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Hitting Probabilities of the Random Covering Sets

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Abstract. Let E be the Dvoretzky random covering sets on the circle. By applying the method of limsup type random fractals, as illustrated in Khoshnevisan, Peres and Xiao [**24**], we determine the hitting probability $\mathbb{P}(E\cap G \neq \emptyset)$ and the packing dimension of the intersection $E \cap G$, where G is an arbitrary Borel set on the circle.

Introduction

We begin with a brief review on random coverings. Let $\{\omega_n\}_{n\geq 1}$ be a sequence of independent random variables on $(\Omega, \mathcal{B}, \mathbb{P})$ which are uniformly distributed over the unit interval $I = [0, 1)$. Let $\{l_n\}_{n\geq 1}$ be a sequence of positive real numbers which is decreasing to zero. For every $n \geq 1$, denote by $I_n := (\omega_n, \omega_n + l_n) (\text{mod } 1)$ the random interval whose starting point and lengthe are determined by ω_n and l_n respectively. Define the *random covering set* as

$$E := \limsup_{n\to\infty} I_n = \{t \in I : t \in I_n \ \text{ for infinitely many } \ n \geq 1\}.$$

The set E consists of the points which are covered by $\{I_n\}$ infinitely often (i.o. for short). The Borel-Cantelli Lemma implies that the Lebesgue measure of the random set E is either 1 or 0 almost surely according to the divergence or convergence of the series $\sum_{n=1}^{\infty} l_n$.

It was Dvoretzky [**5**] who called the attention on study of such random sets; he raised the question that under what condition on $\{l_n\}$ one can have

$$[0, 1) = \limsup_{n\to\infty} I_n \quad \text{a.s.} \tag{0.1}$$

In the literature this is referred to as the Dvoretzky covering problem and had attracted the attention of P. Billard, J.-P. Kahane, B. Mandelbrot, among others, before it was completely solved by L. A. Shepp in 1972. Shepp [**31**] provided a necessary and sufficient condition for (0.1) to hold, namely

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp\Big(l_1 + \cdots + l_n\Big) = \infty. \tag{0.2}$$

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Since then the topic has been under active development and there have been many extensions and refinements. We refer to [**20**, Chapter 11] for a systematic account on the Dvoretzky covering problem and its higher dimensional extensions, to the survey articles [**8**, **22**, **23**] for historical accounts and connections to multiplicative processes, and to [**1**, **4**, **7**–**10**, **14**, **21**] and the references therein for further information. It should also be mentioned that Jonasson and Steif [**16**] (see also [**15**]) have recently extended the Dvoretzky covering model by including time dynamics (In the first variant, they identify $I = [0, 1)$ with the unit circle C and allow the centers of I_n ($n \geq 1$) perform independent Brownian motions on C, each with variance 1. In the second variant, they associate independent Poisson processes with the different intervals.) The work of Jonasson and Steif [**16**] has revealed rich structures in dynamical random coverings and raised more interesting questions about properties of the dynamical random covering sets, including their fractal dimensions and hitting probabilities.

This paper is concerned with the geometric and potential-theoretic properties of the Dvoretzky covering set $E = \limsup_{n\to\infty} I_n$. It is known that the set E is a.s. dense in I and is of second category ([**20**, Chapter 5, Proposition 11]). Thus, the upper box dimension of the set E is 1 almost surely. Several authors have investigated the Hausdorff dimension and other fractal properties of E and/or its complement $F_\infty = I\backslash E$ (which is called the uncovered set). For example, Fan and Wu [**10**] considered the Hausdorff dimension of the set E for the special case $l_n = \frac{a}{n^\gamma}$, where $a > 0$ and $\gamma > 1$ are constants, they proved that $\dim_{\mathrm{H}}(E) = \frac{1}{\gamma}$ a.s., where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. Durand [**4**] considered a general sequence $\{l_n\}$ with $\sum_{n=1}^\infty l_n < \infty$ and proved, among other things,

$$\dim_{\mathrm{H}} E = \alpha \quad \text{and} \quad \dim_{\mathrm{P}} E = 1 \quad \text{a.s.,} \tag{0.3}$$

where α is defined by

$$\alpha := \inf\left\{s > 0 : \sum_{n=1}^\infty l_n^s < \infty\right\} = \sup\left\{s > 0 : \sum_{n=1}^\infty l_n^s = \infty\right\} \tag{0.4}$$

with the convention that $\sup\emptyset = 0$ and $\inf\emptyset = 1$.

The index α defined in (0.4) is known as the *exponent of convergence* of the sequence $\{l_n\}$ and can be calculated by using the following formula

$$\alpha = \limsup_{n\to\infty} \frac{\log n}{-\log l_n} \tag{0.5}$$

(see [**27**] p.285 or [**30**] p.26). Besicovitch and Taylor [**3**] applied the index α (they also introduced another index–the lower index for $\{l_n\}$) to characterize the Hausdorff measure and Hausdorff dimension of a linear compact set K whose complement forms a sequence of open intervals of lengths $\{l_n\}$. Hawkes [**13**] showed that α is the upper box dimension of K, and the lower index of $\{l_n\}$ is the lower box dimension of K. Kahane [**18**, **19**] called α the *upper Besicovitch-Taylor index* of $\{l_n\}$. Some related indices for $\{l_n\}$ were also discussed in [**1**] for studying the Carleson problem and covering numbers for the Dvoretzky covering set E.

The following intersection problem is of intrinsic importance in the study of random coverings and other random fractals. For any given set $A \subset [0, 1)$, we can ask whether or not it is a.s. covered infinitely often by $\{I_n\}$. That is, when does $\mathbb{P}(A \subset E) = 1$ hold? In the case $\sum_{n=1}^{\infty} l_n = \infty$, which is opposite to what we are considering in this paper, the analogous problem for the uncovered set F_∞ has been investigated by several authors. For example, Kahane [**20**] considered the case $l_n = \frac{\beta}{n}, 0 < \beta < 1$ and showed that $\mathbb{P}(A \subset E) = 1$ (equivalently, $\mathbb{P}\big(A \cap F_\infty \neq \emptyset\big) = 0$) if $\dim_{\mathrm{H}}(A) < \beta$, whilst $\mathbb{P}(A \subset E) = 0$ (equivalently, $\mathbb{P}\big(A \cap F_\infty \neq \emptyset\big) = 1$) if $\dim_{\mathrm{H}}(A) > \beta$. For a more general case, Hawkes [**12**] proved that, if the set A satisfies a regularity condition (which in particular requires $\dim_{\mathrm{H}}(A) = \dim_{\mathrm{P}}(A)$), then $\mathbb{P}(A \subset E) = 1$ or 0 according as $\dim_{\mathrm{H}}(A) < \tau$ and $\dim_{\mathrm{H}}(A) > \tau$, where τ is the index of $\{l_n\}$ defined by

$$\tau = \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} l_i}{\log n}.$$

More precise hitting probability results for the Poisson covering (see Mandelbrot [**26**]) have been established by using the connection between F_∞ and the range of a subordinator; see Fitzsimmons, *et al.* [**11**]. However, the problems for determining the hitting probabilities of the Dvoretzky random covering set E had never been studied.

