Strong Local Nondeterminism and Sample Path Properties of Gaussian Random Fields

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Abstract

Sufficient conditions for a real-valued Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ with stationary increments to be strongly locally nondeterministic are proven. As applications, small ball probability estimates, Hausdorff measure of the sample paths, sharp Hölder conditions and tail probability estimates for the local times of Gaussian random fields are established.

Running head: Strong Local Nondeterminism of Gaussian Random Fields

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1 Introduction and definitions of local nondeterminism

The concept of local nondeterminism (LND, in short) of a Gaussian process was first introduced by Berman (1973) to unify and extend his methods for studying the existence and joint continuity of local times of Gaussian processes. Let $X = \{X(t),\, t \in \mathbb{R}_+\}$ be a separable Gaussian process with mean 0 and let $J \subset \mathbb{R}_+$ be an interval. Assume that $\mathbb{E}[X(t)^2] > 0$ for all $t \in J$ and there exists $\delta > 0$ such that

$$\sigma^2(s,t) = \mathbb{E}\big[(X(s) - X(t))^2\big] > 0 \quad \text{for} \;\; s,\ t \in J \text{ with } \; 0 < |s-t| < \delta.$$

Recall from Berman (1973) that X is called *locally nondeterministic* on J if for every integer $m \geq 2$,

$$\lim_{\varepsilon \to 0} \inf_{t_m - t_1 \leq \varepsilon} V_m > 0, \tag{1.1}$$

where V_m is the relative prediction error:

$$V_m = \frac{\mathrm{Var}\big(X(t_m) - X(t_{m-1}) | X(t_1), \ldots, X(t_{m-1})\big)}{\mathrm{Var}\big(X(t_m) - X(t_{m-1})\big)}$$

and the infimum in (1.1) is taken over all ordered points $t_1 < t_2 < \cdots < t_m$ in J with $t_m - t_1 \leq \varepsilon$.

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This definition of LND was extended by Cuzick (1978) who defined local ϕ-nondeterminism by replacing the variance $\sigma^2(t_m, t_{m-1})$ by $\phi(t_m - t_{m-1})$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an arbitrary function with $\phi(0) = 0$. It follows from Berman (1973, Lemma 2.3) that (1.1) is equivalent to the following property: for every integer $m \geq 2$, there exist positive constants c_m and ε (both may depend on m) such that

$$\mathrm{Var}\bigg(\sum_{k=1}^{m} u_k \left(X(t_k) - X(t_{k-1})\right)\bigg) \geq c_m \sum_{k=1}^{m} u_k^2\, \sigma^2(t_{k-1}, t_k) \tag{1.2}$$

for all ordered points $t_1 < t_2 < \cdots < t_m$ in J with $t_m - t_1 < \varepsilon$ and $u_k \in \mathbb{R}$ $(k = 1, \ldots, m)$. Pitt (1978) used (1.2) to define local nondeterminism of a Gaussian random field $X = \{X(t),\, t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$ by introducing a partial order among $t_1, \ldots, t_m \in \mathbb{R}^N$.

Roughly speaking, (1.2) suggests that the increments of X are asymptotically independent so that many of the results on the local times of Brownian motion can be extended to general Gaussian random fields. For example, Berman (1972, 1973), Pitt (1978) have applied LND to prove the joint continuity and Hölder conditions of the local times of a large class of Gaussian processes. See the comprehensive survey of Geman and Horowitz (1980) and the references therein for further information. Moreover, local nondeterminism has also been applied by Cuzick (1978) to study the moments of the zero crossing number of a stationary Gaussian process; by Rosen (1984) and Berman (1991) to study the existence and regularity of intersection local times; by Kahane (1985) to study the geometric properties of the images and level sets of fractional Brownian motion. Because of its various applications, it has been an interesting question to determine when a Gaussian process is locally nondeterministic. Some sufficient conditions for Gaussian processes to be locally nondeterministic can be found in Berman (1973, 1988, 1991), Cuzick (1978), Pitt (1978).

On the other hand, it is known that the local nondeterminism is not enough for establishing fine regularity properties such as the law of the iterated logarithm and the modulus of continuity for the local times of Gaussian processes. For studying these and many other problems on Gaussian processes, the concept of strong local nondeterminism (SLND) has proven to be more appropriate. See Monrad and Pitt (1987), Csörgő et al. (1995), Monrad and Rootzén (1995), Talagrand (1995, 1998), Xiao (1996, 1997a, b, c), Kasahara et al. (1999), Xiao and Zhang (2002), just to mention a few.

The following definition of the strong local ϕ-nondeterminism was essentially given by Cuzick and DuPreez (1982) for Gaussian processes (i.e., $N = 1$). For Gaussian random fields, Definition 1.1 is more general than the definition of *strong local α-nondeterministism* of Monrad and Pitt (1987).

Definition 1.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued random field with $0 < \mathbb{E}[X(t)^2] < \infty$ for $t \in J$, where $J \subseteq \mathbb{R}^N$ is a hyper-rectangle. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function such that $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$. Then X is said to be* strongly locally ϕ-nondeterministic *(SLϕND) on J if there exist positive constants c_0 and r_0 such that for all $t \in J$ and all $0 < r \leq \min\{|t|,\ r_0\}$,*

$$\mathrm{Var}\big(X(t)|X(s) : s \in J,\ r \leq |s - t| \leq r_0\big) \geq c_0\, \phi(r). \tag{1.3}$$

For a stationary Gaussian process $X = \{X(t), t \in \mathbb{R}\}$, Cuzick and DuPreez (1982) have given a sufficient condition for X to be strongly locally ϕ-nondeterministic in terms of its spectral measure F. More precisely, they have proven that if the absolutely continuous part of $dF(\lambda)$ has the property that

$$\frac{dF(\lambda/r)}{\phi(r)} \ge h(\lambda)d\lambda, \qquad \forall 0 < r \le r_0 \tag{1.4}$$

and

$$\int_0^\infty \frac{\log h(\lambda)}{1+\lambda^2}\, d\lambda > -\infty, \tag{1.5}$$

then X is SLϕND. Their proof uses the ideas from Cuzick (1977) and relies on the special properties of stationary Gaussian processes. Cuzick and DuPreez (1982, p. 811) point out that it appears to be difficult to establish conditions under which general Gaussian processes possess the various forms of strong local nondeterminism. There have only been a few known examples of strongly locally nondeterministic Gaussian random fields, one of them is the fractional Brownian motion which has been under extensive investigations in the last decade due to its applications in various areas such as telecommunication networks, hydrology, finance, and so on. A (standard) fractional Brownian motion $B_\alpha = \{B_\alpha(t), t \in \mathbb{R}^N\}$ of index α $(0 < \alpha < 1)$ is a centered, real-valued Gaussian random field with covariance function

$$\mathbb{E}\big(B_\alpha(t)B_\alpha(s)\big) = \frac{1}{2}\big(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\big).$$

The strong local ϕ-nondeterminism of B_α with $\phi(r) = r^{2\alpha}$ follows from Lemma 7.1 of Pitt (1978), where the self-similarity of B_α has played an essential role. Note that when $N = 1$, the strong local $r^{2\alpha}$-nondeterminism of B_α can also be derived from the above result of Cuzick and DuPreez (1982) by using the Lamperti transformation.

In the studies of Gaussian processes $X = \{X(t), t \in \mathbb{R}\}$, due to the simple order structure of $\mathbb{R}$, it is sometimes enough to assume that X is *one-sided* strongly locally ϕ-nondeterministic, namely, for some constant $c_0 > 0$

$$\mathrm{Var}\big(X(t)|X(s) : s \in J,\ r \le t-s \le r_0\big) \ge c_0\, \phi(r); \tag{1.6}$$

see Cuzick (1978), Berman (1972, 1978), Monrad and Rootzén (1995). When $X = \{X(t), t \in \mathbb{R}\}$ is a Gaussian process with stationary increments, some sufficient conditions in terms of the variance function $\sigma^2(h) = \mathbb{E}\big[\big(X(t+h)-X(t)\big)^2\big]$ for the *one-sided* strong local nondeterminism have been obtained earlier. Marcus (1968) and Berman (1978) have proved that if $\sigma(h) \to 0$ as $h \to 0$ and $\sigma^2(h)$ is concave on $(0, \delta)$ for some $\delta > 0$, then X is *one-sided* strongly locally ϕ-nondeterministic for $\phi(r) = \sigma^2(r)$.

The main objective of this paper is to prove sufficient conditions for a Gaussian random field $X = \{X(t),\, t \in \mathbb{R}^N\}$ with stationary increments to be strongly locally ϕ-nondeterministic. In particular, we show that a spectral condition similar to that of Berman (1988) for the ordinary LND of X actually implies that X is strongly locally ϕ-nondeterministic and, moreover, $\phi(r)$ is comparable to the variance function $\sigma^2(h)$ with $|h| = r$; see Theorem 2.5 for details.

Our results on SLϕND have many applications. In Section 3, we apply them to study the sample path properties of Gaussian processes with stationary increments. In particular, we extend the small ball probability estimates of Monrad and Rootzén (1995), Shao and Wang (1995) and Stoltz (1996), the results on the exact Hausdorff measure of Talagrand (1995) and

Xiao (1996, 1997a, b), the local and uniform Hölder conditions and tail probability of the local times of Xiao (1997a) and Kasahara et al. (1999), to more general Gaussian random fields.

We should mention that, in recent years, several authors have applied general Gaussian processes with stationary increments as stochastic models in telecommunications, turbulence, image processing and finance and so on. See, for example, Addie et al. (1999), Anh et al. (1999), Bonami and Estrade (2003), Mannersalo and Norros (2002), Cheridito (2004), Mueller and Tribe (2002). These applications have raised many interesting questions about Gaussian processes with stationary increments. I hope that an appropriate form of strong local nondeterminism and the results in this paper will be useful for studying these questions.

Throughout the rest of this paper, unspecified positive and finite constants will be denoted by K which may have different values from line to line. Specific constants in Section j will be denoted by $K_{j,1}, K_{j,2}, \ldots$. For two non-negative functions f and g on $\mathbb{R}^N$, we denote $f \asymp g$ if there exists a finite constant $K \geq 1$ such that $K^{-1}f(x) \leq g(x) \leq K\, f(x)$ for all x in some neighborhood of 0 or infinity. This will be clear from the context.

2 Spectral conditions for strong local nondeterminism

Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with $X(0) = 0$. We assume that X has stationary increments and continuous covariance function $R(s,t) = \mathbb{E}\big[X(s)X(t)\big]$. According to Yaglom (1957) [see also Dudley (1973)], $R(s,t)$ can be represented as

$$R(s,t) = \int_{\mathbb{R}^N} (e^{i\langle s,\lambda\rangle} - 1)(e^{-i\langle t,\lambda\rangle} - 1)\Delta(d\lambda) + \langle s, Qt\rangle, \tag{2.1}$$

where $\langle x,y\rangle$ is the ordinary scalar product in $\mathbb{R}^N$, Q is an $N\times N$ non-negative definite matrix and $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\mathbb{R}^N\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2}\ \Delta(d\lambda) < \infty. \tag{2.2}$$

The measure Δ is called the *spectral measure* of X.

It follows from (2.1) that X has the following stochastic integral representation:

$$X(t) = \int_{\mathbb{R}^N} (e^{i\langle t,\lambda\rangle} - 1) W(d\lambda) + \langle \mathbf{Y}, t\rangle, \tag{2.3}$$

where $\mathbf{Y}$ is an N-dimensional Gaussian random vector with mean 0 and $W(d\lambda)$ is a centered complex-valued Gaussian random measure which is independent of $\mathbf{Y}$ and satisfies

$$\mathbb{E}\Big(W(A)\overline{W(B)}\Big) = \Delta(A\cap B) \quad \text{and} \quad W(-A) = \overline{W(A)}$$

for all Borel sets $A,\ B \subseteq \mathbb{R}^N$. From now on, we will assume $\mathbf{Y} = 0$. This is equivalent to assuming $Q = 0$ in (2.1). Consequently, we have

$$\sigma^2(h) = \mathbb{E}\big[\big(X(t+h) - X(t)\big)^2\big] = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle h,\lambda\rangle\big)\ \Delta(d\lambda). \tag{2.4}$$

If the function $\sigma^2(h)$ only depends on $|h|$, then X is called an *isotropic* random field. It is important to note that $\sigma^2(h)$ is a negative definite function and can be viewed as the characteristic exponent of a symmetric infinitely divisible distribution; see Berg and Forst (1975) for more information on negative definite functions.

Our main results of this section are Theorems 2.1 and 2.5. Their proofs rely on the ideas from Kahane (1985), Pitt (1975, 1978) and Berman (1988, 1991).

