Fernique-type Inequalities and Exact Moduli of Continuity for Anisotropic Gaussian Random Fields

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Abstract

This paper is concerned with limiting properties of anisotropic Gaussian random fields. We establish Fernique-type inequalities and utilize them to study the exact uniform and local moduli of continuity for a wide class of anisotropic Gaussian random fields. The main theorems are applied to fractional Brownian sheets to improve the existing results in the literature.

Keywords: Gaussian random field; anisotropy; fractional Brownian sheet; exact modulus of continuity; law of the iterated logarithm

2000 AMS Classification Numbers: 60G15, 60G17, 60F10, 60F15

[1]Research supported by NSF grants DMS-0417869, DMS-0803360 and EAR-0823965.
[2]Research supported by NSFC grant 10771070 and NSF grant DMS-0417869.
[3]Research supported by NSF grant DMS-0706728.

1 Introduction

Sample path continuity and Hölder regularity of Gaussian random fields have been studied by many authors. A powerful chaining argument leads to sharp upper bounds for uniform and local moduli of continuity of Gaussian processes in terms of metric entropy or majorizing measures. Here "sharp" means logarithmic correction factors can be obtained. We refer to the recent books of Talagrand (2006), Marcus and Rosen (2006) and Adler and Taylor (2007) for systematic accounts and historical information. Sharp lower bounds for local and uniform moduli of continuity of Gaussian processes are discussed in Marcus and Rosen (2006, Chapter 7). However, except for a few special cases such as certain one-parameter Gaussian processes [cf. Marcus and Rosen (2006)], the Brownian sheet [Orey and Pruitt (1973)] and fractional Brownian motion [Benassi *et al.* (1997)], there have not been many explicit results on sharp lower bounds for uniform and local moduli of Gaussian random fields.

Let us first recall some terminology from Marcus and Rosen (2006). Let (T, τ) be a compact metric or pseudo-metric space and let $X = \{X(t), t \in T\}$ be a centered Gaussian random field with values in $\mathbb{R}$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ is called an *exact uniform modulus of continuity for X on (T, τ)* if

$$\lim_{\varepsilon \to 0} \sup_{s,t \in T,\, \tau(s,t) \le \varepsilon} \frac{|X(s) - X(t)|}{\varphi(\tau(s,t))} = C_1 \qquad \text{a.s.} \tag{1.1}$$

for some constant $C_1 \in (0, \infty)$.

A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(0) = 0$ is called an *exact local modulus of continuity for X at $t_0 \in T$* if

$$\lim_{\varepsilon \to 0} \sup_{s \in T,\, \tau(s,t_0) \le \varepsilon} \frac{|X(s) - X(t_0)|}{\psi(\tau(s,t_0))} = C_2 \qquad \text{a.s.} \tag{1.2}$$

for some constant $C_2 \in (0, \infty)$.

Let $W = \{W(t), t \in \mathbb{R}_+^N\}$ be the real-valued Brownian sheet. Orey and Pruitt (1973) considered increments of W over intervals or between two points and established corresponding exact uniform and local moduli of continuity. In particular, their Theorem 2.4 shows that

$$\lim_{\varepsilon \to 0} \sup_{s,t \in [0,1]^N,\, \delta(s,t) \le \varepsilon} \frac{|W(s) - W(t)|}{\varphi_1(\delta(s,t))} = N^{1/2} \qquad \text{a.s.,} \tag{1.3}$$

where $\varphi_1(r) = \big(2r \log(1/r)\big)^{1/2}$ and $\delta(s,t) = \mathbb{E}\big(W(s) - W(t)\big)^2$ which equals to the Lebesgue measure of the symmetric difference between $[0,t]$ and $[0,s]$. Benassi *et al.* (1997) proved, among other things, similar results on uniform and local moduli of continuity for a class of elliptic Gaussian random fields including fractional Brownian motion.

The present paper is concerned with exact uniform and local moduli of continuity for anisotropic Gaussian random fields considered in Xiao (2009). Typical examples of such Gaussian random fields include fractional Brownian sheets [Kamont (1996), Ayache, *et al.* (2002), Ayache and Xiao (2005)], anisotropic Gaussian random fields with stationary increments [Benassi *et al.* (1997), Bonami and Estrade (2003), Biermé *et al.* (2007), Xiao (2009), Xue and Xiao (2009)] and solutions to stochastic partial different equations [see, e.g., Dalang (1999), Mueller and Trible (2002), Øksendal and Zhang (2000), Hu and Nualart (2009)]. We should also mention that, in recent years, there has been a lot of interest in space-time random fields in spatial statistics and related areas. For the purposes of estimation and prediction, it is important to describe the smoothness properties of space-time models. The results in this paper are applicable to various space-time random fields. See Xue and Xiao (2009) and the references therein for further information along this direction.