The purpose of this paper is to study the hitting probabilities of the Dvoretzky covering set E, as well as fractal dimensions of the intersection $E \cap G$, when it is nonempty. Our main result (Theorem 2.1 below) shows that the hitting probability $\mathbb{P}(E \cap G \neq \emptyset)$ is determined by $\dim_{\mathrm{P}}(G)$, the packing dimension of G (see (0.7) below). This is in contrast with the hitting probability results for the random set F_∞, where Hausdorff dimension plays the natural role. Theorem 2.1 will allow us to determine the packing dimension of $E \cap G$ for any analytic set $G \subseteq [0, 1)$ and provide a refinement (under an extra condition) of (0.3) obtained by Durand [**4**].

Recall that packing dimension was introduced in the early 1980s by Tricot [**33**] as follows. For any $\varepsilon > 0$ and any bounded set $G \subset \mathbb{R}$, let $N(G, \varepsilon)$ be the smallest number of balls of radius ε needed to cover G. The upper box dimension of G is defined as

$$\overline{\dim}_{\mathrm{M}}(G) = \limsup_{\varepsilon\to 0} \frac{\log N(G,\varepsilon)}{-\log \varepsilon} \tag{0.6}$$

and the packing dimension of G is defined as

$$\dim_{\mathrm{P}}(G) = \inf\left\{ \sup_n \overline{\dim}_{\mathrm{M}} G_n : \quad G \subset \bigcup_{n=1}^{\infty} G_n \right\}, \tag{0.7}$$

where the infimum is taken over all countable coverings $\{G_n\}$ of G. It is well known that $0 \le \dim_{\mathrm{H}}(G) \le \dim_{\mathrm{P}}(G) \le \overline{\dim}_{\mathrm{M}}(G) \le 1$ for every set $G \subset \mathbb{R}$. Similarly to Hausdorff dimension, packing dimension has been shown to be a useful tool for characterizing fractal sets and for studying "roughness" of stochastic processes. We refer to Falconer [**6**] and Mattila [**28**] for further properties of packing dimension and to Taylor [**32**] and Xiao [**34**] for extensive surveys on its applications to random fractals.

The rest of this paper is organized as follows. In Section 2 we state the main results and provide some discussions and examples. The proofs of the theorems are given in Section 3 and they rely on the general method on limsup random fractals in Khoshnevisan, Peres and Xiao [**24**]. We remark that our argument extends that in [**24**] and shows that their Theorems 3.1, 3.3 and corollaries still hold if their

Condition 4 is replaced by a weaker condition. Finally Section 4 contains some technical results on the upper Besicovitch-Taylor index. In particular we apply the results in Lapidus and van Frankenhuysen [**25**] to show that every sequence $\{l_n\}$ associated to a self-similar string has its upper Besicovitch-Taylor index equal to the self-similarity dimension and satisfies the condition (C) in this paper.

1. Main results and examples

Throughout this paper we assume $\sum_{n=1}^{\infty} l_n < \infty$. Thus the Lebesgue measure of the Dvoretzky covering set E is 0 almost surely.

Let α be the upper Besicovitch-Taylor index of $\{l_n\}$. Then by Proposition 3.1 below, we have

$$\alpha = \limsup_{k\to\infty} \frac{\log_2 n_k}{k}, \tag{1.1}$$

where $\log_2$ is the logarithm in base 2 and n_k is defined as

$$n_k = \#\Big\{n \in \mathbb{N} : l_n \in [2^{-k+1}, 2^{-k+2})\Big\} \qquad (k \geq 2).$$

Here $\#A$ denotes the cardinality of the set A.

To state the main results of this paper we will make use of the following condition (C):

(C) There exists an increasing sequence of positive integers $\{k_i\}$ such that

$$\lim_{i\to\infty} \frac{k_{i+1}}{k_i} = 1 \tag{1.2}$$

and

$$\lim_{i\to\infty} \frac{\log_2 n_{k_i}}{k_i} = \alpha < 1. \tag{1.3}$$

Theorem 1.1. *Let E be the Dvoretzky covering set associated with the sequence $\{l_n\}$ whose upper Besicovitch-Taylor index is α. If the condition (C) holds, then for every analytic set $G \subset [0,1)$, we have*

$$\mathbb{P}\big(E \cap G \neq \emptyset\big) = \begin{cases} 0 & \textit{if } \dim_{\mathrm{P}}(G) < 1-\alpha, \\ 1 & \textit{if } \dim_{\mathrm{P}}(G) > 1-\alpha. \end{cases}$$

Remark 1.2. Some remarks are in order.

(i) It is clear that if

$$\lim_{k\to\infty} \frac{\log_2 n_k}{k} = \alpha < 1, \tag{1.4}$$

then condition (C) holds. We will give several interesting examples of sequences $\{l_n\}$ that satisfy (1.4).

(ii) If G is regular in the sense that $\underline{\dim}_{\mathrm{M}}(G) = \overline{\dim}_{\mathrm{M}}(G)$, where $\underline{\dim}_{\mathrm{M}}(G)$ is the lower box dimension of G, which is defined by replacing limsup in (0.6) by liminf, then condition (C) is surplus. This follows from the proof of Theorem 1.1 below, in which we can take $\mathfrak{N} = \{k_{i_0}, k_{i_0+1}, \dots\}$ for some $i_0 \geq 1$.

(iii) From the first part of the proof of Theorem 1.1, we see that the conclusion $\dim_{\mathrm{P}}(G) < 1-\alpha$ implies $\mathbb{P}(E \cap G \neq \emptyset) = 0$, even without the condition (C).

(iv) By Proposition 3.3 in Section 4, we see that (1.4) can be replaced by the following: there exists a constant $b \in (1, 2]$ such that

$$\lim_{k\to\infty} \frac{\log_b m_k}{k} = \alpha < 1, \tag{1.5}$$

where $m_k = \#\{n \in \mathbb{N} : l_n \in [b^{-k+1}, b^{-k+2})\}$.

(v) When $\sum_{n=1}^{\infty} l_n = \infty$, but Shepp's condition (0.2) is not satisfied, then $E \neq [0, 1)$. One can consider the random set $F_\infty = [0, 1)\backslash E$ of the uncovered points. The fractal dimension and hitting probabilities have been studied by Hawkes [**12**] (see also Kahane [**20**, Chapter 11]) and have been shown to be very different from Theorem 1.1.

We can extend Theorem 1.1 to the following, which describes the intersection of two independent random covering sets of indices $\alpha, \alpha' < 1$.