Theorem 2.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a mean zero, real-valued Gaussian random field with stationary increments and $X(0) = 0$, and let f be the density function of the absolutely continuous part Δ_c of the spectral measure Δ of X. Assume that there exist two locally bounded functions $\phi(r) : \mathbb{R}_+ \to \mathbb{R}_+$ and $q(\lambda) : \mathbb{R}^N \to \mathbb{R}_+$ satisfying the following conditions: $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$,*

$$\frac{f(\lambda/r)}{\phi(r)} \geq \frac{r^N}{q(\lambda)}, \qquad \forall\, r \in (0,1] \ \ \text{and} \ \ \lambda \in \mathbb{R}^N \tag{2.5}$$

and there exists a positive and finite constant η such that

$$q(\lambda) \leq |\lambda|^\eta, \qquad \forall\ \lambda \in \mathbb{R}^N \ \text{with} \ |\lambda| \ \text{large enough.} \tag{2.6}$$

Then for every $T > 0$, there exists a positive constant $K_{2,1}$ such that for all $t \in [-T, T]^N \backslash \{0\}$ and all $0 < r \leq \min\{1, |t|\}$,

$$\mathrm{Var}\big(X(t) | X(s) :\ s \in I,\ |s - t| \geq r\big) \geq K_{2,1}\, \phi(r). \tag{2.7}$$

In particular, X is strongly locally ϕ-nondeterministic on the hypercube $[-T, T]^N$.

To prove Theorem 2.1, we need the following lemma which implies that SLND of X is determined by the behavior of the spectral measure Δ at infinity.

Lemma 2.2 *Assume the density function f of Δ_c satisfies the conditions (2.5) and (2.6). Then for any fixed constants $T > 0$ and $K_{2,2} > 0$, there exists a positive and finite constant $K_{2,3}$ such that for all functions g of the form*

$$g(\lambda) = \sum_{k=1}^n a_k \big(e^{i\langle s^k, \lambda\rangle} - 1\big), \tag{2.8}$$

where $a_k \in \mathbb{R}$ and $s^k \in [-T, T]^N$, we have

$$\big|g(\lambda)\big| \leq K_{2,3}\, |\lambda| \left(\int_{\mathbb{R}^N} |g(\xi)|^2\, f(\xi)\, d\xi \right)^{1/2} \quad \text{for all} \ \ |\lambda| < K_{2,2}. \tag{2.9}$$

Proof It follows from (2.5) and (2.6) that there exists a positive constant K such that

$$f(\lambda) \geq \frac{K}{|\lambda|^\eta} \quad \text{for all } \lambda \in \mathbb{R}^N \ \text{ with } |\lambda| \ \text{ large.}$$

Hence Proposition 5 of Pitt (1975) implies that for every constant $T > 0$, the measure Δ_c is regular at $[-T, T]^N$. [Pitt (1975, p.304) gives the definition of regularity for finite measures only, an extension to any σ-finite measure is immediate]. Let $\mathcal{G}$ be the collection of the functions

$g(z)$ defined by (2.8) with $a_k \in \mathbb{R}$, $s^k \in [-T,T]^N$ and $z \in \mathbb{C}^N$. Since each $g \in \mathcal{G}$ is an entire function, it follows from Proposition 1 of Pitt (1975) [see also Pitt (1978, p.326)] that

$$K_{2,3} = \sup_{z\in U(0,K_{2,2})} \left\{ \sup_{g\in\mathcal{G}} \left\{ |g(z)| : \int_{\mathbb{R}^N} |g(\lambda)|^2 f(\lambda)\, d\lambda \le 1 \right\} \right\} < \infty,$$

where $U(0,K_{2,2}) = \{z \in \mathbb{C}^N : |z| < K_{2,2}\}$ is the open ball of radius $K_{2,2}$ in $\mathbb{C}^N$. Since $g(0)=0$ and g is analytic in $U(0,K_{2,2})$, Schwartz's lemma implies

$$\big|g(z)\big| \le K_{2,3}\, |z| \Big(\int_{\mathbb{R}^N} |g(\xi)|^2\, f(\xi) d\xi \Big)^{1/2} \qquad \text{for all } \ z \in U(0,K_{2,2}).$$

This proves (2.9). □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1 Working in the Hilbert space setting, the conditional variance in (2.7) is the square of the $L^2(\mathbb{P})$-distance of $X(t)$ from the subspace generated by $\{X(s) : s \in I,\, |s-t| \ge r\}$. Hence it is sufficient to show that there exists a constant $K_{2,1}$ such that for every $t \in [-T,\, T]^N \backslash \{0\}$, $0 < r \le \min\{1, |t|\}$, the inequality

$$\mathbb{E}\Big(X(t) - \sum_{k=1}^n a_k X(s^k) \Big)^2 \ge K_{2,1}\, \phi(r) \tag{2.10}$$

holds for all integers $n \ge 1$, all $a_k \in \mathbb{R}$ and $s^k \in [-T,\, T]^N$ satisfying $|s^k - t| \ge r$, $(k = 1, 2, \ldots, n)$.

It follows from (2.1) or (2.3) that

$$\begin{aligned} \mathbb{E}\Big(X(t) - \sum_{k=1}^n a_k X(s^k) \Big)^2 &= \int_{\mathbb{R}^N} \Big| e^{i\langle t,\lambda\rangle} - 1 - \sum_{k=1}^n a_k \big(e^{i\langle s^k,\lambda\rangle} - 1\big) \Big|^2\, \Delta(d\lambda) \\ &\ge \int_{\mathbb{R}^N} \Big| e^{i\langle t,\lambda\rangle} - \sum_{k=0}^n a_k\, e^{i\langle s^k,\lambda\rangle} \Big|^2\, f(\lambda) d\lambda, \end{aligned} \tag{2.11}$$

where $a_0 = 1 - \sum_{k=1}^n a_k$ and $s_0 = 0$. Now we choose a bump function $\delta(\cdot) \in C^\infty(\mathbb{R}^N)$ with values in $[0,1]$ such that $\delta(0) = 1$ and it vanishes outside the open unit ball. Let $\widehat{\delta}$ be the Fourier transform of δ. It is known that $\widehat{\delta}(\lambda)$ is also in $C^\infty(\mathbb{R}^N)$ and decays rapidly as $\lambda \to \infty$. Let $\delta_r(t) = r^{-N}\delta(t/r)$, then the Fourier inversion formula gives

$$\delta_r(t) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{-i\langle t,\lambda\rangle}\, \widehat{\delta}(r\lambda)\, d\lambda\,.$$

Since $\min\{|s^k - t|,\, 0 \le k \le n\} \ge r$ we have $\delta_r(t - s^k) = 0$ for all $k = 0, 1, \cdots, n$. Hence

$$\begin{aligned} &\int_{\mathbb{R}^N} \Big(e^{i\langle t,\lambda\rangle} - \sum_{k=0}^n a_k\, e^{i\langle s^k,\lambda\rangle} \Big)\, e^{-i\langle t,\lambda\rangle}\, \widehat{\delta}(r\lambda)\, d\lambda \\ &= (2\pi)^N \Big(\delta_r(0) - \sum_{k=0}^n a_k\, \delta_r(t - s^k) \Big) \\ &= (2\pi)^N\, r^{-N}. \end{aligned} \tag{2.12}$$

Now we will make use of the conditions (2.5) and (2.6). We choose the constant $K_{2,2}$ in Lemma 2.2 such that (2.6) holds for all $|\lambda| \ge K_{2,2}$ and split the integral in (2.12) over $\{\lambda : |\lambda| < K_{2,2}\}$ and $\{\lambda : |\lambda| \ge K_{2,2}\}$. Denote the two integrals by I_1 and I_2, respectively. It follows from Lemma 2.2 that

$$
\begin{aligned}
I_1 &\le \int_{|\lambda|<K_{2,2}} \left| e^{i\langle t,\lambda\rangle} - \sum_{k=0}^{n} a_k\, e^{i\langle s^k,\lambda\rangle} \right| |\widehat{\delta}(r\lambda)|\, d\lambda \\
&\le K_{2,3} \left[\int_{\mathbb{R}^N} \left| e^{i\langle t,\lambda\rangle} - \sum_{k=0}^{n} a_k\, e^{i\langle s^k,\lambda\rangle} \right|^2 f(\lambda)\, d\lambda \right]^{1/2} \cdot \int_{|\lambda|<K_{2,2}} |\lambda|\, |\widehat{\delta}(r\lambda)|\, d\lambda \qquad (2.13) \\
&\le K_{2,4} \left[\mathbb{E}\Big(X(t) - \sum_{k=1}^{n} a_k X(s^k) \Big)^2 \right]^{1/2},
\end{aligned}
$$

where the last inequality follows from (2.11) and the boundedness of $\widehat{\delta}$, and where $K_{2,4} > 0$ is a finite constant depending on T and $K_{2,2}$.

On the other hand, by the Cauchy-Schwarz inequality and (2.11), we have

$$
\begin{aligned}
I_2^2 &\le \int_{|\lambda|\ge K_{2,2}} \left| e^{i\langle t,\lambda\rangle} - \sum_{k=0}^{n} a_k\, e^{i\langle s^k,\lambda\rangle} \right|^2 f(\lambda)\, d\lambda \cdot \int_{|\lambda|\ge K_{2,2}} \frac{1}{f(\lambda)} \left| \widehat{\delta}(r\lambda) \right|^2 d\lambda \\
&\le \mathbb{E} \left(X(t) - \sum_{k=1}^{n} a_k X(s^k) \right)^2 \cdot r^{-N} \int_{|\lambda|\ge K_{2,2} r} \frac{1}{f(\lambda/r)} \left| \widehat{\delta}(\lambda) \right|^2 d\lambda.
\end{aligned}
\qquad (2.14)
$$

By using (2.5) and (2.6), we deduce

$$
\begin{aligned}
\int_{|\lambda|\ge K_{2,2} r} \frac{1}{f(\lambda/r)} \left| \widehat{\delta}(\lambda) \right|^2 d\lambda &\le \phi(r)^{-1}\, r^{-N} \int_{|\lambda|\ge K_{2,2} r} q(\lambda) \left| \widehat{\delta}(\lambda) \right|^2 d\lambda \\
&\le K\, \phi(r)^{-1}\, r^{-N} \left\{ \int_{|\lambda|<K_{2,2}} q(\lambda) \left| \widehat{\delta}(\lambda) \right|^2 d\lambda \right. \\
&\qquad \left. + \int_{|\lambda|\ge K_{2,2}} |\lambda|^{\eta} \left| \widehat{\delta}(\lambda) \right|^2 d\lambda \right\} \\
&= K\, \phi(r)^{-1}\, r^{-N}.
\end{aligned}
\qquad (2.15)
$$

Combining (2.14) and (2.15) yields

$$
I_2^2 \le K\, \phi(r)^{-1}\, r^{-2N}\, \mathbb{E}\Big(X(t) - \sum_{k=1}^{n} a_k X(s^k) \Big)^2. \qquad (2.16)
$$

Finally, we square both sides of (2.12) and use (2.13) and (2.16) to obtain

$$
(2\pi)^{2N}\, r^{-2N} \le K_{2,5}\, \phi(r)^{-1}\, r^{-2N}\, \mathbb{E}\Big(X(t) - \sum_{k=1}^{n} a_k X(s^k) \Big)^2.
$$

This implies (2.10) and hence the theorem is proven. □

In order to apply Theorem 2.1 to investigate the sample path properties of the Gaussian random field X, we need to study the relationship between $\phi(|h|)$ and the function $\sigma^2(h)$. In the following, we show that under a condition analogous to that of Berman (1988, 1991), there is a non-decreasing function ϕ such that X is SLϕND and the functions $\phi(|h|)$ and $\sigma^2(h)$ are comparable. More precisely, we assume that the spectral measure Δ is absolutely continuous and its density function $f(\lambda)$ satisfies the following condition [when $N = 1$, this is due to Berman (1988)]:

$$0 < \underline{\alpha} = \frac{1}{2} \liminf_{\lambda \to \infty} \frac{\beta_N \, |\lambda|^N f(\lambda)}{\Delta\{\xi : |\xi| \ge |\lambda|\}} \le \frac{1}{2} \limsup_{\lambda \to \infty} \frac{\beta_N \, |\lambda|^N f(\lambda)}{\Delta\{\xi : |\xi| \ge |\lambda|\}} = \overline{\alpha} < 1, \tag{2.17}$$

where $\beta_1 = 2$ and for $N \ge 2$, $\beta_N = \mu(S^{N-1})$ is the area [i.e., the $(N-1)$-dimensional Lebesgue measure) of S^{N-1}. At the end of this section, we will give several examples of Gaussian random fields satisfying condition (2.17).

In the rest of this section, we define $\phi(r) = \Delta\{\xi : |\xi| \ge r^{-1}\}$ and $\phi(0) = 0$. Then the function ϕ is non-decreasing and continuous on $[0, \infty)$. The following lemma lists some properties of ϕ which will be useful later.

Lemma 2.3 *Assume the condition (2.17) holds. Then for any $\varepsilon \in (0, 2\min\{\underline{\alpha}, 1 - \overline{\alpha}\})$, there exists a constant $r_0 > 0$ such that for all $0 < x \le y \le r_0$,*

$$\left(\frac{x}{y}\right)^{2\overline{\alpha}+\varepsilon} \le \frac{\phi(x)}{\phi(y)} \le \left(\frac{x}{y}\right)^{2\underline{\alpha}-\varepsilon}. \tag{2.18}$$

Consequently, we have

(i). $lim_{r\to 0}\phi(r)/r^2 = \infty$.

