (41). Hence we have

$$\mathbb{P}\Big\{Y(E_1\times E_2)\cap\widetilde{F}\neq\varnothing\Big\} \geqslant \mathbb{P}\big\{\|\nu\|>0\big\} \geqslant \frac{\big[\mathbb{E}(\|\nu\|)\big]^2}{\mathbb{E}\big[\|\nu\|^2\big]} \geqslant c\,\mathcal{C}_{d,d}(E_1\times E_2\times F). \tag{44}$$

This implies the lower bound in (35). □

If we take $E_1 = I_1$ and $E_2 = I_2$, the following corollary gives a sufficient condition and a necessary condition for F to contain intersection points of X^H and X^K. First we need some notation. Let $Q_H = \sum_{j=1}^{N_1} 1/H_j$, $Q_K = \sum_{j=1}^{N_2} 1/K_j$ and $Q = Q_H + Q_K$. Let $\mathcal{H}^\beta$ still denote β-dimensional Hausdorff measure in the Euclidean metric and, for any $F\subset\mathbb{R}^d$, let $\mathcal{C}_{\alpha_1,\alpha_2}(F)$ denote the capacity of order (α_1,α_2) defined by

$$\mathcal{C}_{\alpha_1,\alpha_2}(F) = \bigg[\inf_{\mu\in\mathcal{P}(F)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_{\alpha_1}\big(\|x-y\|\big)f_{\alpha_2}\big(\|x-y\|\big)\mu(dx)\mu(dy)\bigg]^{-1}. \tag{45}$$

Corollary 3.6 *Let $X^H = \{X^H(s), s \in \mathbb{R}^{N_1}\}$ and $X^K = \{X^K(t), t \in \mathbb{R}^{N_2}\}$ be two independent Gaussian random fields with values in $\mathbb{R}^d$ as in Theorem 3.5. Then for any Borel set $F \subseteq \mathbb{R}^d$,*

$$c_{_{14}}^{-1}\,\mathcal{C}_{d-Q_H,d-Q_K}(F) \leqslant \mathbb{P}\Big\{X^H(I_1) \cap X^K(I_2) \cap F \neq \varnothing\Big\} \leqslant c_{_{14}}\,\mathcal{H}^{2d-Q}(F), \tag{46}$$

where $c_{_{14}} \geqslant 1$ is a finite constant which depends only on I_1, I_2, F, H and K.

Proof By (34), it suffices to verify

$$c_{_{14}}^{-1}\,\mathcal{C}_{d-Q_H,d-Q_K}(F) \leqslant \mathcal{C}_{d,d}(I_1 \times I_2 \times F) \quad \text{and} \quad \mathcal{H}^{2d}_{\bar\rho}(I_1 \times I_2 \times F) \leqslant c_{_{14}}\,\mathcal{H}^{2d-Q}(F).$$

Since this is similar to the proof of Corollary 2.2, we omit the details. □

Remark 3.7 It will be helpful to remark that, in general, $f_{\alpha_1} \cdot f_{\alpha_2} \neq f_{\alpha_1+\alpha_2}$. The conditions in (46) may take various interesting forms. For example, if $d = Q_H = Q_K$, then (46) implies that if F has positive Bessel-Riesz capacity with respect to the kernel

$$\big[f_0(\|x-y\|)\big]^2 = \left[\log\left(\frac{e}{\|x-y\| \wedge 1}\right)\right]^2,$$

then F contains the intersection points of X^H and X^K with positive probability. □

Remark 3.8 In this section we only consider intersections of two independent Gaussian random fields. Similar problems can be raised for self-intersections for a single Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ and are more subtle. It seems that some new methods need to be developed in order to solve these problems. □

4 Examples

Our results in Section 3 are not only applicable to the well-known fractional Brownian motion and the Brownian sheet, but also to anisotropic random fields such as fractional Brownian sheets (cf. Kamont [16], Ayache, et al. [1]) and solutions to the stochastic heat equations driven by space-time white noise (cf. Mueller and Tribe [25], Dalang et al. [5, 6]).