Theorem 1.3. *Let E and E' be two independent Dvoretzky covering sets on the same probability space, associated to the sequences $\{l_n\}$ and $\{l'_n\}$ respectively. Suppose both $\{l_n\}$ and $\{l'_n\}$ satisfy the condition (C) with the corresponding upper Besicovitch-Taylor indices α, $\alpha' < 1$ and possibly different subsequences $\{k_i\}$ and $\{k'_i\}$. Then for any analytic set $G \subset [0, 1)$ satisfying $\dim_{\mathrm{P}}(G) > 1 - \min\{\alpha, \alpha'\}$, we have*

$$\mathbb{P}\big(E \cap E' \cap G \neq \emptyset\big) = 1.$$

In particular, if $\dim_{\mathrm{P}}(G) > 1 - \alpha$, then

$$\dim_{\mathrm{P}}(E \cap G) = \dim_{\mathrm{P}}(G) \qquad a.s.$$

In the following we provide an estimate on the Hausdorff dimension of the intersection $E \cap G$ for a given set G.

Theorem 1.4. *Let E be the Dvoretzky covering set associated with the sequence $\{l_n\}$ which satisfies the condition (C). Then for any analytic set $G \subset [0, 1)$, we have*

$$\dim_{\mathrm{H}}(G) - (1-\alpha) \leq \dim_{\mathrm{H}}(E \cap G) \leq \dim_{\mathrm{P}}(G) - (1-\alpha) \quad a.s. \tag{1.6}$$

By taking $G = [0, 1)$ in Theorems 1.3 and 1.4, we obtain $\dim_{\mathrm{H}}(E) = \alpha$ and $\dim_{\mathrm{P}}(E) = 1$ almost surely. This recovers the result (0.3) of Durand [**4**], under the extra condition (C). We remark that our method is different from that of Durand [**4**].

Corollary 1.5. Assume the conditions of Theorem 1.4 hold. For any analytic set $G \subset [0, 1)$ satisfying $\dim_{\mathrm{H}}(G) = \dim_{\mathrm{P}}(G)$, we have

$$\dim_{\mathrm{H}}(E \cap G) = \dim_{\mathrm{H}}(G) - (1-\alpha) \quad a.s.$$

In particular $\dim_{\mathrm{H}}(E) = \alpha$ almost surely.

We end this section with some examples.

Example 1.6. 1. If $l_n \sim c\,n^{-\gamma}$, where $c > 0$ and $\gamma > 1$ are constants and $l_n \sim j_n$ means $\lim\limits_{n\to\infty} \frac{l_n}{j_n} = 1$, then $\{l_n\}$ satisfies (1.4) with $\alpha = \frac{1}{\gamma}$. Hence Theorem 2.1 provides results on hitting probabilities for the associated Dvoretzky covering set E. In particular, we have $\dim_{\mathrm{P}}(E) = 1$. This complements the results in Fan and Wu (2004) on the Hausdorff dimension of E. More generally, we can take $l_n \sim c_1\, n^{-\gamma}$ for even integers n, while $l_n \sim c_2\, n^{-\gamma'}$ for odd integers n, where both constants γ and γ' are larger than 1. Such sequence satisfies (1.4) with $\alpha = \max\{\gamma^{-1}, \gamma'^{-1}\}$.

Thus, by Durand [**4**] or by our Corollary 1.5, $\dim_{\rm H}(E) = \alpha < 1$. On the other hand, by [**4**] or by our Theorem 1.3, $\dim_{\rm P}(E) = 1$.

2. Let $\{l_n\}$ be the sequence corresponding to the complimentary open intervals of the tertiary Cantor set. That is, $l_n = 3^{-m}$ when $\sum_{i=0}^{m-1} 2^i < n \le \sum_{i=0}^{m} 2^i$ ($m = 1, 2, \ldots$). Then it can be verified directly that the upper Besicovitch-Taylor index $\alpha = \log 2/\log 3$ and, moreover, (1.4) holds. Hence our results are applicable to the corresponding random covering set E. Consequently, $\dim_{\rm H}(E) = \log 2/\log 3$ and $\dim_{\rm P}(E) = 1$ almost surely. Similar results hold for the random covering sets associated with more general self-similar sets (or self-similar strings, in the terminology of Lapidus and van Frankenhuysen [**25**]). See Proposition 3.2.

3. If $l_n = a^{-n}$, where $a > 1$ is a constant, then $\{l_n\}$ satisfies the condition (1.5) with $\alpha = 0$, by Remark 1.2 (iv), more generally, Proposition 3.3. Hence Theorem 1.1 holds for such $\{l_n\}$. In particular, we have $\dim_{\rm H} E = 0$ and $\dim_{\rm P}(E) = 1$ almost surely.

4. Finally we provide a simple example of $\{l_n\}$ that satisfies condition (C), but not (1.4). Let $\beta > \log 3/\log 2$ be a constant. We define

$$l_n = \begin{cases} 3^{-m} & \text{if } \sum_{i=0}^{m-1} 2^i < n \le \sum_{i=0}^{m-1} 2^i + 2^{m-1}, \\ n^{-\beta} & \text{if } \sum_{i=0}^{m-1} 2^i + 2^{m-1} < n \le \sum_{i=0}^{m} 2^i. \end{cases}$$

Then we can verify that condition (C) is satisfied with $\alpha = \log 2/\log 3$ and the subsequence $k_i = \lfloor(\log_2 3)i\rfloor$, where $\lfloor x\rfloor$ denotes the largest integer $\le x$. However, (1.4) fails. Nevertheless, the theorems in this section are applicable to the corresponding Dvoretzky covering set E.

2. Proofs of the theorems

In this section we prove Theorems 1.1, 1.3 and Theorem 1.4. It will be clear that the method for studying limsup random fractals in Khoshnevisan, Peres and Xiao [**24**] plays an essential role in our proofs. We remark that, even though the second half of the proof of Theorem 1.1 is a modification of the proof of Theorem 3.1 in [**24**], our argument is more general and proves that Theorems 3.1 and 3.2 and their corollaries in [**24**] still hold if their **Condition 4** is replaced by

Condition 4′: For some constant $\gamma > 0$,

$$\limsup_{k\to\infty} \frac{\log_2 p_k}{k} = -\gamma$$

and there exists an increasing sequence of positive integers $\{k_i\}$ satisfying (1.2) such that

$$\lim_{i\to\infty} \frac{\log_2 p_{k_i}}{k_i} = -\gamma.$$

For proving Theorem 1.1 (and for extending the results in [**24**]) we will use the following elementary lemma on upper box dimension.

Lemma 2.1. *Let $\{k_i\}$ be an increasing sequence of positive integers which satisfies (1.2). Then for any bounded set $G \subset \mathbb{R}$,*

$$\overline{\dim}_{\rm M}(G) = \limsup_{i\to\infty} \frac{\log_2 N(G, 2^{-k_i})}{k_i}. \tag{2.1}$$

PROOF. For any $\varepsilon > 0$, there is an integer i such that $2^{-k_{i+1}} < \varepsilon \le 2^{-k_i}$. Thus for any bounded set $G \subset \mathbb{R}$ we have

$$N(G, 2^{-k_i}) \le N(G, \varepsilon) \le N(G, 2^{-k_{i+1}}).$$

This implies that

$$\frac{\log N(G, 2^{-k_i})}{k_i \log 2} \frac{k_i}{k_{i+1}} \le \frac{\log N(G, \varepsilon)}{-\log \varepsilon} \le \frac{\log N(G, 2^{-k_{i+1}})}{k_{i+1} \log 2} \frac{k_{i+1}}{k_i}.$$

It is clear that (2.1) follows from the above and (1.2). □

Now we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Firstly we show that $\dim_{\mathrm{P}}(G) < 1-\alpha$ implies $\mathbb{P}(E \cap G \neq \emptyset) = 0$. By (0.7), it suffices to show that whenever $\overline{\dim}_{\mathrm{M}}(G) < 1 - \alpha$, then $E \cap G = \emptyset$, a.s.