(ii). The function ϕ has the following doubling property: there exists a constant $K_{2,6} > 0$ such that for all $0 < r < r_0/2$,

$$\phi(2r) \le K_{2,6}\, \phi(r). \tag{2.19}$$

Proof For $N = 1$, (2.18) was proved by Berman (1988). Extension to $N > 1$ is easy and a proof is included for completeness. Denote $G(r) = \Delta\{\xi : |\xi| \ge r\}$. Then we can write

$$G(r) = \int_r^\infty \rho^{N-1} \int_{S^{N-1}} f(\rho\,\theta)\mu(d\theta)\, d\rho, \tag{2.20}$$

where μ is the surface measure on the unit sphere S^{N-1}. It follows that

$$\frac{d}{dr}\big[\log G(r)\big] = -\frac{r^{N-1} \int_{S^{N-1}} f(r\theta)\mu(d\theta)}{G(r)}.$$

Thus we derive the identity

$$\frac{G(x)}{G(y)} = \exp\left(\int_x^y \frac{r^N \int_{S^{N-1}} f(r\theta)\mu(d\theta)}{G(r)} \frac{dr}{r}\right) \qquad \text{for all } \; x, y > 0. \tag{2.21}$$

Note that the condition (2.17) and Fatou's lemma imply that

$$0 < \underline{\alpha} \le \frac{1}{2} \liminf_{r\to\infty} \frac{r^N \int_{S^{N-1}} f(r\theta)\mu(d\theta)}{G(r)} \le \frac{1}{2} \limsup_{r\to\infty} \frac{r^N \int_{S^{N-1}} f(r\theta)\mu(d\theta)}{G(r)} \le \overline{\alpha} < 1.$$

Hence for any $\varepsilon \in (0, 2\min\{\underline{\alpha},\, 1-\overline{\alpha}\})$, there exists $r_0 > 0$ such that for all $r \ge r_0^{-1}$, we have

$$2\underline{\alpha} - \varepsilon < \frac{r^N \int_{S^{N-1}} f(r\theta)\mu(d\theta)}{G(r)} < 2\overline{\alpha} + \varepsilon. \tag{2.22}$$

Therefore, (2.18) follows from (2.21) and (2.22). □

Remark 2.4 The equation (2.18) shows that, under the assumption that the spectral measure Δ has a density $f(\lambda)$, Condition (2.17) is more general than assuming ϕ is regularly varying at 0. Using the terminology of Bingham et al. (1987, pp.65-67), (2.18) implies that ϕ is extended regularly varying at 0 with upper and lower Karamata indices $2\overline{\alpha}$ and $2\underline{\alpha}$, respectively. Under (2.17), a necessary and sufficient condition for $\phi(r)$ to be regularly varying at 0 of index 2α is that the limit

$$\alpha = \frac{1}{2} \lim_{r\to\infty} \frac{r^N \int_{S^{N-1}} f(r\theta)\mu(d\theta)}{\Delta\{\xi : |\xi| \ge r\}}$$

exists; this follows from Theorem 2.1.1 in Bingham et al. (1987) and (2.22). □

The following theorem shows that the assumption (2.17) implies that X is SLϕND and $\phi(r)$ is comparable with $\sigma^2(h)$ with $|h| = r$ near 0. In Section 3, we will show that it is often more convenient to use the function ϕ to characterize the probabilistic and geometric properties of X.

Theorem 2.5 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a mean zero, real-valued Gaussian random field with stationary increments and $X(0) = 0$. Assume that the spectral measure Δ of X has a density function f that satisfies (2.17). Then*

$$0 < \liminf_{h\to 0} \frac{\sigma^2(h)}{\phi(|h|)} \le \limsup_{h\to 0} \frac{\sigma^2(h)}{\phi(|h|)} < \infty. \tag{2.23}$$

Moreover, for every $T > 0$, X is strongly locally ϕ-nondeterministic on the hypercube $[-T, T]^N$.

Proof The proof of (2.23) is based on the proof of Theorem 3.1 of Berman (1991) which deals with the case of $N = 1$. Let $T, \tau > 0$ be any constants with $T\tau < 1$. By (2.4), we can write $\sigma^2(h)$ as

$$\begin{aligned}\sigma^2(h) = 2\int_{|\lambda|\le T} \big(1-\cos\langle h,\lambda\rangle\big) f(\lambda)d\lambda + 2\int_{T<|\lambda|\le 1/\tau} \big(1-\cos\langle h,\lambda\rangle\big) f(\lambda)d\lambda \\ + 2\int_{|\lambda|>1/\tau} \big(1-\cos\langle h,\lambda\rangle\big) f(\lambda)d\lambda \\ := 2(J_1 + J_2 + J_3).\end{aligned} \tag{2.24}$$

First we prove the left inequality in (2.23). Let $0 < \varepsilon < 2\min\big\{\underline{\alpha}, 1-\overline{\alpha}\big\}$ be fixed. Condition (2.17) implies the existence of a $\tau_0 \in (0, r_0)$ [r_0 is given in Lemma 2.3] such that

$$2\underline{\alpha} - \varepsilon \le \frac{\beta_N\, |\lambda|^N f(\lambda)}{\phi(1/|\lambda|)} \le 2\overline{\alpha} + \varepsilon \quad \text{for all} \quad \lambda \in \mathbb{R}^N \text{ with } |\lambda| \ge 1/\tau_0. \tag{2.25}$$

It follows from (2.25) and Lemma 2.3 that for $\tau = |h| < \tau_0$ in (2.24),

$$\begin{aligned}
\frac{J_3}{\phi(|h|)} &\ge \frac{2\underline{\alpha} - \varepsilon}{\beta_N} \int_{|\lambda|>1/|h|} \big(1 - \cos\langle h, \lambda\rangle\big)\, \frac{\phi(1/|\lambda|)}{\phi(|h|)}\, \frac{d\lambda}{|\lambda|^N} \\
&\ge \frac{2\underline{\alpha} - \varepsilon}{\beta_N} \int_{|\lambda|>1/|h|} \big(1 - \cos\langle h, \lambda\rangle\big)\, \frac{1}{(|\lambda||h|)^{2\overline{\alpha}+\varepsilon}}\, \frac{d\lambda}{|\lambda|^N} \\
&= \frac{2\underline{\alpha} - \varepsilon}{\beta_N} \int_{|\xi|>1} \Big(1 - \cos\langle \frac{h}{|h|}, \xi\rangle\Big)\, \frac{d\xi}{|\xi|^{N+2\overline{\alpha}+\varepsilon}} \\
&\ge K_{2,7},
\end{aligned} \tag{2.26}$$

where $K_{2,7}$ is a positive constant. In the above, the equality follows from a change of variable and the last inequality follows from Lemma 3.3 in Xiao (2003). It is clear that (2.24) and (2.26) imply the left inequality in (2.23).

In order to prove the right inequality in (2.23), we estimate J_1, J_2 and J_3 separately. Since $1 - \cos\langle h, \lambda\rangle \le |h|^2|\lambda|^2$, we have

$$\begin{aligned}
\frac{J_1}{\phi(|h|)} &\le \int_{|\lambda|\le T} |h|^2|\lambda|^2\, \frac{f(\lambda)}{\phi(|h|)}\, d\lambda \\
&\le K\, \frac{|h|^2}{\phi(|h|)} \to 0 \qquad\quad \text{as } h \to 0,
\end{aligned} \tag{2.27}$$

by (2.2) and Lemma 2.3. Next, suppose we have chosen the constant $T > 1/\tau_0$ so that (2.25) holds for all $\lambda \in \mathbb{R}^N$ with $|\lambda| > T$. Thus by taking $\tau = |h| < \tau_0$ in (2.24), we derive

$$\begin{aligned}
\frac{J_2}{\phi(|h|)} &\le \frac{2\overline{\alpha} + \varepsilon}{\beta_N} \int_{T<|\lambda|\le 1/|h|} \big(1 - \cos\langle h, \lambda\rangle\big)\, \frac{\phi(1/|\lambda|)}{\phi(|h|)}\, \frac{d\lambda}{|\lambda|^N} \\
&\le \frac{2\overline{\alpha} + \varepsilon}{\beta_N} \int_{T<|\lambda|\le 1/|h|} \big(1 - \cos\langle h, \lambda\rangle\big)\, \frac{1}{\big(|h||\lambda|\big)^{2\overline{\alpha}+\varepsilon}}\, \frac{d\lambda}{|\lambda|^N} \\
&\le \frac{2\overline{\alpha} + \varepsilon}{\beta_N} \int_{|\xi|\le 1} \frac{d\xi}{|\xi|^{N-2(1-\overline{\alpha})+\varepsilon}} < \infty.
\end{aligned} \tag{2.28}$$

Similar to (2.28), we use (2.25), Lemma 2.3 and the inequality $1 - \cos\langle h, \lambda\rangle \le 2$ to deduce

$$\begin{aligned}
\frac{J_3}{\phi(|h|)} &\le \frac{2\overline{\alpha} + \varepsilon}{\beta_N} \int_{|\lambda|\ge 1/|h|} \big(1 - \cos\langle h, \lambda\rangle\big)\, \frac{\phi(1/|\lambda|)}{\phi(|h|)}\, \frac{d\lambda}{|\lambda|^N} \\
&\le \frac{2\overline{\alpha} + \varepsilon}{\beta_N} \int_{|\lambda|\ge 1/|h|} \big(1 - \cos\langle h, \lambda\rangle\big)\, \frac{1}{\big(|h||\lambda|\big)^{2\underline{\alpha}-\varepsilon}}\, \frac{d\lambda}{|\lambda|^N} \\
&\le \frac{2(2\overline{\alpha} + \varepsilon)}{\beta_N} \int_{|\xi|\ge 1} \frac{d\xi}{|\xi|^{N+2\underline{\alpha}-\varepsilon}} < \infty.
\end{aligned} \tag{2.29}$$

Therefore the right inequality in (2.23) follows from (2.27), (2.28) and (2.29). This finishes the proof of (2.23).

Finally, note that Condition (2.17), together with Lemma 2.3, implies that (2.5) and (2.6) hold with $q(r) = K_{2,8}\, r^{N+2\overline{\alpha}+\varepsilon}$. Therefore, for any $T > 0$, the strong local ϕ-nondeterminism of X on $I = [-T, T]^N$ follows from Theorem 2.1. □

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a stationary random field with mean 0, variance 1 and spectral measure Δ. Then X can be represented as

$$X(t) = \int_{\mathbb{R}^N} e^{i\langle t,\lambda\rangle} W(d\lambda), \quad \forall t \in \mathbb{R}^N. \tag{2.30}$$

Clearly, Theorems 2.1 and 2.5 are applicable to the Gaussian random field $Y = \{Y(t), t \in \mathbb{R}^N\}$ defined by $Y(t) = X(t) - X(0)$. Furthermore, we remark that the proofs of Theorems 2.1 and 2.5 remain effective for X itself. Either way we have the following partial extension of the result of Cuzick and DuPreez (1982) mentioned in the Introduction to $N > 1$. It is not known to me whether (2.6) can be replaced by the weaker condition (1.5).

Corollary 2.6 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a stationary Gaussian random field with mean 0 and variance 1.*

(i). If the spectral measure Δ of X has an absolutely continuous part with density f satisfying (2.5) and (2.6), then for every $T > 0$, X is strongly locally ϕ-nondeterministic on the hypercube $[-T, T]^N$.

(ii). If the spectral density of X satisfies (2.17), then (2.23) holds and X is SLϕND on the hypercube $[-T, T]^N$.

We end this section with some examples of Gaussian random fields whose SLND can be determined.

Example 2.7 Let $B_\alpha = \{B_\alpha(t),\, t \in \mathbb{R}^N\}$ be an N-parameter fractional Brownian motion in $\mathbb{R}$ with Hurst index $\alpha \in (0, 1)$, then its spectral density is given by

$$f_\alpha(\lambda) = c(\alpha, N) \frac{1}{|\lambda|^{2\alpha+N}},$$

where $c(\alpha, N) > 0$ is a normalizing constant such that $\sigma^2(h) = |h|^{2\alpha}$; see e.g., Kahane (1985). Clearly, the condition (2.17) holds with $\underline{\alpha} = \overline{\alpha} = \alpha$. As mentioned earlier, the strong local nondeterminism of B_α was first proved by Pitt (1978). □

Example 2.8 Consider the mean zero Gaussian random field $X = \{X(t),\, t \in \mathbb{R}^N\}$ with stationary increments and spectral density

$$f_{\gamma,\beta}(\lambda) = \frac{c(\gamma, \beta, N)}{|\lambda|^{2\gamma}(1 + |\lambda|^2)^\beta},$$

where γ and β are constants satisfying

$$\beta + \gamma > \frac{N}{2}, \quad 0 < \gamma < 1 + \frac{N}{2}$$

and $c(\gamma, \beta, N) > 0$ is a normalizing constant. Since the spectral density $f_{\gamma,\beta}$ involves both the Fourier transforms of the Riesz kernel and the Bessel kernel, Anh et al. (1999) call the corresponding Gaussian random field the fractional Riesz-Bessel motion with indices β and γ; and they have shown that these Gaussian random fields can be used for modelling simultaneously long range dependence and intermittency.