4.1 Nonlinear stochastic heat equations

Let $\dot W = \big(\dot W_1, \ldots, \dot W_d\big)$ be a space-time white noise in $\mathbb{R}^d$. That is, the components $\dot W_1(x,s), \ldots, \dot W_d(x,s)$ of $\dot W(x,s)$ are independent space-time white noises, which are generalized Gaussian processes with covariance given by

$$\mathbb{E}\big[\dot W_i(s,x)\dot W_i(t,y)\big] = \delta(x-y)\delta(s-t), \quad (i = 1, \ldots, d),$$

where $\delta(\cdot)$ is the Dirac delta function. For all $1 \leqslant j \leqslant d$, let $b_j : \mathbb{R}^d \to \mathbb{R}$ be globally Lipschitz and bounded functions, and let $\sigma := (\sigma_{i,j})$ be a deterministic $d \times d$ invertible matrix.

Consider the system of stochastic partial differential equations

$$\frac{\partial u_i}{\partial s}(s,x) = \frac{\partial^2 u_i}{\partial x^2}(s,x) + \sum_{j=1}^{d} \sigma_{i,j}\dot W_j(s,x) + b_i\big(u(s,x)\big) \tag{47}$$

for $1 \leqslant i \leqslant d$, $s \in [0,T]$ and $x \in [0,1]$, with the initial conditions $u(0,x) = 0$ for all $x \in [0,1]$, and the Neumann boundary conditions

$$\frac{\partial u_i}{\partial x}(s,0) = \frac{\partial u_i}{\partial x}(s,1) = 0, \qquad 0 \leqslant s \leqslant T. \tag{48}$$

where $u(s,x) = \big(u_1(s,x),\ldots,u_d(s,x)\big)$. Equation (4.1) can be interpreted rigorously as in Dalang et al. [5].

The linear form of (47) [i.e., $b \equiv 0$ and $\sigma \equiv I_d$] is studied by Mueller and Tribe [25] and Wu and Xiao [34]. More generally, Dalang et al. [5] and Biermé et al. [3] studied hitting probabilities and the Hausdorff dimensions of the range and inverse image sets for the *non-linear* equation (47).

Let $u = \big\{u(s,x), s \in [0,T],\, x \in [0,1]\big\}$ and $v = \big\{v(t,y), t \in [0,T],\, y \in [0,1]\big\}$ be the solutions to the two systems of stochastic heat equations of the form (4.1), driven by two independent space-time white noises $\dot W$ and $\dot{\widetilde{W}}$, respectively. In the following, let $E_1, E_2 \subseteq [0,T]\times[0,1]$ and $F \subseteq \mathbb{R}^d$ be given Borel sets and we apply results in Section 3 to provide necessary conditions and sufficient conditions for $\mathbb{P}\{u(E_1)\cap v(E_2)\cap F \neq \varnothing\} > 0$.

As shown by Proposition 5.1 in Dalang et al. [5], it is sufficient to consider these problems for the solution of equation (4.1) in the following drift-free case [i.e., $b \equiv 0$]:

$$\frac{\partial u}{\partial s}(s,x) = \frac{\partial^2 u}{\partial x^2}(s,x) + \sigma\, \dot W. \tag{49}$$

The solution of (4.3) is the mean zero Gaussian random field $u = \{u(s,x), s\in[0,T], x\in[0,1]\}$ with values in $\mathbb{R}^d$ defined by

$$u(s,x) = \int_0^s \int_0^1 G_{s-r}(x,y)\, \sigma W(dr,dy), \qquad \forall\ s\in[0,T],\ x\in[0,1], \tag{50}$$

where $G_t(x,y)$ is the Green kernel for the heat equation with Neumann boundary conditions (see Walsh [33])

The following is a consequence of Lemmas 4.2 and 4.3 of Dalang, et al. [5] or Lemma 4.1 of Biermé, et al. [3].