Fix an arbitrary but small $\eta > 0$ such that $\overline{\dim}_{\mathrm{M}}(G) < 1 - \alpha - \eta$. For any $r > 0$, denote by $\mathcal{C}_r = \mathcal{C}_r(G)$ a collection of the smallest number of the intervals with length r that cover the set G. Let $\mathcal{N}_r(G) = \#\mathcal{C}_r$. Since

$$\limsup_{n\to\infty} \frac{\log \mathcal{N}_{l_n}(G)}{-\log l_n} \le \limsup_{r\to 0} \frac{\log \mathcal{N}_r(G)}{-\log r} = \overline{\dim}_{\mathrm{M}}(G) < 1 - \alpha - \eta,$$

there exists an integer $n_0 \in \mathbb{N}$ such that

$$\mathcal{N}_{l_n}(G) < l_n^{-(1-\alpha-\eta)} \tag{2.2}$$

for all $n \ge n_0$. For any interval J in $[0, 1)$ with length l_n, since ω_n is uniformly distributed on $[0, 1)$, we have

$$\mathbb{P}\{I_n \cap J \neq \emptyset\} \le 3l_n.$$

Note that

$$\{I_n \cap G \neq \emptyset\} \subset \bigcup_{J \in \mathcal{C}_{l_n}} \{I_n \cap J \neq \emptyset\},$$

we derive from this and (2.2) that

$$\mathbb{P}\{I_n \cap G \neq \emptyset\} \le \sum_{J \in \mathcal{C}_{l_n}} \mathbb{P}\{I_n \cap J \neq \emptyset\} \le \mathcal{N}_{l_n}(G) \cdot 3l_n < 3l_n^{\alpha+\eta}$$

for all $n \ge n_0$. Hence the series $\sum_{n=1}^{\infty} \mathbb{P}\{I_n \cap G \neq \emptyset\}$ converges by the definition of α and $\eta > 0$. By the Borel-Cantelli Lemma, we have

$$\mathbb{P}\{I_n \cap G \neq \emptyset \text{ i.o.}\} = 0.$$

That is, $\mathbb{P}\{\exists n_0, \text{ s.t. } \forall n \ge n_0, I_n \cap G = \emptyset\} = 1$. Therefore, $E \cap G = \emptyset$ a.s.

In the following, we prove that if $\dim_{\mathrm{P}}(G) > 1 - \alpha$, then

$$\mathbb{P}(E \cap G \neq \emptyset) = 1.$$

For this purpose, we construct a random subset $E_* \subset E$ and show that $\mathbb{P}(E_* \cap G \neq \emptyset) = 1$. The random subset E_* is a discrete limsup random fractal as in [**24**]. Our proof below is a modification and extension of the method in their Section 3 and is divided into two steps.

(i) **Construction**. For any $k \ge 2$, let $\mathscr{D}_k$ be the collection of dyadic intervals of the form $(\frac{i}{2^k}, \frac{i+1}{2^k})$, $i = 3, 4, \ldots, 2^k - 1$. Denote by $\mathfrak{T}_k = \{n \in \mathbb{N} : l_n \in [2^{-k+1}, 2^{-k+2})\}$ and let $n_k = \#\mathfrak{T}_k$.

For every $J \in \mathscr{D}_k$, define

$$Z_k(J) = \begin{cases} 1 & \text{if } \exists\, n \in \mathfrak{T}_k \text{ such that } J \subset I_n = (\omega_n, \omega_n + l_n), \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$A(k) = \bigcup_{\substack{J \in \mathscr{D}_k \\ Z_k(J)=1}} J$$

be the union of open dyadic intervals of order k that are contained in some I_n with length $l_n \in [2^{-k+1}, 2^{-k+2})$. Observe that

$$A(k) \subset \bigcup_{n \in \mathfrak{T}_k} I_n.$$

We define $E_* := \limsup_{k\to\infty} A(k)$. From the above, we have $E_* \subset E$.

(ii) **Hitting probability**. Now let $G \subset [0,1)$ be an analytic set such that $\dim_{\mathrm{P}}(G) > 1-\alpha$. Then by Joyce and Preiss [**17**], we can find a closed set $G_* \subset G$, such that for all open set V, we have $\overline{\dim}_{\mathrm{M}}(G_* \cap V) > 1-\alpha$, whenever $G_* \cap V \neq \emptyset$.

In the following, we show $\mathbb{P}\big(E_* \cap G_* \neq \emptyset\big) = 1$. Our method is a modification and extension of that in Section 3 of [**24**].

For every $J \in \mathscr{D}_k$, the probability

$$\mathbb{P}\big(Z_k(J) = 1\big) = \mathbb{P}\big\{\exists\, n \in \mathfrak{T}_k \ \text{ such that } \ J \subset (\omega_n, \omega_n + l_n)\big\}$$

does not depend on J due to our assumption on $\{\omega_n\}$ and our definition of $\mathscr{D}_k$. Denote the above probability by P_k. Then

$$P_k \le n_k(l_n - 2^{-k}) \le 3\, n_k 2^{-k}. \tag{2.3}$$

On the other hand,

$$\begin{aligned} P_k = \mathbb{P}\left(\bigcup_{n\in\mathfrak{T}_k} \{J \subset I_n\}\right) &\ge \sum_{n\in\mathfrak{T}_k} \mathbb{P}(I_n \supset J) - \sum_{m\in\mathfrak{T}_k} \sum_{\substack{n\in\mathfrak{T}_k \\ n\neq m}} \mathbb{P}\big(I_m \supset J,\, I_n \supset J\big) \\ &\ge n_k\big(l_n - 2^{-k}\big) - 9n_k^2\, 2^{-2k} \\ &\ge n_k\, 2^{-k}(1 - 9n_k 2^{-k}). \end{aligned} \tag{2.4}$$

In the above, we have used the independence of I_m and I_n $(m \neq n)$ to derive the second inequality. Combining (2.3) and (2.4), together with (1.1) and Condition (C), we derive that

$$\limsup_{k\to\infty} \frac{\log_2 P_k}{k} = -(1-\alpha) \tag{2.5}$$

and there is an increasing sequence of integers $\{k_i\}$ that satisfies (1.2) such that

$$\lim_{i\to\infty} \frac{\log_2 P_{k_i}}{k_i} = -(1-\alpha). \tag{2.6}$$

Hence E_* is a limsup random fractals which satisfies **Condition 4$'$** with $\gamma = 1-\alpha$ (which is weaker than **Condition 4** in Khoshnevisan, Peres and Xiao [**24**, p.11]). Still using their terminology, we call E_* a limsup random fractal of index $1-\alpha$.