It is easy to check that Condition (2.17) is satisfied with $\underline{\alpha} = \overline{\alpha} = \gamma + \beta - \frac{N}{2}$. Moreover, since the spectral density $f_{\gamma,\beta}(x)$ is regularly varying at infinity of order $2(\beta + \gamma) > N$, by a result of Pitman (1968) we know that, if $\gamma + \beta - \frac{N}{2} < 1$, then $\sigma(h)$ is regularly varying at 0 of order $\gamma + \beta - N/2$ and

$$\sigma(h) \sim |h|^{\gamma+\beta-N/2} \qquad \text{as } \ h \to 0.$$

Theorem 2.5 implies that X is SLND with respect to $\sigma^2(h)$. Hence, many sample path properties of the d-dimensional fractional Riesz-Bessel motion X with indices β and γ can be can be derived from the results in Section 3. □

Example 2.9 Let $0 < \alpha < 1$ and $0 < c_1 < c_2$ be constants such that $(\alpha c_2)/c_1 < 1$. For any increasing sequence $\{b_n, \, n \geq 0\}$ of real numbers such that $b_0 = 0$ and $b_n \to \infty$, define the function f on $\mathbb{R}^N$ by

$$f(\lambda) = \begin{cases} c_1 \, |\lambda|^{-(2\alpha+N)} & \text{if } \ |\lambda| \in (b_{2k}, b_{2k+1}], \\ c_2 \, |\lambda|^{-(2\alpha+N)} & \text{if } \ |\lambda| \in (b_{2k+1}, b_{2k+2}]. \end{cases} \tag{2.31}$$

Some elementary calculation shows that, when $\lim_{n\to\infty} b_{n+1}/b_n = \infty$, Condition (2.17) is satisfied with $\underline{\alpha} = (\alpha c_1)/c_2 < \overline{\alpha} = (\alpha c_2)/c_1$. Note that in this case,

$$\frac{c_1}{c(\alpha, N)} \, |h|^{2\alpha} \leq \sigma^2(h) \leq \frac{c_2}{c(\alpha, N)} \, |h|^{2\alpha}, \qquad \forall \ h \in \mathbb{R}^N,$$

where $c(\alpha, N)$ is the constant in Example 2.7, and

$$\frac{c_1 \, \beta_N}{2\alpha} \, r^{2\alpha} \leq \phi(r) \leq \frac{c_2 \, \beta_N}{2\alpha} \, r^{2\alpha}, \qquad \forall \ r > 0.$$

However, both functions are not regularly varying at the origin. □

Next, we present a class of Gaussian random fields for which (2.17) does not hold, but Theorem 2.1 is still applicable.

Example 2.10 For any given constants $0 < \alpha_1 < \alpha_2 < 1$ and any increasing sequence $\{b_n, \, n \geq 0\}$ of real numbers such that $b_0 = 0$ and $b_n \to \infty$, define the function f on $\mathbb{R}^N$ by

$$f(\lambda) = \begin{cases} |\lambda|^{-(2\alpha_1+N)} & \text{if } \ |\lambda| \in (b_{2k}, b_{2k+1}], \\ |\lambda|^{-(2\alpha_2+N)} & \text{if } \ |\lambda| \in (b_{2k+1}, b_{2k+2}]. \end{cases} \tag{2.32}$$

Using such functions f as spectral densities, we obtain a quite large class of Gaussian random fields with stationary increments that are significantly different from the fractional Brownian motion. If X is such a Gaussian random field, then it follows from (2.4) and (2.20) that there exist positive constants $K_{2,9}$ and $K_{2,10} \geq 1$ such that

$$K_{2,9}^{-1} \, |h|^{2\alpha_2} \leq \sigma^2(h) \leq K_{2,9} \, |h|^{2\alpha_1}, \qquad \forall \, h \in \mathbb{R}^N \text{ with } \ |h| \leq 1 \tag{2.33}$$

and

$$K_{2,10}^{-1}\, r^{2\alpha_2} \le \phi(r) \le K_{2,10}\, r^{2\alpha_1}, \qquad \forall\, 0 < r \le 1. \tag{2.34}$$

Now we choose a strictly increasing sequence $\{b_n\}$ such that for all $k \ge 1$,

$$b_{2k+1}^{2\alpha_2}\Big(b_{2k}^{-2\alpha_1} - b_{2k+1}^{-2\alpha_1}\Big) \le 1, \tag{2.35}$$

$$b_{2k+1}^{2-2\alpha_1} - b_{2k}^{2-2\alpha_1} \le b_{2k}^{2-2\alpha_2}\, \frac{1}{k^2} \tag{2.36}$$

and

$$\frac{b_{2k+2}}{b_{2k+1}} \ge (k+1)^{1/\alpha_2}. \tag{2.37}$$

This can be done inductively: choose b_{2k+1} close to b_{2k} so that both (2.35) and (2.36) hold; then choose b_{2k+2} so that (2.37) holds.

We claim that the following properties hold:

(i) $\phi(r) \asymp r^{2\alpha_2}$ for $r \in (0,1)$.

(ii) $\sigma^2(h) \asymp |h|^{2\alpha_2}$ for all $h \in \mathbb{R}^N$ with $|h| \le 1$.

(iii) Condition (2.17) is not satisfied.

(iv) the corresponding Gaussian random field X is SLϕND on all hypercubes $I = [-T, T]^N$.

In order to verify (i), by (2.34), we only need to show $\phi(r) \le K_{2,11}\, r^{2\alpha_2}$ for some finite constant $K_{2,11}$. For any $r > 0$ small, there exists an integer $k_0 > 0$ such that either $r^{-1} \in [b_{2k_0},\, b_{2k_0+1})$ or $r^{-1} \in [b_{2k_0+1},\, b_{2k_0+2})$. In the first case

$$\begin{aligned}
\phi(r) &\le \int_{b_{2k_0} \le |\lambda| \le b_{2k_0+1}} |\lambda|^{-(2\alpha_1+N)} d\lambda + \int_{b_{2k_0+1} \le |\lambda| \le b_{2k_0+2}} |\lambda|^{-(2\alpha_2+N)} d\lambda + \cdots \\
&= \frac{K}{2\alpha_1}\Big[b_{2k_0}^{-2\alpha_1} - b_{2k_0+1}^{-2\alpha_1} + b_{2k_0+2}^{-2\alpha_1} - b_{2k_0+3}^{-2\alpha_1} + \cdots\Big] \\
&\qquad + \frac{K}{2\alpha_2}\Big[b_{2k_0+1}^{-2\alpha_2} - b_{2k_0+2}^{-2\alpha_2} + b_{2k_0+3}^{-2\alpha_2} - b_{2k_0+4}^{-2\alpha_2} + \cdots\Big].
\end{aligned} \tag{2.38}$$

Clearly, the second sum is bounded above by $K\, r^{2\alpha_2}$. It follows from (2.35) and (2.37) that the first sum is bounded above by $K\, r^{2\alpha_2}$ as well. This verifies (i) when $r^{-1} \in [b_{2k_0},\, b_{2k_0+1})$.

In the second case when $r^{-1} \in [b_{2k_0+1},\, b_{2k_0+2})$, we have

$$\begin{aligned}
\phi(r) &\le \int_{r^{-1} \le |\lambda| \le b_{2k_0+2}} |\lambda|^{-(2\alpha_2+N)} d\lambda + \int_{b_{2k_0+2} \le |\lambda| \le b_{2k_0+3}} |\lambda|^{-(2\alpha_1+N)} d\lambda + \cdots \\
&= \frac{K}{2\alpha_2}\Big[r^{2\alpha_2} - b_{2k_0+2}^{-2\alpha_2} + b_{2k_0+3}^{-2\alpha_2} - b_{2k_0+4}^{-2\alpha_2} + \cdots\Big] \\
&\qquad + \frac{K}{2\alpha_1}\Big[b_{2k_0+2}^{-2\alpha_1} - b_{2k_0+3}^{-2\alpha_1} + b_{2k_0+4}^{-2\alpha_1} - b_{2k_0+5}^{-2\alpha_1} + \cdots\Big] \\
&\le K\, r^{2\alpha_2},
\end{aligned} \tag{2.39}$$

where the last inequality follows from (2.35) and (2.37).

Next we verify (ii). Because of (2.33), we only need to show $\sigma^2(h) \le K_{2,12}\,|h|^{2\alpha_2}$ for all $h \in \mathbb{R}^N$ with $|h| \le 1$. Fix such an $h \in \mathbb{R}^N$, let k_1 be the integer such that $|h|^{-1} \in [b_{2k_1},\, b_{2k_1+1})$ or $|h|^{-1} \in [b_{2k_1+1},\, b_{2k_1+2})$. In both cases, (2.35) and (2.36) imply $|\langle h, \lambda\rangle| \le |h|\,|\lambda| \le 2$ for all $|\lambda| \in [b_{2k},\, b_{2k+1})$ and all $k \le k_1$. For such λ, $1 - \cos\langle h, \lambda\rangle \le (|h|\,|\lambda|)^2$. It follows from (2.4) that

$$\begin{aligned}\sigma^2(h) \le 2 \int_{\mathbb{R}^N} (1-\cos\langle h, \lambda\rangle)\, \frac{d\lambda}{|\lambda|^{2\alpha_2+N}} + 2\sum_{k=0}^{k_1} \int_{b_{2k}\le|\lambda|\le b_{2k+1}} (|h|\,|\lambda|)^2 \frac{d\lambda}{|\lambda|^{2\alpha_1+N}} \\ + 2 \sum_{k=k_1+1}^{\infty} \int_{b_{2k}\le|\lambda|\le b_{2k+1}} \frac{d\lambda}{|\lambda|^{2\alpha_1+N}}.\end{aligned} \tag{2.40}$$

By Example 2.7, the first integral equals $K\,|h|^{2\alpha_2}$. Moreover, a few lines of elementary calculation using (2.35) and (2.36) show that both sums in (2.40) are at most $K\,|h|^{2\alpha_2}$. This proves (ii).

It follows from (2.38) that

$$\limsup_{\lambda\to\infty} \frac{\beta_N\,|\lambda|^N f(\lambda)}{\Delta\{\xi : |\xi| \ge |\lambda|\}} = \infty.$$

Thus (2.17) is not satisfied.

Finally, (2.32) and (i) above imply that (2.5) and (2.6) hold with $q(\lambda) = |\lambda|^{N+2\alpha_2}$. Therefore, Theorem 2.1 implies that the Gaussian random field X with spectral density (2.32) is SLϕND. □

Remark 2.11 In this paper, we have not considered SLND for Gaussian random fields with stationary increments and discrete spectral measures. A systematic treatment for such Gaussian random fields will be done elsewhere. An example of stationary Gaussian processes with discrete spectrum that is (two-sided) strongly locally nondeterministic can be found in Shieh and Xiao (2005). □

3 Sample path properties of Gaussian random fields

In the studies of Gaussian random fields with stationary increments, the variance function $\sigma^2(h)$ has played a significant role and it is typically assumed to be regularly varying at 0 and/or monotone in $|h|$. See Csörgő et al. (1995), Kasahara et al. (1999), Monrad and Rootzén (1995), Talagrand (1995, 1998), Xiao (1996, 1997a, b, 2003) and the references therein. In this section, we show that the regularly varying assumption on $\sigma^2(h)$ can be significantly weakened and the monotonicity assumption can be removed.

We will consider the Gaussian random field $X = \{X(t),\, t \in \mathbb{R}^N\}$ in $\mathbb{R}^d$ defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \quad \forall t \in \mathbb{R}^N, \tag{3.1}$$

where $X_1, \ldots, X_d$ are independent copies of a real-valued, centered Gaussian random field $Y = \{Y(t), t \in \mathbb{R}^N\}$. We call Y the associated random field. In the rest of this paper, we will often assume that Y satisfies the following Condition (C):

(C1) there exist positive constants δ_0, $K_{3,1}$, $K_{3,2}$ and a non-decreasing, right continuous function $\phi : [0, \delta_0) \to [0, \infty)$ such that $\phi(0) = 0$ and

$$\frac{\phi(2r)}{\phi(r)} \le K_{3,1} \qquad \forall\ r \in [0, \delta_0/2) \tag{3.2}$$

and for all $t \in \mathbb{R}^N$ and $h \in \mathbb{R}^N$ with $|h| \le \delta_0$,

$$K_{3,2}^{-1}\phi(|h|) \le \mathbb{E}\big[\big(Y(t+h) - Y(t)\big)^2\big] \le K_{3,2}\phi(|h|). \tag{3.3}$$

(C2) For any $T > 0$, Y is strongly locally ϕ-nondeterministic on $[-T, T]^N$.

It follows from Theorem 2.5 that for any Gaussian random field Y with stationary increments and spectral density satisfying (2.17), Condition (C) is satisfied. We point out that the setting of this section is more general than that of Section 2. In particular, our results in this section are applicable to Gaussian random fields with stationary increments and discrete spectral measures.

3.1 Small ball probability

In recent years, there has been much interest in studying the small ball probability of Gaussian processes. We refer to Li and Shao (2001) and Lifshits (1999) for extensive surveys on small ball probabilities, their applications and open problems.

Our next theorem gives estimates on the small ball probability of Gaussian random fields satisfying the condition (C). In particular, the upper bound in (3.4) confirms a conjecture of Shao and Wang (1995), under a much weaker condition.

Theorem 3.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field in $\mathbb{R}$ satisfying the condition (C). Then there exist positive constants $K_{3,3}$ and $K_{3,4}$ such that for all $x \in (0, 1)$,*

$$\exp\left(-\frac{K_{3,3}}{[\phi^{-1}(x^2)]^N}\right) \le \mathbb{P}\left\{\max_{t\in[0,1]^N} |X(t)| \le x\right\} \le \exp\left(-\frac{K_{3,4}}{[\phi^{-1}(x^2)]^N}\right), \tag{3.4}$$

where $\phi^{-1}(x) = \inf\{y : \phi(y) > x\}$ is the right-continuous inverse function of ϕ.