Lemma 4.1 *Let $u = \{u(s,x), s\in[0,T], x\in[0,1]\}$ be the solution of (4.3). Then for every $\varepsilon \in (0,T)$, there exist positive and finite constants $c_{15},\ldots,c_{19}$ such that the following hold:*

(i) For all $(s,x),(t,y) \in I = [\varepsilon,T]\times[0,1]$, we have $c_{15} \leqslant \mathbb{E}[u(s,x)^2] \leqslant c_{16}$ and

$$\begin{aligned} c_{17}\left(|s-t|^{1/4} + |x-y|^{1/2}\right)^2 &\leqslant \mathbb{E}\Big[\big(u(s,x)-u(t,y)\big)^2\Big] \\ &\leqslant c_{18}\left(|s-t|^{1/4} + |x-y|^{1/2}\right)^2. \end{aligned} \tag{51}$$

(ii) For all (s,x), $(t,y) \in I$,

$$\mathrm{Var}\big(u(t,y)\big|u(s,x)\big) \geqslant c_{19}\left(|s-t|^{1/4} + |x-y|^{1/2}\right)^2. \tag{52}$$

In other words, Lemma 4.1 states that the Gaussian random field u in (50) satisfies Conditions (C1) and (C2) with $H_1 = 1/4$ and $H_2 = 1/2$.

Hence we can apply Theorems 3.1 and 3.5 to two solutions to SPDE (4.3) driven by two independent space-time white noises $\dot W$ and $\dot{\widetilde{W}}$, respectively. The following theorem complements the previous results in Mueller and Tribe [25], Wu and Xiao [34], Dalang, et al. [5] and Biermé, et al. [3].

Theorem 4.2 *Let $u = \big\{u(s,x), s \in [0,T],\ x \in [0,1]\big\}$ and $v = \big\{v(t,y), t \in [0,T],\ y \in [0,1]\big\}$ be two independent solutions to* (4.1) *and let $F \subseteq \mathbb{R}^d$ be a Borel set. Then the following conclusions hold.*

(i) If $E_i \subseteq [\varepsilon, T] \times [0,1]$ $(i = 1,2)$ are Borel sets, then

$$c^{-1}\, \mathcal{C}_{\rho,d}(E_1 \times E_2) \leqslant \mathbb{P}\Big\{u(E_1) \cap v(E_2) \neq \varnothing\Big\} \leqslant c\, \mathcal{H}^d_\rho(E_1 \times E_2), \tag{53}$$

where $c \geqslant 1$ is a constant and ρ is the metric on $([0,T] \times [0,1])^2$ defined by

$$\rho((s,x;t,y),(s',x';t',y')) = |s-s'|^{1/4} + |x-x'|^{1/2} + |t-t'|^{1/4} + |y-y'|^{1/2}. \tag{54}$$

(ii) If, in addition, $F \subseteq \mathbb{R}^d$ is a Borel set, then

$$c^{-1}\, \mathcal{C}_{d,d}(E_1 \times E_2 \times F) \leqslant \mathbb{P}\Big\{u(E_1) \cap v(E_2) \cap F \neq \varnothing\Big\} \leqslant c\, \mathcal{H}^{2d}_{\overline{\rho}}(E_1 \times E_2 \times F), \tag{55}$$

where $c \geqslant 1$ is a constant and $\overline{\rho}$ is the metric on $([0,T] \times [0,1])^2 \times \mathbb{R}^d$ defined by

$$\overline{\rho}\big((s,x;t,y;w),(s',x';t',y';w)\big) = \max\Big\{\rho\big((s,x;t,y),(s',x';t',y')\big),\ \|w-w'\|\Big\}.$$

Here the metric ρ is defined in (54). In particular, we have

$$c^{-1}\, \mathcal{C}_{d-6,d-6}(F) \leqslant \mathbb{P}\Big\{u([\varepsilon,T] \times [0,1]) \cap v([\varepsilon,T] \times [0,1]) \cap F \neq \varnothing\Big\} \leqslant c\, \mathcal{H}^{2d-12}(F). \tag{56}$$

Proof The conclusions follows directly from Lemma 4.1, Theorems 3.1 and 3.5. □

4.2 Fractional Brownian sheets

In this final section, we consider the intersections of two independent fractional Brownian sheets.