Next we verify their **Condition 5** regarding the correlation of $Z_k(J)$ and $Z_k(J')$ in [**24**]. Given J and $J' \in \mathscr{D}_k$ such that the distance $d(J,J') \geq 2^{-k+2}$. Since

$$\begin{aligned} \mathrm{Cov}\big(Z_k(J), Z_k(J')\big) &= \mathbb{E}\big(Z_k(J)Z_k(J')\big) - \mathbb{E}\big(Z_k(J)\big)\mathbb{E}\big(Z_k(J')\big) \\ &= \mathbb{E}\big(Z_k(J)Z_k(J')\big) - P_k^2, \end{aligned} \tag{2.7}$$

we estimate $\mathbb{E}(Z_k(J)Z_k(J'))$ first,

$$\begin{aligned} \mathbb{E}(Z_k(J)Z_k(J')) &= \mathbb{P}\big(Z_k(J) = 1,\ Z_k(J') = 1\big) \\ &= \mathbb{P}\big\{\exists\ m, n \in \mathfrak{T}_k \text{ such that } I_m \supset J \ \text{ and } \ I_n \supset J'\big\} \\ &\leq \sum_{m \in \mathfrak{T}_k} \sum_{\substack{n \in \mathfrak{T}_k, \\ n \neq m}} \mathbb{P}\big(I_m \supset J, I_n \supset J'\big) \\ &= \Big(\sum_{m \in \mathfrak{T}_k} \mathbb{P}\big(I_m \supset J\big)\Big)\Big(\sum_{\substack{n \in \mathfrak{T}_k, \\ n \neq m}} \mathbb{P}\big(I_n \supset J'\big)\Big). \end{aligned} \tag{2.8}$$

By (2.7), (2.8) and the first inequality in (2.4) we derive

$$\begin{aligned} \mathrm{Cov}(Z_k(J), Z_k(J')) &\leq 2\Big(\sum_{m \in \mathfrak{T}_k} \mathbb{P}\big(I_m \supset J\big)\Big) \cdot \sum_{m \in \mathfrak{T}_k} \sum_{\substack{n \in \mathfrak{T}_k \\ n \neq m}} \mathbb{P}\big(I_m \supset J, I_n \supset J'\big) \\ &\leq C\,\big(n_k\, 2^{-k}\big)\mathbb{E}\big(Z_k(J)\big)\mathbb{E}\big(Z_k(J')\big), \end{aligned} \tag{2.9}$$

where the last inequality follows from (2.3) and $C > 0$ is a finite constant. It follows from (2.9) and (1.1) that for any $\varepsilon > 0$

$$\mathrm{Cov}\big(Z_k(J), Z_k(J')\big) < \varepsilon\, \mathbb{E}\big(Z_k(J)\big)\mathbb{E}\big(Z_k(J')\big)$$

for all k large enough. This implies that $f(k,\varepsilon) \leq 8$, where

$$f(k,\varepsilon) = \max_{J \in \mathscr{D}_k} \#\big\{J' \in \mathscr{D}_k : \mathrm{Cov}(Z_k(J),\ Z_k(J')) \geq \varepsilon \mathbb{E}(Z_k(J))\mathbb{E}(Z_k(J'))\big\}.$$

In particular,

$$\lim_{k \to \infty} \frac{\log f(k,\varepsilon)}{k} = 0.$$

Thus we have shown that **Condition 5** in [**24**] is satisfied with $\delta = 0$.

The rest of the proof follows a similar line as in the proof of Theorem 3.1 in [**24**]. For convenience of the reader, we give it below. Notice that our set $\mathfrak{N}$ is determined by Condition (C) and may be different from that in [**24**].

Fix an open set $V \subset [0,1)$ such that $G_* \cap V \neq \emptyset$. Let $\mathcal{N}_k$ be the number of dyadic intervals $J \in \mathscr{D}_k$ such that

$$J \cap G_* \cap V \neq \emptyset. \tag{2.10}$$

Since $\overline{\dim}_{\mathrm{M}}(G_* \cap V) > 1 - \alpha$, we use Lemma 2.1 to derive that, for any $\beta \in \big(1-\alpha, \overline{\dim}_{\mathrm{M}}(G_* \cap V)\big)$, $\mathcal{N}_{k_i} \geq 2^{k_i \beta}$ for infinitely many integers i. This implies the set $\mathfrak{N}$ defined as

$$\mathfrak{N} := \{i \geq 1 : \mathcal{N}_{k_i} \geq 2^{k_i \beta}\} \tag{2.11}$$

satisfies $\#\mathfrak{N} = \infty$. Similarly to [**24**], we define

$$S_i = \sum_{\substack{J \in \mathscr{D}_{k_i} \\ J \cap G_* \cap V \neq \emptyset}} Z_{k_i}(J).$$

Namely, S_i is the total number of intervals $J \in \mathscr{D}_{k_i}$ such that

$$J \cap G_* \cap V \cap A(k_i) \neq \emptyset.$$

We now show $\mathbb{P}\big(S_i > 0 \text{ i.o.}\big) = 1$.

To this end, we estimate

$$\mathrm{Var}(S_i) = \sum_{\substack{J \in \mathscr{D}_{k_i} \\ J \cap G_* \cap V \neq \emptyset}} \sum_{\substack{J' \in \mathscr{D}_{k_i} \\ J' \cap G_* \cap V \neq \emptyset}} \mathrm{Cov}\big(Z_{k_i}(J), Z_{k_i}(J')\big).$$

Fix $\varepsilon > 0$, for each $J \in \mathscr{D}_{k_i}$ which satisfies (2.10), let $\mathscr{G}_{k_i}(J)$ be the collection of all $J' \in \mathscr{D}_{k_i}$ such that

(i) $J' \cap G_* \cap V \neq \emptyset$, and

(ii) $\mathrm{Cov}\big(Z_{k_i}(J), Z_{k_i}(J')\big) \leq \varepsilon P_{k_i}^2$.

If $J' \in \mathscr{D}_{k_i}$ satisfies (i), but not (ii), then we say $J' \in \mathscr{B}_{k_i}(J)$. Thus

$$\begin{aligned} \mathrm{Var}(S_i) &\leq \varepsilon \mathcal{N}_{k_i}^2 P_{k_i}^2 + \sum_{\substack{J \in \mathscr{D}_{k_i} \\ J' \in \mathscr{B}_{k_i}(J)}} \mathrm{Cov}(Z_{k_i}(J), Z_{k_i}(J')) \\ &\leq \varepsilon \mathcal{N}_{k_i}^2 P_{k_i}^2 + \mathcal{N}_{k_i} \max_{J \in \mathscr{D}_{k_i}} \#\mathscr{B}_{k_i}(J) P_{k_i}, \end{aligned}$$

where the last term comes from the fact that $\mathrm{Cov}(Z_k(J), Z_k(J')) \leq \mathbb{E}\big(Z_k(J)\big) = P_k$. Since we have shown $\max_{J \in \mathscr{D}_k} \#\mathscr{B}_k(J) \leq 8$ for all k large enough, the above implies

$$\limsup_{\substack{i \to \infty \\ i \in \mathfrak{N}}} \frac{\mathrm{Var}(S_i)}{[\mathbb{E}(S_i)]^2} \leq \varepsilon + \limsup_{\substack{i \to \infty \\ i \in \mathfrak{N}}} \frac{\max_{J \in \mathscr{D}_{k_i}} \#\mathscr{B}_{k_i}(J)}{\mathcal{N}_{k_i} P_{k_i}} = \varepsilon.$$