Proof Equip $I = [0, 1]^N$ with the Dudley metric

$$d(s, t) = (\mathbb{E}|X(s) - X(t)|^2)^{1/2}, \quad s,\, t \in I$$

and denote by $N_d(I, \varepsilon)$ the smallest number of d-balls of radius $\varepsilon > 0$ needed to cover I. Then it is easy to see from (C1) that for all $\varepsilon \in (0, 1)$,

$$N_d(I, \varepsilon) \le K \left(\frac{1}{\phi^{-1}(\varepsilon^2)}\right)^N := \Psi(\varepsilon).$$

Moreover, it follows from Condition (C1) that Ψ has the doubling property, i.e., $\Psi(\varepsilon) \le \Psi(\varepsilon/2) \le K\,\Psi(\varepsilon)$. Hence the lower bound in (3.4) follows from a result of Talagrand (1993); see also Ledoux (1996, p.257).

The proof of the upper bound in (3.4) is based on Condition (C2) and an argument in Monrad and Rootzén (1995). For any integer $n \ge 2$, we choose n^N points $t_{n,\mathbf{i}} \in [0,1]^N$, where

$$t_{n,\mathbf{i}} = \Big(\frac{i_1}{n}, \ldots, \frac{i_N}{n}\Big), \qquad \mathbf{i} = (i_1, \ldots, i_N) \in \{1, \ldots, n\}^N,$$

and denote them [in any order] by $t_{n,k}$ $(k = 1, 2, \ldots, n^N)$. Then

$$\mathbb{P}\left\{\max_{t\in[0,1]^N} |X(t)| \le x\right\} \le \mathbb{P}\left\{\max_{1\le k\le n^N} |X(t_{n,k})| \le x\right\}. \tag{3.5}$$

By Anderson's inequality for Gaussian measures and the SLϕND of X, we derive the following upper bound for the conditional probabilities

$$\mathbb{P}\Big\{|X(t_{n,k})| \le x \big| X(t_{n,j}),\ 1 \le j \le k-1\Big\} \le \Phi\left(\frac{K\,x}{\phi^{1/2}(n^{-1})}\right), \tag{3.6}$$

where $\Phi(x)$ is the distribution function of a standard normal random variable. It follows from (3.5) and (3.6) that

$$\mathbb{P}\left\{\max_{t\in[0,1]^N} |X(t)| \le x\right\} \le \left[\Phi\left(\frac{K\,x}{\phi^{1/2}(n^{-1})}\right)\right]^{n^N}. \tag{3.7}$$

By taking n to be the smallest integer $\ge \big[\phi^{-1}(x^2)\big]^{-1}$, we obtain the upper bound in (3.4). □

Combining Theorem 3.1 with Theorem 7.1 in Li and Shao (2001) yields the following Chung's law of the iterated logarithm. When σ is regularly varying, this is also obtained in Xiao (1997a).

Corollary 3.2 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field defined by (3.1). Suppose that the associated Gaussian random field Y has stationary increments and spectral measure Δ. If Y satisfies Condition (C) and its spectral measure Δ satisfies*

$$\liminf_{\lambda\to\infty} |\lambda|^{N+2}\Delta\big(B(\lambda, r)\big) > 0, \tag{3.8}$$

where $B(\lambda, r) = \{x \in \mathbb{R}^N : |x - \lambda| \le r\}$. Then there exists a positive and finite constant $K_{3,5}$ such that

$$\liminf_{r\to 0} \frac{\sup_{t\in[0,r]^N} |X(t)|}{\phi^{1/2}\big(r/(\log\log(1/r))^{1/N}\big)} = K_{3,5} \qquad a.s. \tag{3.9}$$

Proof By applying Theorem 3.1 and slightly modifying the proof of Theorem 7.1 in Li and Shao (2001) to each component X_k $(k = 1, \ldots, d)$ of X, we derive that there exists a positive constant $K_{3,6} \ge 1$ such that

$$K_{3,6}^{-1} \le \liminf_{r\to 0} \frac{\sup_{t\in[0,r]^N} |X(t)|}{\phi^{1/2}\big(r/(\log\log(1/r))^{1/N}\big)} \le K_{3,6} \quad a.s. \tag{3.10}$$

Since the components of X are independent, (3.8) implies that the zero-one law of Pitt and Tran (1979) holds for X at $t = 0$. Hence (3.9) follows from this and (3.10). □

Remark 3.3 When Δ has a density function which satisfies (2.17), (3.8) follow easily from (2.25) and Lemma 2.3. Hence (3.9) holds. □

We can also consider the small ball probability of Gaussian random fields under the Hölder-type norm. Let $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and non-decreasing function such that $\kappa(r) > 0$ for all $r > 0$. For any function $y \in C_0([0,1]^N)$, we consider the functional

$$\|y\|_\kappa = \sup_{s,t\in[0,1]^N, s\neq t} \frac{|y(s) - y(t)|}{\kappa(|s-t|)}. \tag{3.11}$$

When $\kappa(r) = r^\alpha$, $\|\cdot\|_\kappa$ is the α-Hölder norm on $C_0([0,1]^N)$ and is denoted by $\|\cdot\|_\alpha$.

The following theorem uses SLϕND to improve the results of Stolz (1996). We mention that the conditions of Theorem 2.1 of Kuelbs, Li and Shao (1995) can be weakened in a similar way.

Theorem 3.4 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field in $\mathbb{R}$ satisfying the condition (C). If for some constant $\beta > 0$,*

$$\frac{\phi^{1/2}(r)}{\kappa(r)} \asymp r^\beta, \quad \forall r \in (0,1). \tag{3.12}$$

Then there exist positive constants $K_{3,7}$ and $K_{3,8}$ such that for all $\varepsilon \in (0,1)$,

$$\exp\Big(-K_{3,7}\,\varepsilon^{-N/\beta}\Big) \le \mathbb{P}\Big\{\|X\|_\kappa \le \varepsilon\Big\} \le \exp\Big(-K_{3,8}\,\varepsilon^{-N/\beta}\Big). \tag{3.13}$$

Proof The lower bound in (3.13) follows directly from Theorem 1.1 of Stolz (1996). The proof of the upper bound in (3.13) is a modification of the proof of Theorem 1.3 of Stolz (1996), using (C2) in place of Lemma 7.1 of Pitt (1978). We leave it to the interested reader. □

3.2 Hausdorff dimension and Hausdorff measure of the range

In this section we consider the fractal properties of the range $X([0,1]^N)$ and graph $\mathrm{Gr}X([0,1]^N) = \{(t, X(t)) : t \in [0,1]^N\}$ of the Gaussian random field in $\mathbb{R}^d$ defined by (3.1). In particular, we will show that the Hausdorff dimension of $X([0,1]^N)$ and $\mathrm{Gr}X([0,1]^N)$ can be determined mainly by the upper index of ϕ at 0 defined by

$$\alpha^* = \inf\Big\{\beta \ge 0 : \lim_{r\to 0} \frac{\phi(r)}{r^{2\beta}} = \infty\Big\} \tag{3.14}$$

with the convention $\inf\emptyset = \infty$. Analogously, we can define the lower index of ϕ at 0 by

$$\alpha_* = \sup\Big\{\beta \ge 0 : \lim_{r\to 0} \frac{\phi(r)}{r^{2\beta}} = 0\Big\}. \tag{3.15}$$

Clearly, $0 \le \alpha_* \le \alpha^* \le \infty$. When the real-valued Gaussian random field $Y = \{Y(t), t \in \mathbb{R}^N\}$ associated with (3.1) has stationary increments and a continuous covariance function. Then the above upper and lower indices α^* and α_* coincide with the upper and lower indices of $\sigma(h)$, where

$$\sigma^2(h) = \mathbb{E}\big[\big(Y(t+h) - Y(t)\big)^2\big], \qquad \forall\, h \in \mathbb{R}^N. \tag{3.16}$$

In this case, we also call α^* and α_* the upper and lower indices of Y. See Adler (1981) for more information.

The following example shows that it is possible to have $\alpha^* = \infty$.

Example 3.5 Let $N \geq 2$ and let Δ be a Borel measure on $\mathbb{R}^N$ with support in a linear subspace L of $\mathbb{R}^N$ and satisfying (2.2). If Y is a Gaussian random field with stationary increments and spectrum measure Δ, then for all h in the linear subspace of $\mathbb{R}^N$ that is orthogonal to L, we have $\sigma^2(h) = 0$. Thus $\alpha^* = \infty$. □

Lemma 3.6 below shows that under quite general conditions the inequality $\alpha^* \leq 1$ holds. In particular, this is true for all the Gaussian random fields considered in Section 2.

Lemma 3.6 *Let $Y = \{Y(t), t \in \mathbb{R}^N\}$ be a Gaussian random field in $\mathbb{R}$ with stationary increments and spectrum measure Δ. If $N = 1$, or $N \geq 2$ and Δ has an absolutely continuous part with density $f(\lambda)$. Then $\alpha^* \leq 1$.*

Proof It follows from (2.4) that

$$\begin{aligned} \sigma^2(h) &\geq \int_{|\lambda| \leq |h|^{-1}} (1 - \cos\langle h, \lambda\rangle)\Delta(d\lambda) \\ &\geq K|h|^2 \int_{|\lambda| \leq |h|^{-1}} \langle \frac{h}{|h|}, \lambda\rangle^2\, \Delta(d\lambda). \end{aligned} \tag{3.17}$$

It is clear that when $N = 1$, we have $\sigma^2(h) \geq K|h|^2$ for all $h \in \mathbb{R}$ with $|h|$ small enough. This implies $\alpha^* \leq 1$ whenever $N = 1$.

Now we assume that $N \geq 2$. It follows from (3.17) that

$$\begin{aligned} \sigma^2(h) &\geq K|h|^2 \int_{|\lambda| \leq |h|^{-1}} \langle \frac{h}{|h|}, \lambda\rangle^2\, f(\lambda)\, d\lambda \\ &\geq K\,|h|^2 \int_{S^{N-1}} \langle \frac{h}{|h|}, \theta\rangle^2\, \mu(d\theta) \int_0^{|h|^{-1}} \rho^{N+1} f(\rho\theta)\, d\rho. \end{aligned} \tag{3.18}$$

Since $f(\lambda) > 0$ on a set of positive N-dimensional Lebesgue measure, we see that for all $h \in \mathbb{R}^N$ with $|h|$ small enough,

$$\int_{S^{N-1}} \langle \frac{h}{|h|}, \theta\rangle^2\, \mu(d\theta) \int_0^{|h|^{-1}} \rho^{N+1} f(\rho\theta)\, d\rho \geq K_{3,9} \tag{3.19}$$

for some constant $K_{3,9} > 0$. Hence we have $\alpha^* \leq 1$. □

Remark 3.7 It follows from Lemma 2.3 that if the spectral measure Δ has a density function that satisfies the condition (2.17), then $\underline{\alpha} \leq \alpha^* \leq \alpha_* \leq \overline{\alpha}$. Example 2.9 shows that it is possible to have $\underline{\alpha} < \alpha^* = \alpha_* < \overline{\alpha}$. □

The following result gives general formulas for the Hausdorff dimensions of $X([0,1]^N)$ and $\mathrm{Gr}X([0,1]^N) = \{(t, X(t)) : t \in [0,1]^N\}$ in terms of the upper index α^*. When $\alpha_* = \alpha^* \in (0,1)$, (3.20) and (3.21) are essentially due to Adler (1981); otherwise, they seem to be new.

Theorem 3.8 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be the Gaussian random field in $\mathbb{R}^d$ defined by (3.1). If the associated random field Y satisfies Condition (C1) and $0 < \alpha_* \le \alpha^* < 1$, then*

$$\dim_{\mathrm{H}} X([0,1]^N) = \min\Big\{d, \frac{N}{\alpha^*}\Big\} \quad a.s. \tag{3.20}$$

$$\dim_{\mathrm{H}} \mathrm{Gr}X([0,1]^N) = \min\Big\{N + (1-\alpha^*)d, \frac{N}{\alpha^*}\Big\} \quad a.s., \tag{3.21}$$

where $\dim_{\mathrm{H}}$ *denotes Hausdorff dimension.*

Remark 3.9 (a). We can allow the components $X_1, \ldots, X_d$ in (3.1) to have different distributions. If the upper index of X_i is α_i^*, then the formulae for $\dim_{\mathrm{H}} X([0,1]^N)$ and $\dim_{\mathrm{H}} \mathrm{Gr}X([0,1]^N)$ analogous to those in Theorem 2.1 in Xiao (1995) hold.

(b). The proof of Theorem 3.8 also gives $\dim_{\mathrm{H}} X(E)$ and $\dim_{\mathrm{H}} \mathrm{Gr}X(E)$ for all Borel sets $E \subset \mathbb{R}^N$ with $\dim_{\mathrm{H}} E = \dim_{\mathrm{P}} E$. However, the question of determining $\dim_{\mathrm{H}} X(E)$ and $\dim_{\mathrm{H}} \mathrm{Gr}X(E)$ for an arbitrary Borel set $E \subset \mathbb{R}^N$ remains to be open. □

For the proof of Theorem 3.8 as well as the proofs of Theorems 3.11 and 3.14 below, we need the following lemma on the modulus of continuity of Y, which is reminiscent to Corollary 2.3 or Theorem 2.10 of Dudley (1973), and on the tail probability of the supremum of Y.