In order to study exact uniform and local moduli of continuity for anisotropic Gaussian random fields, we first prove some Fernique-type inequalities for anisotropic Gaussian random fields, which may be of interest beyond the scope of the present paper. Then we utilize these inequalities to establish exact uniform modulus of continuity for Gaussian random fields which have the property of *sectorial local nondeterminism*. See Section 2 for its definition.

The rest of this paper is organized as follows. Section 2 specifies the Gaussian random fields under investigation in the present paper. In particular, the properties of sectorial local nondeterminism and strong local nondeterminism are discussed. They will play an important role in proving lower bounds for uniform modulus of continuity. In Section 3, we prove some Fernique-type inequalities for anisotropic Gaussian random fields. Uniform modulus of continuity and local modulus of continuity are treated in Sections 4 and 5 respectively. The main results characterize precisely the anisotropic nature (from an analytic point of view) of Gaussian random fields explicitly in terms of their Hurst parameters. Finally in Section 6, we apply the main results in Sections 4 and 5 to fractional Brownian sheets and some anisotropic Gaussian random fields with stationary increments.

We end the Introduction with some notation. The parameter space is $\mathbb{R}^N$ or $\mathbb{R}^N_+ = [0,\infty)^N$, endowed with the Euclidean norm $\|\cdot\|$. A typical parameter ("time point") is $t = (t_1, ..., t_N)$, sometimes also written as $\langle t_i \rangle$, or $\langle c \rangle$, if $t_1 = \cdots = t_N = c$. The inner product of s, $t \in \mathbb{R}^N$ is denoted by $\langle s, t\rangle$. Given two points $s = \langle s_i \rangle$, $t = \langle t_i \rangle \in \mathbb{R}^N_+$, $s \le t$ (resp. $s < t$) means that $s_i \le t_i$ (resp. $s_i < t_i$) for all $1 \le i \le N$. When $s \le t$, we use $[s, t]$ to denote the N-dimensional *interval* $[s,t] = \times_{i=1}^N [s_i, t_i]$. For $x \in \mathbb{R}_+$, let $\log x = \ln(x \vee e)$, $\log\log x = \ln((\ln x) \vee e)$. Throughout this paper we will use c to denote an unspecified positive and finite constant which may be different in each occurrence. More specific constants in Section i are numbered as $c_{i,1}, c_{i,2}, \ldots$.

2 General assumptions

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}$. Let $I \subset \mathbb{R}^N$ be a fixed compact N-dimensional interval, and our goal is to determine the exact uniform and local moduli of continuity of $X(t)$ when $t \in I$. Typically this paper, we will take $I = [0,1]^N$ or $I = [a,1]^N$, where $a \in (0,1)$ is a fixed constant.

Many sample path properties of the Gaussian random field X can be determined by the function:

$$\sigma^2(s,t) = \mathbb{E}\big(X(s) - X(t)\big)^2, \qquad \forall s, t \in \mathbb{R}^N.$$

In order to describe the anisotropy of X, we make use of the following metric on $\mathbb{R}^N$:

$$\rho(s,t) = \sum_{j=1}^N |s_j - t_j|^{H_j}, \qquad \forall s, t \in \mathbb{R}^N,$$

where $H = (H_1, \ldots, H_N) \in (0,1]^N$ is a fixed vector.

We will assume that the Gaussian random field X satisfies some of the following conditions. Condition (A1) is essential for establishing Fernique-type inequalities, uniform and local moduli of continuity of X. Condition (A2) is useful for the conditioning argument in the proof of the uniform modulus of continuity of X. Condition (A3) is listed here mainly for comparison purpose.

(A1). There exist positive and finite constants $c_{2,1}$ and $c_{2,2}$ such that

$$c_{2,1}\rho(s,t)^2 \le \sigma^2(s,t) \le c_{2,2}\rho(s,t)^2$$

for all $s, t \in I$.

(A2). [Sectorial local nondeterminism] There exists a constant $c_{2,3} > 0$ such that for all integers $n \ge 1$, all $u, t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\Big(X(u)\big|X(t^1), \ldots, X(t^n)\Big) \ge c_{2,3} \sum_{j=1}^{N} \min_{0\le k\le n} |u_j - t_j^k|^{2H_j},$$

where $t_j^0 = 0$ for every $j = 1, \ldots, N$.

(A3). [Strong local nondeterminism in metric ρ] There exists a constant $c_{2,4} > 0$ such that for all integers $n \ge 1$, and all $u, t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\Big(X(u)\big|X(t^1), \ldots, X(t^n)\Big) \ge c_{2,4} \min_{0\le k\le n} \rho(u, t^k)^2,$$

where $t^0 = 0$.