In the above, we have used that facts that $\mathbb{E}(S_i) = \mathcal{N}_{k_i} P_{k_i}$ and $\mathcal{N}_{k_i} P_{k_i} \to \infty$ if $i \in \mathfrak{N}$ and $i \to \infty$ (recall (2.6) and (2.11)). Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{\substack{i \to \infty \\ i \in \mathfrak{N}}} \frac{\mathrm{Var}(S_i)}{[\mathbb{E}(S_i)]^2} = 0. \tag{2.12}$$

It follows from the Paley-Zygmund inequality ([**20**, p.8]) that

$$\begin{aligned} \mathbb{P}\big(S_i > 0\big) &\geq \frac{(\mathbb{E}(S_i))^2}{\mathbb{E}(S_i^2)} \\ &= 1 - \frac{\mathrm{Var}(S_i)}{\mathbb{E}(S_i^2)} \geq 1 - \frac{\mathrm{Var}(S_i)}{\big[\mathbb{E}(S_i)\big]^2}. \end{aligned}$$

Combining the above inequality, (2.12) and Fatou's Lemma, we derive

$$\mathbb{P}\big(S_i > 0 \ \text{ i.o.}\big) \geq \limsup_{i \to \infty} \mathbb{P}\big(S_i > 0\big) = 1. \tag{2.13}$$

It follows from (2.13) that

$$\mathbb{P}\left\{ \left(\bigcup_{k=n}^{\infty} A(k) \right) \cap G_* \cap V \neq \emptyset, \quad \forall n \geq 1 \right\} = 1$$

for every open set V with $G_* \cap V \neq \emptyset$. Letting V run over all open interval with rational endpoints, we obtain that $(\cup_{k=n}^{\infty} A(k)) \cap G_*$ is a.s. dense in G_* for all $n \geq 1$. Since

$$\left(\bigcup_{k=n}^{\infty} A(k) \right) \cap G_*$$

is an open set in G_* and G_* is a complete metric space, by Baire's category theorem (see Munkres [**29**]), we know $\cap_{n=1}^{\infty}(\cup_{k=n}^{\infty}A(k)) \cap G_*$ is a.s. dense in G_*, that is, $E_* \cap G_*$ is a.s. dense in G_*. In particular, $E_* \cap G_* \neq \emptyset$ a.s. This finishes the proof of Theorem 1.1. □

Proof of Theorem 1.3. We use the same method as in the proof of Theorem 3.2 in [**24**]. Let G_* be the closed subset of G described in the proof of Theorem 1.1. Suppose $\dim_{\mathrm{P}}(G) > 1 - \min\{\alpha, \alpha'\}$, the proof of Theorem 1.1 shows that for any open set V such that $V \cap G_* \neq \emptyset$ we have

$$\mathbb{P}\Big(\Big(\bigcup_{k=n}^{\infty} I_k\Big) \cap V \cap G_* \neq \emptyset, \quad \forall n \geq 1\Big) = \mathbb{P}\Big(\Big(\bigcup_{k=n}^{\infty} I'_k\Big) \cap V \cap G_* \neq \emptyset, \quad \forall n \geq 1\Big) = 1.$$

By independence, there exists a single null probability event outside which for all open intervals V with rational endpoints satisfying $V \cap G_* \neq \emptyset$, we have

$$\Big(\bigcup_{k=n}^{\infty} I_k\Big) \cap V \cap G_* \neq \emptyset \quad \text{and} \quad \Big(\bigcup_{k=n}^{\infty} I'_k\Big) \cap V \cap G_* \neq \emptyset \ \text{ for all } \ n \geq 1.$$

That is, $\big\{\big(\cup_{k=n}^{\infty} I_k\big) \cap G_*\}_{n\geq 1} \cup \big\{\big(\cup_{k=n}^{\infty} I'_k\big) \cap G_*\big\}_{n\geq 1}$ is a countable collection of open, dense subsets of the complete metric space G_*. Again, Baire's theorem implies that

$$\mathbb{P}\big(E \cap E' \cap G_* \ \text{ is dense in } G_*\big) = 1.$$

In particular, $E \cap E' \cap G_* \neq \emptyset$ a.s. That is, $\mathbb{P}(E \cap E' \cap G \neq \emptyset) = 1$. This proves the first part of Theorem 1.3.

In order to prove the second half, we regard the set $E \cap G$ as the target set with respect to the random covering set E'. By Theorem 1.1, we know that $\mathbb{P}(E' \cap E \cap G \neq \emptyset) = 1$ implies $\dim_{\mathrm{P}}(E \cap G) \geq 1 - \alpha'$ a.s. Therefore, from the above we see that $\dim_{\mathrm{P}}(G) > 1 - \min\{\alpha, \alpha'\}$ implies $\dim_{\mathrm{P}}(E \cap G) \geq 1 - \alpha'$ a.s.

Now we assume $\dim_{\mathrm{P}}(G) > 1 - \alpha$. For any α' with $1 - \dim_{\mathrm{P}}(G) < \alpha' < \alpha$, that is, $\dim_{\mathrm{P}}(G) > 1 - \min\{\alpha, \alpha'\}$, we have $\dim_{\mathrm{P}}(E \cap G) \geq 1 - \alpha'$ a.s. Letting α' tend to $1 - \dim_{\mathrm{P}}(G)$ along rational numbers, we obtain

$$\dim_{\mathrm{P}}(E \cap G) \geq \dim_{\mathrm{P}}(G) \quad \text{a.s.}$$

Therefore, $\dim_{\mathrm{P}}(E \cap G) = \dim_{\mathrm{P}}(G)$ a.s. □

Proof of Theorem 1.4. Firstly, we prove the right-hand inequality in (1.6). By (0.7), it suffices to prove that

$$\dim_{\mathrm{H}}(E \cap G) \leq \overline{\dim}_{\mathrm{M}}(G) - (1 - \alpha) \quad a.s. \tag{2.14}$$

Denote by $\mathcal{C}_{l_n}$ a collection of the smallest number of the intervals with length l_n, the union of such intervals covers the set G. Let $\mathcal{N}_{l_n}(G) = \#\mathcal{C}_{l_n}$. Since $\xi := \overline{\dim}_{\mathrm{M}}(G) \geq \limsup_{n\to\infty} \frac{\log \mathcal{N}_{l_n}(G)}{-\log l_n}$, we have

$$\mathcal{N}_{l_n}(G) < l_n^{-(\xi+\varepsilon)}$$

as n large enough, say $n \geq n_1(\varepsilon)$, where $\varepsilon > 0$ is an arbitrary small real number. Let $\mathscr{G}_n$ be the collection of the intervals $J \in \mathcal{C}_{l_n}$ such that $J \cap I_n \neq \emptyset$ and denote $T_n = \#\mathscr{G}_n$. For any $J \in \mathscr{G}_n$, $\mathbb{P}(I_n \cap J \neq \emptyset) \leq 3l_n$. Thus

$$\mathbb{E}(T_n) \leq \sum_{J\in\mathscr{G}_n} \mathbb{P}(I_n \cap J \neq \emptyset) \leq 3\mathcal{N}_{l_n}(G)l_n \leq 3l_n^{1-\xi-\varepsilon}.$$