Lemma 3.10 *Assume that the Gaussian random field $Y = \{Y(t), t \in \mathbb{R}^N\}$ in $\mathbb{R}$ satisfies Conditions (C1) and $0 < \alpha_* \le \alpha^* < 1$. Let*

$$\omega_Y(\delta) = \sup_{\substack{t,\ t+s \in [0,1]^N \\ |s| \le \delta}} |Y(t+s) - Y(t)|$$

be the uniform modulus of continuity of $Y(t)$ on $[0,1]^N$. Then there exists a finite constant $K_{3,10} > 0$ such that

$$\limsup_{\delta \to 0} \frac{\omega_Y(\delta)}{\sqrt{\phi(\delta) \log \frac{1}{\delta}}} \le K_{3,10}, \quad a.s. \tag{3.22}$$

If, in addition, there is a constant $K_{3,11} > 0$ such that

$$\int_1^\infty \Big(\frac{\phi(a e^{-u^2})}{\phi(a)}\Big)^{1/2} du \le K_{3,11} \quad \text{for all } a \in [0, \delta_0). \tag{3.23}$$

Then there exist positive constants $K_{3,12}$ and $K_{3,13}$ such that for all $r > 0$ small enough and $u \ge K_{3,12}\, \phi^{1/2}(r)$, we have

$$\mathbb{P}\Big\{\sup_{|t| \le r} |Y(t)| \ge u\Big\} \le \exp\Big(-\frac{u^2}{K_{3,13}\, \phi(r)}\Big). \tag{3.24}$$

Proof Because $0 < \alpha_* \le \alpha^* < 1$, the first part, i.e., (3.22), follows from Corollary 2.3 in Dudley (1973). The proof of the second part is based on the Gaussian isopermetric inequality [cf. Talagrand (1995)] and is standard. We include it for completeness.

Let $r < \delta_0$ and $S = \{t: |t| \le r\}$. It follows from (3.3) that $d(s,t) \le K_{3,2}^{1/2}\,\phi^{1/2}(|t-s|)$, we have

$$D := \sup\{d(s,t); s,\ t \in S\} \le K_{3,2}^{1/2}\,\phi^{1/2}(r)$$

and

$$N_d(S,\varepsilon) \le K\Big(\frac{r}{\phi^{-1}\big(\varepsilon^2/K_{3,2}\big)}\Big)^N,$$

where ϕ^{-1} the inverse function of ϕ defined as in Theorem 3.1. Since $\alpha_* > 0$, there exists $\eta > 0$ such that

$$\sigma(r) \le r^{\eta} \quad \text{for all} \quad r \in [0, \delta_0). \tag{3.25}$$

Some simple calculations and (3.3) yield

$$\begin{aligned}
\int_0^D \sqrt{\log N_d(S,\varepsilon)}\; d\varepsilon &\le K \int_0^{K_{3,2}^{1/2}\phi^{1/2}(r)} \sqrt{\log\Big(\frac{r}{\phi^{-1}\big(\varepsilon^2/K_{3,2}\big)}\Big)}\; d\varepsilon \\
&\le K \int_0^r \sqrt{\log(r/t)}\;\; d\phi^{1/2}(t) \\
&= K \int_0^1 \frac{1}{u\sqrt{\log(1/u)}}\,\phi^{1/2}(ur)du \\
&\le K \,\Big(\phi^{1/2}(r) + \int_1^\infty \phi^{1/2}(re^{-u^2})du\Big) \\
&\le K_{3,14}\,\phi^{1/2}(r)\ ,
\end{aligned}$$

where the last inequality follows from (3.23). It follows from Lemma 2.1 in Talagrand (1995) that for all $u \ge K_{3,14}\,\phi^{1/2}(r)$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{|t|\le r}\ |Y(t)| \ge 2\,u\Big\} \\
&\le \mathbb{P}\Big\{\sup_{|t|\le r}|Y(t)| \ge u + \int_0^D \sqrt{\log N_d(S,\varepsilon)}\; d\varepsilon\Big\} \\
&\le \exp\Big(-\frac{u^2}{K_{3,15}\phi(r)}\Big).
\end{aligned} \tag{3.26}$$

This proves (3.24) and the lemma. □

Proof of Theorem 3.8 The proofs of the lower bounds in (3.20) and (3.21) using a standard capacity argument are the same as in Adler (1981) or Kahane (1985), which also complete the proof of Theorem 3.8 when $\alpha^* = 0$.

To prove the upper bound in (3.20), we only need to show $\dim_{\rm H} X([0,1]^N) \le N/\alpha^*$ a.s. Note that for any $\gamma' < \gamma < \alpha^*$, it follows from (3.14) that there exists a sequence $r_n \to 0$ such that $\phi(r_n) \le r_n^{2\gamma}$. For each fixed $n \ge 1$, divide $[0,1]^N$ into r_n^{-N} subcubes $C_{n,i}$ $(i = 1, \ldots, r_n^{-N})$

of side-length r_n. It follows from (3.22) in Lemma 3.10 that a.s. for n large enough, each $X(C_{n,i})$ can be covered by a ball of radius $r_n^{\gamma'}$ in $\mathbb{R}^d$. This implies that $\dim_{\rm H} X([0,1]^N) \le N/\gamma'$ a.s. Since $\gamma' < \alpha^*$ is arbitrary, we have $\dim_{\rm H} X([0,1]^N) \le \min\{d, N/\alpha^*\}$ a.s. The proof of the upper bound in (3.21) is similar and hence omitted. □

Now we consider the exact Hausdofff measure of the image $X([0,1]^N)$.

Theorem 3.11 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field defined in (3.1) with the associated real-valued Gaussian random Y that has stationary increments and satisfies Condition (C). In addition, we assume that the function ϕ satisfies (3.23) and there exists a constant $K_{3,16} > 0$ such that*

$$\int_1^{\delta_0/a} \Big(\frac{\phi(a)}{\phi(ax)}\Big)^{d/2} x^{N-1} dx \le K_{3,16} \qquad \text{for all} \;\; a \in (0, \delta_0), \tag{3.27}$$

then

$$0 < \varphi_1\text{-}m\big(X([0,1]^N)\big) < \infty \quad a.s., \tag{3.28}$$

where $\varphi_1(r) = \big[\phi^{-1}(r^2)\big]^N \log\log 1/r$.

Remark 3.12 If Y has stationary increments with spectrum density f satisfying Condition (2.17), then it follows from Lemma 2.3 that (3.23) always holds and, moreover, (3.27) holds whenever $N < \underline{\alpha}\, d$. □

Proof Since the proof of (3.28) is similar to that in Xiao (1996), we will only point out places where modifications have to be made.

To prove the lower bound in (3.28), we follow the standard procedure of using the density theorem of Rogers and Taylor (1961). For notational convenience, we assume $\delta_0 = 1$ and define the sojourn time

$$T(r) = \int_{[0,1]^N} \mathbb{1}_{B(0,r)}(X(t))\, dt$$

of X in the closed ball $B(0,r)$. Then it is sufficient to show the following estimate of the moments: there exists a constant $K_{3,17}$ such that for all integers $n \ge 1$,

$$\mathbb{E}\big[T(r)^n\big] \le K_{3,17}^n\, n!\, \big[\phi^{-1}(r^2)\big]^{Nn}. \tag{3.29}$$

This can be proved by using induction, which is where the strong local nondeterminism will be needed. The details are given in Xiao (1996) and we only check (3.29) for $n = 1$. Denote $\psi(r) = \phi^{-1}(r^2)$. Note that $\phi(\psi(r)) \ge r^2$. Hence by (3.3), a change of variables and (3.27), we derive

$$
\begin{aligned}
\mathbb{E}\big[T(r)\big] &\le \int_{[0,1]^N} \min\Big\{1,\ K\Big(\frac{r}{\phi^{1/2}(|t|)}\Big)^d\Big\}\, dt \\
&\le K \int_0^1 \min\Big\{1,\ K\Big(\frac{r^2}{\phi(\rho)}\Big)^{d/2}\Big\}\, \rho^{N-1}\, d\rho \\
&\le K \int_0^{\psi(r)} \rho^{N-1} d\rho + K \int_{\psi(r)}^1 \Big(\frac{r^2}{\phi(\rho)}\Big)^{d/2} \rho^{N-1} d\rho \\
&\le K\, \psi(r)^N + K\, \psi(r)^N \int_1^{1/\psi(r)} \Big(\frac{\phi(\psi(r))}{\phi(\psi(r)x)}\Big)^{d/2} x^{N-1} dx \\
&\le K\, \psi(r)^N < \infty.
\end{aligned}
\tag{3.30}
$$

In order to prove the upper bound in (3.28), we need to construct a sequence of economical coverings $\{B_n, n \ge 1\}$ for $X([0,1]^N)$ such that $\mathrm{diam} B_n \to 0$ as $n \to \infty$ and almost surely $\sum_n \varphi_1(\mathrm{diam} B_n) < \infty$. This has been done by Talagrand (1995) for fractional Brownian motion and by Xiao (1996) for any Gaussian random field with stationary increments such that its variance function $\sigma^2(h)$ is regularly varying at 0. By examining carefully the proofs in Xiao (1996), we see that the key ingredient for the construction of the desired coverings of $X([0,1]^N)$, Proposition 3.1 in Xiao (1996), is still valid under the present conditions. The rest of the proof is the same as in Xiao (1996). Therefore (3.28) is proven. □

Remark 3.13 In a similar way, the results on the exact Hausdorff measure of the graph set $\mathrm{Gr}X([0,1]^N)$ in Xiao (1997a, c) can be extended to Gaussian random fields in this paper. □

3.3 Local times and level sets of Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with stationary increments in $\mathbb{R}^d$ defined by (3.1). Suppose the associated real-valued random field Y satisfies (3.3) and for some $\varepsilon > 0$,

$$
\int_{[0,1]^N} \frac{dh}{\sigma^{d+\varepsilon}(h)} < \infty.
$$

If Y is locally nondeterministic on a cube $I \subset \mathbb{R}^N$, say, $I = [0,1]^N$, then it follows from Theorem 26.1 in Geman and Horowitz (1980) [see also Berman (1973) and Pitt (1978)] that X has a jointly continuous local time $L(x,t) := L(x,\ [0,t])$ for $(x,t) \in \mathbb{R}^d \times I$ and satisfies certain Hölder conditions in the time and space variables, respectively.

Under the assumptions of strong local nondeterminism and regular variation of $\sigma^2(h)$, Xiao (1997a) has established sharp local and uniform Hölder conditions for the local time $L(x,t)$ in the time variable t. Besides interest in their own right, such results are also useful in studying the fractal properties of the sample paths of X. In the following, we show that the results in Xiao (1997a) and Kasahara et al. (1999) still hold under the more general Condition (C). For simplicity, we will only consider the case $N = 1$.

Theorem 3.14 *Let $X = \{X(t), t \in \mathbb{R}\}$ be a mean zero Gaussian process in $\mathbb{R}^d$ defined by (3.1) satisfying Condition (C). In addition, we assume that the function ϕ satisfies (3.23) and there exist constants $\gamma_0 \in (0,1)$ and $K_{3,18} > 0$ such that*

$$\int_0^1 \Big(\frac{\phi(a)}{\phi(as)}\Big)^{\frac{d}{2}+\gamma_0} ds \le K_{3,18} \qquad \textit{for all} \;\; a \in (0, \delta_0). \tag{3.31}$$

Then the following properties hold:

(i) X has a local time $L(x,t)$ that is jointly continuous in (x,t) almost surely.

(ii) For any $B \in \mathcal{B}(\mathbb{R})$ define $L^*(B) = \sup_{x\in\mathbb{R}^d} L(x,B)$ be the maximum local time. Then there exists a positive constant $K_{3,19}$ such that for all $t \in \mathbb{R}$,

$$\limsup_{r\to 0} \frac{L^*(B(t,r))}{\varphi_2(r)} \le K_{3,19} \qquad a.s. \tag{3.32}$$

and for all intervals $I \subseteq \mathbb{R}$, there exists a positive finite constant $K_{3,20}$ such that

$$\limsup_{r\to 0} \sup_{t\in I} \frac{L^*(B(t,r))}{\varphi_3(r)} \le K_{3,20} \qquad a.s., \tag{3.33}$$

where $B(t,r) = (t-r, t+r)$,

$$\varphi_2(r) = \frac{r}{\phi(r(\log\log 1/r)^{-1})^{d/2}} \qquad \textit{and} \qquad \varphi_3(r) = \frac{r}{\phi(r(\log 1/r)^{-1})^{d/2}}.$$

Remark 3.15 If X has stationary increments and its spectral measure satisfies (2.17) then (3.23) always holds. Moreover, if $1 > \overline{\alpha}\, d$, then Lemma 2.3 implies that (3.31) is satisfied for any $\gamma_0 \in (0, (1-\overline{\alpha}d)/(2\overline{\alpha}))$. □

The following states that the local Hölder condition for the maximum local time is sharp.