Remark 2.1. The following are some remarks about the above conditions.

- Under condition (A1), X has a version which has continuous sample functions on I almost surely. Henceforth we will assume without loss of generality that the Gaussian random field X has continuous sample paths. Moreover, there are several ways to derive sharp upper bounds for the uniform and local moduli of continuity of X; see for example Marcus and Rosen (2006) or Xiao (2009).

- Under condition (A1), $\sigma(s,t)$ is also a metric on I. The purpose of this paper is to determine the exact uniform and local moduli of continuity of X over the metric spaces (I,σ) and (I,ρ).

- If $H_1, \ldots, H_N$ are not the same, then $X(t)$ is anisotropic in parameter t and may have different behavior along different directions. It is known that many sample path properties of X are different from those of isotropic Gaussian random fields such as fractional Brownian motion. We are concerned with characterizing analytic and geometric properties of X in terms of $H = (H_1, \ldots, H_N)$, which is sometimes called the generalized Hurst indices of X.

- The term "sectorial local nondeterminism" was given by Khoshnevisan and Xiao (2007). They proved that, for every $\varepsilon > 0$, the Brownian sheet W satisfies (A2) with $H = \langle 1/2 \rangle$ for all intervals $I \subset [\varepsilon, \infty)^N$. Wu and Xiao (2007) showed that (A2) is also satisfied by a fractional Brownian sheet with Hurst indices $(H_1, \ldots, H_N) \in (0,1)^N$.

- Condition (A3) implies Condition (A2). It can be verified that the converse does not even hold for the Brownian sheet. Fractional Brownian motion with Hurst index $\alpha \in (0,1)$ satisfies (A3) with $H_1 = \cdots = H_N = \alpha$ on all intervals $I \subset \mathbb{R}^N$. This follows from a result of Pitt (1978). See Xiao (2009) and Xue and Xiao (2009) for examples of anisotropic Gaussian random fields satisfying (A3).

For any Gaussian random field that satisfies Condition (A1), we have the following 0-1 laws for uniform and local moduli of continuity.

Lemma 2.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}$. If Condition (A1) holds, then*

$$\lim_{\varepsilon\to 0} \sup_{t\in I,\, \sigma(s,t)\le\varepsilon} \frac{|X(s)-X(t)|}{\sigma(s,t)\sqrt{\log(1+\sigma(s,t)^{-1})}} = c_{2,5} \qquad a.s. \tag{2.1}$$

for some constant $0 \le c_{2,5} < \infty$; and for every fixed $t_0 \in I$,

$$\lim_{\varepsilon\to 0} \sup_{s\in I,\, \sigma(s,t_0)\le\varepsilon} \frac{|X(s)-X(t_0)|}{\sigma(s,t_0)\sqrt{\log\log(1+\sigma(s,t_0)^{-1})}} = c_{2,6} \qquad a.s. \tag{2.2}$$

for some constant $0 \le c_{2,6} < \infty$. Here $c_{2,6}$ may depend on t_0. The above conclusions still hold if $\sigma(s,t)$ is replaced by $\rho(s,t)$ everywhere.

Proof. Note that due to monotonicity the limits in the left hand side of (2.1) and (2.2) exist almost surely. The key point of the lemma is that these limits are non-random.

It follows from (69), (72) in Xiao (2009) and the monotonicity of the functions $r \mapsto r\sqrt{\log(1+r^{-1})}$ and $r \mapsto r\sqrt{\log\log(1+r^{-1})}$ that, under Condition (A1), we have

$$\limsup_{\varepsilon\to 0} \sup_{t\in I,\, \sigma(s,t)\le\varepsilon} \frac{|X(s)-X(t)|}{\sigma(s,t)\sqrt{\log(1+\sigma(s,t)^{-1})}} \le c_{2,7}, \qquad \text{a.s.} \tag{2.3}$$

and for every $t_0 \in I$,

$$\limsup_{\varepsilon\to 0} \sup_{s\in I,\, \sigma(s,t_0)\le\varepsilon} \frac{|X(s)-X(t_0)|}{\sigma(s,t_0)\sqrt{\log\log(1+\sigma(s,t_0)^{-1})}} \le c_{2,8}, \qquad \text{a.s.} \tag{2.4}$$

In the above, $c_{2,7}$ and $c_{2,8}$ are positive finite constants depending on $c_{2,2}$, I and H only. Hence (2.1) and (2.2) follow from Lemma 7.1.1 in Marcus and Rosen (2006). The same proof yields the last conclusion of the lemma. □

The main purpose of this paper is to show that $c_{2,5} > 0$ under Conditions (A1) and (A2); and $c_{2,6} > 0$ for a large class of Gaussian random fields with stationary increments and fractional Brownian sheets. Using the terminology of Marcus and Rosen (2006), we will show that, under Condition (A2), the function $r \to r\sqrt{\log(1+r^{-1})}$ is an exact uniform modulus of continuity for X on (I,σ) and (I,ρ); and the $r \to r\sqrt{\log\log(1+r^{-1})}$ is an exact local modulus of continuity for X on (I,σ) and (I,ρ).