For any $\theta > \xi - (1-\alpha)$, we choose $\varepsilon > 0$ such that $2\varepsilon < \theta - \xi + (1-\alpha)$, then

$$\mathbb{E}\bigg(\sum_{n=n_1(\varepsilon)}^{\infty} T_n\, l_n^{\theta}\bigg) < 3 \sum_{n=n_1(\varepsilon)}^{\infty} l_n^{1-\xi-\varepsilon} l_n^{\xi-(1-\alpha)+2\varepsilon} = 3 \sum_{n=n_1(\varepsilon)}^{\infty} l_n^{\alpha+\varepsilon} < \infty.$$

Thus $\mathbb{E}(\sum_{n=1}^{\infty} T_n\, l_n^{\theta}) < \infty$. It follows that $\sum_{n=1}^{\infty} T_n\, l_n^{\theta} < \infty$ a.s.

For any $m \geq 1$, the collection $\{J \in \mathscr{G}_n\}_{n\geq m}$ is a covering of the set $E \cap G$, then

$$\mathcal{H}^{\theta}(E \cap G) \leq \sum_{n=m}^{\infty} T_n\, l_n^{\theta} < \infty \quad a.s.,$$

which implies $\dim_{\mathrm{H}}(E \cap G) \leq \theta$ a.s. Since $\theta > \overline{\dim}_{\mathrm{M}}(G) - (1-\alpha)$ is arbitrary, this proves that (2.14) holds.

The left-hand inequality in (1.6) can be derived from Theorem 1.1 and the following Lemma, due to Khoshnevisan, Peres, and Xiao [**24**] (Lemma 3.4 with $N = 1$ and $\gamma = 1 - \alpha$). The proof of Theorem 1.3 completed. □

LEMMA 2.2. *Equip* $[0, 1]$ *with the Borel* σ*-field. Suppose* $E = E(\omega)$ *is a random set in* $[0, 1]$ *(i.e., the indicator function* $\chi_{E(\omega)}(x)$ *is jointly measurable) such that for any compact set* $F \subset [0, 1]$ *with* $\dim_{\mathrm{H}}(F) > \gamma$*, we have* $\mathbb{P}(E \cap F \neq \emptyset) = 1$*. Then, for any analytic set* $F \subset [0, 1]$*,*

$$\dim_{\mathrm{H}}(F) - \gamma \leq \dim_{\mathrm{H}}(E \cap F) \quad a.s.$$

3. Technical results

The upper Besicovitch-Taylor index (or the convergence exponent) of $\{l_n\}$ plays an essential role in this paper. In this section we provide some equivalent characterizations for this index and elaborate more on the condition (C) and (1.4).

First we show that

PROPOSITION 3.1. For any constant $a > 1$, let $n_k = \#\{n \in \mathbb{N} : l_n \in [a^{-k+1}, a^{-k+2})\}$. Then

$$\alpha = \limsup_{k\to\infty} \frac{\log_a n_k}{k}. \tag{3.1}$$

PROOF. For any $\gamma > \alpha$, we have $\sum_{n=1}^{\infty} l_n^{\gamma} < \infty$ or $\sum_{k=1}^{\infty} n_k a^{-\gamma(k-1)} < \infty$. Thus

$$n_k a^{-\gamma(k-1)} \leq 1$$

for all k large, which implies

$$\limsup_{k\to\infty} \frac{\log_a n_k}{k} \leq \gamma.$$

Hence we have

$$\limsup_{k\to\infty} \frac{\log_a n_k}{k} \leq \alpha.$$

On the other hand, if $\gamma > \limsup_{k\to\infty} \frac{\log_a n_k}{k}$, we choose γ' such that

$$\limsup_{k\to\infty} \frac{\log_a n_k}{k} < \gamma' < \gamma.$$

This implies $n_k \leq a^{k\gamma'}$ for all k large enough, say $k \geq k_0$. Hence

$$\sum_{k=k_0}^{\infty} n_k a^{-k\gamma} \leq \sum_{k=k_0}^{\infty} a^{-k(\gamma-\gamma')} < \infty.$$

This implies $\alpha \leq \gamma$, which proves $\alpha \leq \limsup_{k\to\infty} \frac{\log_a n_k}{k}$. Therefore (3.1) holds. □

For any decreasing sequence $\{l_n\}$ of positive numbers such that $\sum_{n=1}^{\infty} l_n < \infty$, one can associate the following Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} l_n^s = \sum_{n=1}^{\infty} e^{-s\ln(l_n^{-1})},$$

which is called the geometric zeta function in Lapidus and van Franhenhuysen [**25**]. Then the upper Besicovitch-Taylor index α defined by (0.4) is the abscissa of convergence of the above Dirichlet series. On the other hand, denote by $N(x)$ the counting function defined by

$$N(x) = \#\Big\{n : l_n^{-1} \leq x\Big\},$$

see [**25**, p.8]. Then n_k in Proposition 3.1 can be written as $n_k = N(a^{k-1}) - N(a^{k-2})$ for $a > 1$, hence the index α can also be determined by $N(x)$ (we take $a = 2$):

$$\alpha = \limsup_{k\to\infty} \frac{\log_2\left(N(2^{k-1}) - N(2^{k-2})\right)}{k}. \tag{3.2}$$

Thanks to the above we can also apply the results in [**25**] to calculate the upper Besicovitch-Taylor index of a sequence $\{l_n\}$. In the following we focus on sequences which are associated to self-similar sets (or self-similar strings in [**25**]).

Given an integer $M \geq 2$ and constants $r_1, \ldots, r_M \in (0, 1)$ such that

$$1 \geq r_1 \geq r_2 \geq \cdots \geq r_M > 0 \quad \text{and} \quad R = \sum_{i=1}^{M} r_i < 1,$$

one can construct self-similar sets in [0, 1] with scaling ratios $r_1, \ldots, r_M$ (cf. [**6**, **25**, **28**]). Similarly to the tertiary Cantor set in Section 2, we denote the corresponding sequence by $\{l_n\}$, where each l_n is of the form

$$r_1^{k_1} \cdots r_M^{k_M}, \qquad \text{where } k_1, \ldots, k_M \in \mathbb{N}.$$

It can be verified that the multiplicity of the length $r_1^{k_1} \cdots r_M^{k_M}$ in $\{l_n\}$ is the multinomial coefficient $\binom{q}{k_1 \cdots k_M}$, where $q = \sum_{i=1}^{M} k_i$; see [**25**, p.24].