Remark 3.16 By the definition of local times, we have that for any interval $Q \subseteq \mathbb{R}$,

$$\begin{aligned} |Q| &= \int_{\overline{X(Q)}} L(x,Q)\, dx \\ &\le L^*(Q) \cdot \Big(\sup_{s,t\in Q} |X(s) - X(t)|\Big)^d. \end{aligned} \tag{3.34}$$

If X has stationary increments and satisfies the conditions of Theorem 3.14, then Theorem 3.1 and the proof of Theorem 7.1 in Li and Shao (2001) imply the existence of a constant $K_{3,21} \ge 1$ such that for every $t \in \mathbb{R}$,

$$K_{3,21}^{-1} \le \liminf_{r\to 0} \frac{\sup_{s\in B(t,r)} |X(s) - X(t)|}{\phi^{1/2}\big(r/(\log\log(1/r))^{1/N}\big)} \le K_{3,21} \quad a.s. \tag{3.35}$$

By taking $Q = B(t,r)$ in (3.34) and using the upper bound in (3.35), we derive the lower bound in the following

$$K_{3,22} \le \limsup_{r\to 0} \frac{L^*(B(t,r))}{\varphi_2(r)} \le K_{3,19} \qquad a.s., \tag{3.36}$$

where $K_{3,22} > 0$ is a constant and the upper bound is given by (3.32). A similar lower bound for (3.33) could also be established by using (3.34), if one proves that for every interval $I \subseteq \mathbb{R}$,

$$\liminf_{r\to 0} \inf_{t\in I} \sup_{s\in B(t,r)} \frac{|X(s)-X(t)|}{\phi^{1/2}(r/(\log 1/r)^{1/N})} \le K_{3,23} \qquad a.s. \tag{3.37}$$

This is left to the interested reader. □

The proof Theorem 3.14 is similar to Xiao (1997a) which is based on getting sharp moment estimates for $L(x, B)$ and $L(x + y, B) - L(x, B)$ and on a chaining argument. We will not reproduce all the details. Instead, we give a simplified proof of the following key estimates.

Lemma 3.17 *Under the conditions of Theorem 3.14, there exist positive constants $K_{3,24}$ and $K_{3,25}$ such that for all integers $n \ge 1$, $r(0, \delta_0)$, $x \in \mathbb{R}^d$ and $0 < \gamma < \gamma_0$, we have*

$$\mathbb{E}\big[L(x,r)^n\big] \le \frac{K_{3,24}^n\, r^n}{\phi(r/n)^{nd/2}} \tag{3.38}$$

and

$$\mathbb{E}\big[L(x+y,r) - L(x,r)\big]^n \le \frac{K_{3,25}^n |y|^{n\gamma}\, r^n}{\big[\phi(r/n)\big]^{(d+2\gamma)n/2}}\,(n!)^\gamma. \tag{3.39}$$

For the proof of Lemma 3.17, we will need several lemmas. Lemma 3.18 is essentially due to Cuzick and DuPreez (1982) and Lemma 3.19 extends Lemma 3 of Kasahara et al. (1999).

Lemma 3.18 *Let $Z_1, \cdots, Z_n$ be mean zero Gaussian variables which are linearly independent. Then for any measurable function $g : \mathbb{R} \to \mathbb{R}_+$,*

$$\int_{\mathbb{R}^n} g(v_1) e^{-\frac{1}{2}\mathrm{Var}(\sum_{j=1}^n v_j Z_j)} dv_1 \cdots dv_n = \frac{(2\pi)^{n-1}}{(\mathrm{detCov}(Z_1,\cdots,Z_n))^{1/2}} \int_{\mathbb{R}} g\Big(\frac{v}{\sigma_1}\Big) e^{-v^2/2} dv, \tag{3.40}$$

where $\sigma_1^2 = Var(Z_1|Z_2,\cdots,Z_n)$ is the conditional variance of Z_1 given $Z_2, \cdots, Z_n$ and $\mathrm{detCov}(Z_1,\cdots,Z_n)$ *is the determinant of the covariance matrix of $(Z_1,\cdots,Z_n)$.*

Lemma 3.19 *Let $U(x)$ be a right continuous, non-decreasing function on $\mathbb{R}_+$ with $U(0) = 0$. If there exists a constant $K_{3,26} > 0$ such that $U(2t) \le K_{3,26} U(t)$ for all $t > 0$, then*

$$\Big[\int_{\mathbb{R}_+^n \cap \{0 < t_1+t_2+\cdots+t_n \le 1\}} dU(t_1)\cdots dU(t_n)\Big]^{1/n} \asymp U(1/n), \qquad as\ \ n \to \infty. \tag{3.41}$$

Proof The lower bound follows easily from

$$\begin{aligned} \int_{\mathbb{R}_+^n \cap \{0 < t_1+t_2+\cdots+t_n \le 1\}} dU(t_1)\cdots dU(t_n) \ &\ge\ \int_0^{1/n} \cdots \int_0^{1/n} dU(t_1)\cdots dU(t_n) \\ &=\ \big[U(1/n)\big]^n. \end{aligned} \tag{3.42}$$

To prove the upper bound, we follow the argument of Kasahara et al. (1999) and define the distribution functions F_n on $\mathbb{R}_+$ by

$$F_n(t) = \int_{\mathbb{R}_+^n \cap \{0 \le t_1 + t_2 + \cdots + t_n \le t\}} dU(t_1) \cdots dU(t_n).$$

Then the integral on the left-hand side of (3.42) is $F_n(1)$. Note that the Laplace transform of F_n can be written as

$$\int_0^\infty e^{-st} dF_n(t) = \left(\int_0^\infty e^{-st}\, dU(t) \right)^n.$$

Hence we have

$$F_n(1) \le e^n \int_0^1 e^{-nt} dF_n(t) \le e^n \int_0^\infty e^{-nt} dF_n(t) \le \left(e \int_0^\infty e^{-nt} dU(t) \right)^n.$$

It follows that

$$\limsup_{n\to\infty} \frac{1}{U(1/n)} \big[F_n(1)\big]^{1/n} \le \limsup_{n\to\infty} \frac{e}{U(1/n)} \int_0^\infty e^{-nt} dU(t). \tag{3.43}$$

Now we split the last integral over the intervals $[0, n^{-1})$ and $[n^{-1}2^{k-1}, n^{-1}2^k)$ $(k \ge 1)$, which gives

$$\begin{aligned} \int_0^\infty e^{-nt} dU(t) &\le U(1/n) + \sum_{k=1}^\infty e^{-2^{k-1}} U(2^k/n) \\ &\le U(1/n) \left[1 + \sum_{k=1}^\infty e^{-2^{k-1}} K_{3,26}^k \right] \\ &= K_{3,27}\, U(1/n), \end{aligned} \tag{3.44}$$

where in deriving the second inequality, we have made use of the doubling property of U. Therefore, the upper bound in (3.41) follows from (3.43) and (3.44). □

Proof of Lemma 3.17 It follows from (25.5) and (25.7) in Geman and Horowitz (1980) [see also Pitt (1978)] that for any $x, y \in \mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R})$ and any integer $n \ge 1$, we have

$$\begin{aligned} \mathbb{E}\big[L(x,B)\big]^n = (2\pi)^{-nd} \int_{B^n} \int_{\mathbb{R}^{nd}} &\exp\left(-i \sum_{j=1}^n \langle u_j, x\rangle \right) \\ &\times \mathbb{E} \exp\left(i \sum_{j=1}^n \langle u_j, X(t_j)\rangle \right) d\bar{u}\; d\bar{t} \end{aligned} \tag{3.45}$$

and for any even integer $n \ge 2$,

$$\begin{aligned} \mathbb{E}\big[L(x+y,B) - L(x,B)\big]^n = (2\pi)^{-nd} \int_{B^n} \int_{\mathbb{R}^{nd}} &\prod_{j=1}^n \left(e^{-i\langle u_j, x+y\rangle} - e^{-i\langle u_j, x\rangle} \right) \\ &\times \mathbb{E} \exp\left(i \sum_{j=1}^n \langle u_j, X(t_j)\rangle \right) d\bar{u}\; d\bar{t}, \end{aligned} \tag{3.46}$$

where $\overline{u} = (u_1, \cdots, u_n)$, $\bar{t} = (t_1, \cdots, t_n)$, and each $u_j \in \mathbb{R}^d$, $t_j \in \mathbb{R}$. In the coordinate notation we then write $u_j = (u_j^1, \cdots, u_j^d)$.

Take $B = [0, r]$. It follows from (3.45) and the proof of Lemma 2.5 in Xiao (1997a) that for all integers $n \ge 1$,

$$\mathbb{E}\big[L(x,r)^n\big] \le (2\pi)^{-nd/2} \int_{[0,\,r]^n} \frac{1}{\big[\det C_n(t_1, \cdots, t_n)\big]^{d/2}}\, dt_1 \cdots dt_n, \tag{3.47}$$

where $C_n(t_1, \cdots, t_n)$ denotes the covariance matrix of the Gaussian variables $X_1(t_1), \ldots, X_1(t_n)$. It is well known that

$$\det\big(C_n(t_1, \cdots, t_n)\big) = \mathrm{Var}\big(X_1(t_1)\big) \prod_{j=2}^{n} \mathrm{Var}\big(X_1(t_j) | X_1(t_1), \cdots, X_1(t_{j-1})\big). \tag{3.48}$$

We apply (C2) to derive that for any $0 < t_1 < t_2 < \ldots < t_n$,

$$K^n \prod_{j=1}^{n} \phi(t_j - t_{j-1}) \le \det\big(C_n(t_1, \cdots, t_n)\big) \le \prod_{j=1}^{n} \phi(t_j - t_{j-1}), \tag{3.49}$$

where $t_0 = 0$. By (3.47)–(3.49) and a simple substitution, we deduce that

$$\begin{aligned} \mathbb{E}\big[L(x,r)^n\big] &\le K^n\, n! \int_{0<t_1<t_2<\ldots<t_n \le r} \prod_{j=1}^{n} \frac{1}{\big(\phi(t_j - t_{j-1})\big)^{d/2}}\, dt_1 \cdots dt_n \\ &\le K^n\, n!\, r^n \int_{0<s_1+s_2+\cdots+s_n \le 1} dU_1(s_1) \cdots dU_1(s_n), \end{aligned} \tag{3.50}$$

where the function $U_1(t)$ is defined by

$$U_1(t) = \int_0^{\min\{t,1\}} \frac{ds}{(\phi(rs))^{d/2}} \qquad \text{for all} \quad t \ge 0.$$

Since ϕ is non-decreasing, we see that $U_1(2t) \le 2U_1(t)$ for all $t \ge 0$. Hence it follows from Lemma 3.19 that

$$\mathbb{E}\big[L(x,r)^n\big] \le K_{3,28}^n\, n!\, r^n \big[U_1(1/n)\big]^n. \tag{3.51}$$

On the other hand, by (3.31), we derive that

$$U_1(t) \le K_{3,29} \frac{t}{\big(\phi(rt)\big)^{d/2}} \qquad \text{for all } 0 \le t \le 1. \tag{3.52}$$

Therefore, (3.38) follows from (3.51), (3.52) and Stirling's formula.

Now we turn to the proof of (3.39). By (3.46) and the elementary inequality

$$|e^{iu} - 1| \le 2^{1-\gamma} |u|^\gamma \qquad \text{for all} \quad u \in \mathbb{R},\ 0 < \gamma < 1,$$

we see that for all even integers $n \geq 2$ and any $0 < \gamma < 1$,

$$\mathbb{E}\big[L(x+y,r) - L(x,r)\big]^n \leq (2\pi)^{-nd} 2^{(1-\gamma)n} |y|^{n\gamma} \int_{[0,\,r]^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |u_j|^\gamma \times \exp\left(-\frac{1}{2} \mathrm{Var}\Big(\sum_{j=1}^n \langle u_j,\ X(t_j) \rangle \Big) \right) d\overline{u}\ d\overline{t}. \tag{3.53}$$

Since for any $0 < \gamma < 1$, $|a+b|^\gamma \leq |a|^\gamma + |b|^\gamma$, we have

$$\prod_{j=1}^n |u_j|^\gamma \leq \sum{}' \prod_{j=1}^n |u_j^{k_j}|^\gamma, \tag{3.54}$$

where the summation $\sum{}'$ is taken over all $(k_1, \cdots, k_n) \in \{1, \cdots, d\}^n$. Fix such a sequence $(k_1, \cdots, k_n)$ and fix n points $0 < t_1 < \cdots < t_n \leq r$, we consider the integral

$$I_3 := \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |u_j^{k_j}|^\gamma\ \exp\left(-\frac{1}{2} \mathrm{Var}\Big(\sum_{j=1}^n \langle u_j,\ X(t_j) \rangle \Big) \right) d\overline{u}.$$

It follows from (C2) that the Gaussian random variables $X_l(t_j)$ $(l = 1, \cdots, d,\ j = 1, \cdots, n)$ are linearly independent. Hence we use the generalized Hölder's inequality and Lemma 3.18 to deduce that I_3 is at most

$$\begin{aligned}
&\prod_{j=1}^n \left\{ \int_{\mathbb{R}^{nd}} |u_j^{k_j}|^{n\gamma} \exp\left[-\frac{1}{2} \mathrm{Var}\Big(\sum_{j=1}^n \sum_{l=1}^d u_j^l X_l(t_j) \Big) \right] d\overline{u} \right\}^{1/n} \\
&= \frac{(2\pi)^{nd-1}}{\big[\mathrm{detCov}(X_l(t_j),\ 1 \leq l \leq d,\ 1 \leq j \leq n)\big]^{1/2}} \int_{\mathbb{R}} |v|^{n\gamma} \exp\Big(-\frac{v^2}{2} \Big) dv \prod_{j=1}^n \frac{1}{\sigma_j^\gamma} \qquad (3.55) \\
&\leq \frac{K^n\ (n!)^\gamma}{\big[\mathrm{detCov}(Y(t_1), \cdots, Y(t_n))\big]^{d/2}} \prod_{j=1}^n \frac{1}{\sigma_j^\gamma},
\end{aligned}$$

where σ_j^2 is the conditional variance of $X_{k_j}(t_j)$ given $X_l(t_i)$ $(l \neq k_j$ or $l = k_j,\ i \neq j)$ and the last inequality follows from Stirling's formula.