Recall that, for a given vector $\gamma = (\gamma_1, \ldots, \gamma_p) \in (0,1)^p$, a real-valued fractional Brownian sheet $B_0^\gamma = \{B_0^\gamma(t),\ t \in \mathbb{R}^p\}$ with index γ is a centered Gaussian random field with covariance function given by

$$\mathbb{E}[B_0^\gamma(s) B_0^\gamma(t)] = \prod_{\ell=1}^p \frac{1}{2}\big(|s_\ell|^{2\gamma_\ell} + |t_\ell|^{2\gamma_\ell} - |s_\ell - t_\ell|^{2\gamma_\ell}\big), \quad s,\ t \in \mathbb{R}^p. \tag{57}$$

We associate with B_0^γ a Gaussian random field $B^\gamma = \{B^\gamma(t),\ t \in \mathbb{R}^p\}$ with valued in $\mathbb{R}^q$ by

$$B^\gamma(t) = (B_1^\gamma(t), \ldots, B_q^\gamma(t)), \tag{58}$$

where $B_1^\gamma, \ldots, B_q^\gamma$ are independent copies of B_0^γ. We call B^γ the (p,q)-fractional Brownian sheet with Hurst index $\gamma = (\gamma_1, \ldots, \gamma_p)$.

Fractional Brownian sheets were first introduced by Kamont [16] as an extension of the Brownian sheet and fractional Brownian motion. It is an important example of anisotropic Gaussian random fields and their sample path properties have been studied by many authors. We refer to Ayache et al. [1], Mason and Shi [23], Xiao and Zhang [41], Ayache and Xiao [2], Chen [4], Wu and Xiao [35, 37], Xiao [40] and the references therein for further information.

It follows from Ayache and Xiao [2, Lemma 8] and Wu and Xiao [35, Theorem 1] that, for any constants $0 < \varepsilon < T < \infty$, B^γ satisfies Conditions (C1) and (C2) on $I = [\varepsilon, T]^N$ with indices γ. [Wu and Xiao [35] proves that B^γ has the property of sectorial local nondeterminism, which is stronger than (C2).] Hence, we can derive the following theorem for fractional Brownian sheets from the results in Section 3.

Theorem 4.3 *Given two vectors $H = (H_1, \ldots, H_{N_1}) \in (0,1)^{N_1}$ and $K = (K_1, \ldots, K_{N_2}) \in (0,1)^{N_2}$, let $B^H = \{B^H(s),\, s \in \mathbb{R}^{N_1}\}$ and $B^K = \{B^K(t),\, t \in \mathbb{R}^{N_2}\}$ be two independent fractional Brownian sheets with valued in $\mathbb{R}^d$. The following conclusions hold.*

(i) If $E_i \subseteq [\varepsilon, T]^{N_i}$ $(i = 1, 2)$ are Borel sets, then

$$c^{-1}\,\mathcal{C}_{\rho,d}(E_1 \times E_2) \leqslant \mathbb{P}\Big\{B^H(E_1) \cap B^K(E_2) \neq \varnothing\Big\} \leqslant c\,\mathcal{H}^d_\rho(E_1 \times E_2), \tag{59}$$

where $c \geqslant 1$ is a constant and ρ is the metric on $\mathbb{R}^{N_1+N_2}$ defined by (26). In particular, we have

$$\mathbb{P}\Big\{B^H([\varepsilon,T]^{N_1}) \cap B^K([\varepsilon,T]^{N_2}) \neq \varnothing\Big\} \begin{cases} = 0, & \text{if } d > Q; \\ > 0 & \text{if } d < Q. \end{cases}$$

Here $Q = \sum_{j=1}^{N_1} 1/H_j + \sum_{j=1}^{N_2} 1/K_j := Q_H + Q_K$.

(ii) Let $F \subseteq \mathbb{R}^d$ be a compact set. Then

$$c^{-1}\,\mathcal{C}_{d-Q_H,d-Q_K}(F) \leqslant \mathbb{P}\Big\{B^H([\varepsilon,T]^{N_1}) \cap B^K([\varepsilon,T]^{N_2}) \cap F \neq \varnothing\Big\} \leqslant c\,\mathcal{H}^{2d-Q}(F), \tag{60}$$

where $c \geqslant 1$ is a constant.

Theorem 4.3 extends partially the results in Rosen [28], Khoshnevisan [17], Dalang, et al. [7] for the Brownian sheet.

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