By the proof of Theorem 2.3 in [**25**] we see that the geometric zeta function of $\{l_n\}$ is

$$\zeta(s) = \sum_{q=0}^{\infty} \left(\sum_{i=1}^{M} r_i^s\right)^q, \qquad \forall s \in \mathbb{C}. \tag{3.3}$$

This, together with (0.4), implies the first assertion of Proposition 3.2 below.

The asymptotic behavior of the counting function $N(x)$ for a sequence $\{l_n\}$ associated to a self-similar set has been studied in [**25**] (see also the references therein for further information). We notice that the zeta function $\zeta(s)$ in (3.3) satisfies conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2')$ in [**25**, p.80] with $\kappa = 0$ and $A = r_M$ (see [**25**, pp.121–122]). Hence we can apply Theorem 4.8 in [**25**, p.88] to obtain that

$$N(x) = \sum_{\omega \in \mathcal{D}(\mathbb{C})} \operatorname{res}\left(\frac{x^s \zeta(s)}{s}; \omega\right) + \text{constant} \tag{3.4}$$

for all $x > r_M$. In the above $\mathcal{D}(\mathbb{C})$ denotes the set of complex dimensions of $\{l_n\}$ (i.e., the set of poles of $\zeta(s)$ or equivalently the set of solutions of the equation $\sum_{i=1}^M r_i^\omega = 1$) and $\operatorname{res}(g(s); \omega)$ denotes the residue of a meromorphic function $g(s)$ at $s = \omega$.

To obtain more explicit information about the terms on the right hand side of (3.4), we distinguish two cases:

Nonlattice case: The additive group generated by $\log r_1, \dots, \log r_M$ is dense in $\mathbb{R}$.

Lattice case: There exists some number $\delta > 0$ such that $\log r_1, \dots, \log r_M \in \delta\mathbb{Z}$. The largest such δ is called the additive generator and is denoted by r [**25**, p.34]. The positive constant $\mathbf{p} = (2\pi)/(\log r^{-1})$ is called the oscillatory period.

In the nonlattice case, it follows from (5.44) in [**25**, p.126] that

$$N(x) = \operatorname{res}(\zeta; \alpha)\frac{x^\alpha}{\alpha} + o(x^\alpha), \qquad \text{as } x \to \infty. \tag{3.5}$$

The lattice case is much simpler since the complex dimension of $\{l_n\}$ are located on finitely many vertical lines [**25**, Theorem 2.13]. It follows from (5.33) and (5.34) in [**25**, pp.122-123] that

$$N(x) = \operatorname{res}(\zeta; \alpha)\frac{b^{1-\{u\}}}{b-1}\,\frac{2\pi}{\mathbf{p}}\,x^\alpha + o(x^\alpha), \qquad \text{as } x \to \infty, \tag{3.6}$$

where $\log b = 2\pi\alpha/\mathbf{p}$, $u = \mathbf{p}\log x/2\pi$, $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.

By (3.5), (3.6) and (3.2) we derive

$$\lim_{k\to\infty} \frac{\log_2\left(N(2^{k-1}) - N(2^{k-2})\right)}{k} = \alpha. \tag{3.7}$$

In other words, (1.4) always holds for a self-similar sequence $\{l_n\}$.

Hence we have proved the following proposition.

Proposition 3.2. Let $\{l_n\}$ be the sequence associated to a self-similar set with scaling ratios $r_1, \dots, r_M$. Then the upper Besicovitch-Taylor index α of $\{l_n\}$ coincides with the self-similarity dimension D, which is the unique constant satisfying

$$\sum_{i=1}^M r_i^D = 1.$$

Moreover, (1.4) holds.

As an example, we mention the Fibonacci sequence, which is obtained by taking $M = 2$, $r_1 = 1/2$ and $r_2 = 1/4$. Then it can be verified directly that $\alpha = \log_2 \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and its geometric counting function is given by

$$N_{\text{Fib}}(x) = \frac{3+4\phi}{5}\phi^{-\{\log_2 x\}}x^\alpha - 1 + \frac{7-4\phi}{5}\phi^{\{\log_2 x\}}x^{-\alpha}(-1)^{\lfloor \log_2 x \rfloor},$$

see [**25**, p.124]. It can be verified directly that (1.4) holds.

Finally we show that condition (1.4) can be replaced by (1.5), as stated in Remark 1.2 (iv).

PROPOSITION 3.3. For any constants $a > b > 1$, let $m_k = \#\{n : l_n \in [b^{-k+1}, b^{-k+2})\}$ and let $n_k = \#\{n : l_n \in [a^{-k+1}, a^{-k+2})\}$. If $\lim_{k\to\infty} \frac{\log_b m_k}{k} = \alpha$, then

$$\lim_{k\to\infty} \frac{\log_a n_k}{k} = \alpha.$$

PROOF. We state the elementary fact that if $\lim_{k\to\infty} \frac{\log_b m_k}{k} = \alpha$, then for any fixed integer $\tau_0 \geq 1$, we have

$$\lim_{k\to\infty} \frac{\log_b(m_k + m_{k+1} + \cdots + m_{k+\tau_0})}{k} = \alpha. \tag{3.8}$$

This can be verified by the fact that $b^{(\alpha-\epsilon)k} < m_k < b^{(\alpha+\epsilon)k}$ for all k large implies

$$b^{(\alpha-\epsilon)k} < m_k + m_{k+1} + \cdots + m_{k+\tau_0} < (\tau_0 + 1)b^{(\alpha+\epsilon)(k+\tau_0)}$$

for all k large.

To prove the lemma, observe that

$$l_n \in [a^{-k+1},\, a^{-k+2}) \iff l_n \in [b^{-(\log_b a)(k-1)},\, b^{-(\log_b a)(k-2)}).$$

Hence

$$\begin{aligned} n_k &\leq \#\{n : l_n \in [b^{-\lfloor(\log_b a)(k-1)\rfloor-1},\, b^{-\lfloor(\log_b a)(k-2)\rfloor})\} \\ &= m_{\lfloor(\log_b a)(k-1)\rfloor+2} + \cdots + m_{\lfloor(\log_b a)(k-2)\rfloor+2}, \end{aligned} \tag{3.9}$$

where $\lfloor x \rfloor$ denotes the largest integer $\leq x$, and note that $a > b$, we have

$$\begin{aligned} n_k &\geq \#\{n : l_n \in [b^{-\lfloor(\log_b a)(k-1)\rfloor},\, b^{-\lfloor(\log_b a)(k-2)\rfloor-1})\} \\ &= m_{\lfloor(\log_b a)(k-1)\rfloor+1} + \cdots + m_{\lfloor(\log_b a)(k-2)\rfloor+3}. \end{aligned} \tag{3.10}$$

Since $\lim_{k\to\infty} \frac{\log_b m_{\lfloor(\log_b a)k\rfloor}}{(\log_b a)k} = \alpha$, we derive from (3.8), (3.9) and (3.10) that

$$\lim_{k\to\infty} \frac{\log_a n_k}{k} = \lim_{k\to\infty} \frac{\log_b n_k}{(\log_b a)k} = \alpha.$$

This proves the lemma. $\square$

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