It follows from the independence of the Gaussian random fields $X_1, \cdots, X_n$ and Condition (C) that

$$\sigma_j^2 \geq K \min\big\{ \phi(t_j - t_{j-1}),\ \phi(t_{j+1} - t_j) \big\},$$

where $t_0 := 0$. Hence

$$\prod_{j=1}^n \frac{1}{\sigma_j^\gamma} \leq K^n \prod_{j=1}^n \frac{1}{\big[\phi(t_j - t_{j-1})\big]^\gamma} \tag{3.56}$$

Combining (3.55), (3.48), (3.49) and (3.56), we obtain

$$I_3 \leq K^n\ (n!)^\gamma \prod_{j=1}^n \frac{1}{\big[\phi(t_j - t_{j-1})\big]^{(d+2\gamma)/2}} \tag{3.57}$$

It follows from (3.53), (3.54) and (3.57) that

$$
\begin{aligned}
\mathbb{E}\big[L(x+y,r)-L(x,r)\big]^n &\le K^n |y|^{n\gamma} (n!)^{1+\gamma} \\
&\quad \times \int_{0<t_1<\cdots<t_n\le r} \prod_{j=1}^{n} \frac{1}{\big[\phi(t_j-t_{j-1})\big]^{(d+2\gamma)/2}}\, dt_1 \ldots dt_n \\
&\le K^n |y|^{n\gamma} (n!)^{1+\gamma}\, r^n \int_{0<s_1+s_2+\cdots+s_n\le 1} dU_2(s_1)\cdots dU_2(s_n)
\end{aligned}
\tag{3.58}
$$

where the function $U_2(t)$ is defined by

$$
U_2(t) = \int_0^{\min\{t,1\}} \frac{ds}{(\phi(rs))^{(d+2\gamma)/2}}, \qquad t \ge 0.
$$

Again U_2 has the doubling property. Hence it follows from Lemma 3.19 that

$$
\mathbb{E}\big[L(x+y,r)-L(x,r)\big]^n \le K_{3,30}^n\, |y|^{n\gamma} (n!)^{1+\gamma}\, r^n \,\big[U_2(1/n)\big]^n. \tag{3.59}
$$

Finally, (3.31) implies that

$$
U_2(t) \le K_{3,31}\, \frac{t}{\big[\phi(rt)\big]^{(d+2\gamma)/2}} \qquad \text{for all } \ t\in[0,1]. \tag{3.60}
$$

Therefore, (3.39) follows from (3.59), (3.60) and Stirling's formula. □

Theorem 3.14 can be applied to determine the Hausdorff dimension and Hausdorff measure of the level set $X^{-1}(x) = \{t \in \mathbb{R} : X(t) = x\}$, where $x \in \mathbb{R}^d$. See Berman (1970, 1972), Adler (1981), Monrad and Pitt (1987) and Xiao (1997a). In the following theorem we prove a uniform Hausdorff dimension result for the level sets of the Gaussian process X, extending the previous results of Berman (1972), Monrad and Pitt (1987).

Theorem 3.20 *Let $X = \{X(t), t \in \mathbb{R}\}$ be a Gaussian process in $\mathbb{R}^d$ defined by (3.1) satisfying the conditions of Theorem 3.14. Then with probability one,*

$$
\dim_{\rm H} X^{-1}(x) = 1-\alpha^* d \quad \textit{for all } \ x \in \mathcal{O}, \tag{3.61}
$$

where $\mathcal{O}$ is the random open set defined by

$$
\mathcal{O} = \bigcup_{s,t\in\mathbb{Q};\, s<t} \Big\{ x\in\mathbb{R}^d:\ L(x,[s,t])>0\Big\}.
$$

Proof Let Ω_0 be the event on which the modulus of continuity for X [cf. (3.22)] and Theorem 3.14 hold. Clearly, $\mathbb{P}(\Omega_0)=1$. Now we choose and fix an $\omega \in \Omega_0$, and prove our conclusion for the sample path $X(\cdot,\omega)$.

To prove the upper bound in (3.61), it is sufficient to show that almost surely,

$$
\dim_{\rm H}\big(X^{-1}(x)\cap[0,1]\big) \le 1-\alpha^* d \quad \text{for all } \ x\in\mathbb{R}^d. \tag{3.62}
$$

For any integer $n \geq 1$, we divide the interval $[0,1]$ into 2^n subintervals $I_{n,k} = [(k-1)2^{-n}, k2^{-n}]$ $(k = 1, \ldots, 2^n)$. For every $x \in \mathbb{R}^d$, denote by $N(n,x)$ the number of k's such that $x \in X(I_{n,k})$. The modulus of continuity of X in Lemma 3.10 implies that if $x \in X(I_{n,k})$ then $X(I_{n,k}) \subseteq B(x, \rho_n)$, where $\rho_n = K\sqrt{\phi(2^{-n})\log 2^n}$. Since the local time $L(y,1)$ is a continuous in y, it is bounded on $B(x,\rho_n)$. Hence we have

$$N(n,x)\, 2^{-n} \leq \int_{B(x,\rho_n)} L(y,1)dy \leq K_{3,32}\, \rho_n^d, \tag{3.63}$$

where $K_{3,32}$ depends on ω. This gives

$$N(n,x) \leq K_{3,32}\, 2^n\, \rho_n^d. \tag{3.64}$$

Hence for every $0 < \gamma < \alpha^*$, there exists a sequence $\{n_k\}$ of positive integers such that $N(n_k, x) \leq K_{3,32}\, 2^{n_k(1-\gamma d)}$. This implies (3.62).

To prove the lower bound in (3.61), we note that the jointly continuous local time $L(x,t)$ of X can be extended to become a random Borel measure, denoted by $L(x,\cdot)$, on $X^{-1}(x)$; see Adler (1981). Moreover, for every $x \in \mathcal{O}$, $L(x,\cdot)$ is a positive measure.

Now for any $\gamma > \alpha^*$, (3.33) of Theorem 3.14 implies that almost surely, $L(x, B(t,r)) \leq K\, r^{1-\gamma d}$ for all $x \in \mathbb{R}^d$, all $t \in [0,1]$ and $r > 0$ small. By the Frostman lemma [cf. Kahane (1985)], we have almost surely $\dim_{\rm H} X^{-1}(x) \geq 1 - \gamma d$ for all $x \in \mathcal{O}$. Since $\gamma > \alpha^*$ is arbitrary, this proves the lower bound in (3.61) and hence the theorem. □

Remark 3.21 It is an interesting question to characterize the random open set $\mathcal{O}$. Monrad and Pitt (1987) have given a real-valued *periodic* stationary Gaussian process X for which $\mathcal{O}$ is a proper subset of $\mathbb{R}$ [because the range of X is a.s. bounded]. They have shown a sufficient condition in terms of the spectral measure of a stationary (N,d)-Gaussian random field X so that $\mathcal{O} = \mathbb{R}^d$ holds. Monrad and Pitt (1987) also point out that the self-similarity of an (N,d)-fractional Brownian motion B_α implies that if $N > \alpha d$ then $\mathcal{O} = \mathbb{R}^d$ almost surely. However, we do not know whether $\mathcal{O} = \mathbb{R}^d$ is true for the (N,d)-Gaussian random fields satisfying the conditions of Theorem 2.5. □

The local time $L(0,1)$ [i.e., $L(x,1)$ at $x=0$] of a Gaussian process X sometimes appears as a limit in some limit theorems on the occupation measure of X; see, for example, Kasahara and Ogawa (1999) and the references therein. Since there is little knowledge on the explicit distribution of $L(0,1)$, it is of interest in estimating the tail probability $\mathbb{P}\{L(0,1) > x\}$ as $x \to \infty$. This problem has been considered by Kasahara et al. (1999) under some extra conditions on the Gaussian process X. The next theorem is an extension of their main result.

Theorem 3.22 *Let $X = \{X(t) : t \in \mathbb{R}\}$ be a mean 0 Gaussian process in $\mathbb{R}^d$ defined by (3.1). We assume that the associated Gaussian process Y satisfies Condition (C) and the condition (3.31) with $\gamma_0 = 0$. Then for $x > 0$ large enough,*

$$-\log \mathbb{P}\big\{L(0,1) > x\big\} \asymp \frac{1}{\phi^{-1}(1/x^2)}, \tag{3.65}$$

where ϕ^{-1} is the inverse function of ϕ as defined in Theorem 3.1.

Theorem 3.22 follows from the moment estimates for $L(0,1)$ in Lemma 3.24 and the following lemma on the tail probability of nonnegative random variables. When ψ is a power function or a regularly varying function, Lemma 3.23 is well known.

Lemma 3.23 *Let ξ be a non-negative random variable and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function having the doubling property. If there exist positive constants $K_{3,33}$ and $K_{3,34}$ such that*

$$K_{3,33}^n\, \psi(n)^n \le \mathbb{E}(\xi^n) \le K_{3,34}^n\, \psi(n)^n$$

for all n large enough, then there exist positive constants $K_{3,35} > K_{3,34}$, $K_{3,36}$ and $K_{3,37}$ such that for all $x > 0$ large enough,

$$e^{-K_{3,36}x} \le \mathbb{P}\big\{\xi \ge K_{3,35}\, \psi(x)\big\} \le e^{-K_{3,37}x}. \tag{3.66}$$

Proof The upper bound in (3.66) follows easily from Chebyshev's inequality and a monotonicity argument. In order to prove the lower bound, we follow the elementary argument of Talagrand (1998). By applying the Paley-Zygmund inequality [cf. Kahane (1985), p.8] to ξ^n, we have

$$\begin{aligned}\mathbb{P}\Big\{\xi \ge \frac{K_{3,33}}{2}\, \psi(n)\Big\} &\ge \mathbb{P}\Big\{\xi^n \ge \frac{1}{2}\, \mathbb{E}(\xi^n)\Big\}\\ &\ge \frac{1}{4}\, \frac{\big[\mathbb{E}(\xi^n)\big]^2}{\mathbb{E}(\xi^{2n})}\\ &\ge \frac{1}{4}\, \frac{K_{3,33}^{2n}\, \psi(n)^{2n}}{K_{3,34}^{2n}\, \psi(2n)^{2n}}.\end{aligned}$$

Now it is clear that the lower bound in (3.66) follows from the doubling property of ψ and a standard monotonicity argument. □

Lemma 3.24 *There exist positive and finite constants $K_{3,38}$ and $K_{3,39}$ such that for all integers $n \ge 1$*

$$\frac{K_{3,38}^n}{\phi(1/n)^{nd/2}} \le \mathbb{E}\big[L(0,1)^n\big] \le \frac{K_{3,39}^n}{\phi(1/n)^{nd/2}}. \tag{3.67}$$

Proof As in the proofs of Lemma 2.5 in Xiao (1997a) or Lemma 1 in Kasahara et al. (1999), we derive from (3.45) that for any integer $n \ge 1$,

$$\mathbb{E}\big[L(0,1)^n\big] = (2\pi)^{-nd/2} \int_{[0,1]^n} \frac{1}{\big[\det C_n(t_1, \cdots, t_n)\big]^{d/2}}\, dt_1 \cdots dt_n. \tag{3.68}$$

It follows from (3.48) and (3.49) that

$$\begin{aligned}\Big[\mathbb{E}\big(L(0,1)^n\big)\Big]^{1/n} &\asymp \left[n! \int_{0<t_1<t_2<\ldots<t_n\le 1} \prod_{j=1}^n \frac{1}{\sigma^d(t_j - t_{j-1})}\, dt_1 \cdots dt_n\right]^{1/n}\\ &\asymp \left[n! \int_{0<s_1+s_2+\cdots+s_n\le 1} dU_3(s_1) \cdots dU_3(s_n)\right]^{1/n},\end{aligned} \tag{3.69}$$

where the function $U_3(t)$ is defined by

$$U_3(t) = \int_0^{\min\{t,1\}} \frac{ds}{(\phi(s))^{d/2}} \qquad \forall\ t \ge 0.$$

Since ϕ is non-decreasing, we see that $U_3(2t) \le 2U_3(t)$ for all $t \ge 0$. Hence it follows from Lemma 3.19 that

$$\Big[\mathbb{E}\big(L(0,1)^n\big)\Big]^{1/n} \asymp (n!)^{1/n} U_3(1/n). \tag{3.70}$$

Therefore, (3.67) follows from (3.52) and Stirling's formula as in the proof of Lemma 3.17. □

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