3 Fernique-type inequalities

The aim of this section is to establish Fernique-type inequalities for anisotropic Gaussian random fields. We start with the following lemma which is a consequence of the results in Fernique (1974) and Berman (1985).

Lemma 3.1 *Suppose that $Y = \{Y(t), t \in \mathbb{R}^N\}$ is a centered Gaussian random field with values in $\mathbb{R}$ and denote*

$$d(s,t) := d_Y(s,t) = \big(\mathbb{E}|Y(t) - Y(s)|^2\big)^{1/2}, \qquad s, t \in \mathbb{R}^N.$$

Let S be a closed cube in $\mathbb{R}^N$ of edge-length δ and let $\sigma^2 = \sup_{t \in S} \mathbb{E}(Y(t)^2)$. For any $h > 0$, $\varepsilon > 0$, define

$$\gamma(\varepsilon) = \sup_{s,t \in S, \|s-t\| \le \varepsilon} d(s,t)$$

and

$$Q(h) = (2 + \sqrt{2}) \int_1^\infty \gamma\big(h\, 2^{-y^2}\big)\, dy.$$

Then for all $x > 0$ which satisfy $x \ge (1 + 4N \log 2)^{1/2}(\sigma + x^{-1})$,

$$\mathbb{P}\Big\{\sup_{t \in S} |Y(t)| > x\Big\} \le 2^{2N+2} \Big(\frac{\delta}{Q^{-1}(1/x)} + 1\Big)^N \frac{\sigma + x^{-1}}{x} \exp\Big(-\frac{x^2}{2(\sigma + x^{-1})^2}\Big), \tag{3.1}$$

where $Q^{-1}(x) = \sup\{y : Q(y) \le x\}$. Particularly, from (3.1) it follows that for any $\varepsilon > 0$ there exist positive constants $x_0 = x_0(\varepsilon, \sigma)$ and $c_{3,1} = c_{3,1}(\varepsilon, \sigma, N)$ such that for any $x \ge x_0$,

$$\mathbb{P}\Big\{\sup_{t \in S} |Y(t)| > x\Big\} \le c_{3,1} \Big(\frac{\delta}{Q^{-1}(1/x)} + 1\Big)^N \exp\Big(-\frac{x^2}{(2+\varepsilon)\sigma^2}\Big). \tag{3.2}$$

Proof. For every $h \in (0, \delta]$, S can be covered by $(\lfloor \delta/h \rfloor + 1)^N$ closed subcubes $\{S_i\}$ of side-length h, where $\lfloor u \rfloor$ denote the largest integer $\le u$. Hence

$$\mathbb{P}\Big\{\sup_{t \in S} |Y(t)| > x\Big\} \le \Big(\lfloor \frac{\delta}{h} \rfloor + 1\Big)^N \max_i \mathbb{P}\Big\{\sup_{t \in S_i} |Y(t)| > x\Big\}. \tag{3.3}$$

Take $h = Q^{-1}(1/x) \wedge \delta$. It follows from the Fernique inequality [with $p = 2$] in Section 4.1.3 of Fernique (1974) or (4.2) in Berman (1985) that for every subcube S_i, we have

$$\mathbb{P}\Big\{\sup_{t \in S_i} |Y(t)| > x\Big\} \le 5\sqrt{2\pi}\, 2^{2N-1}\, \Psi\Big(\frac{x}{\sigma + Q(h)}\Big) \tag{3.4}$$

for all $x \ge (1 + 4N \log 2)^{1/2}(\sigma + Q(h))$, where $\Psi(u) = \mathbb{P}\{N(0,1) > u\}$ is the tail probability of a standard normal random variable. In deriving (3.4), we have also used the fact that $\Psi(u)$ is decreasing.

Notice that $Q(h) \le x^{-1}$ and $\Psi(u) \le (2\pi)^{-\frac{1}{2}} u^{-1} e^{-\frac{u^2}{2}}$ for all $u > 0$, we can write the inequality (3.4) as

$$\mathbb{P}\Big\{\sup_{t \in S_i} |Y(t)| > x\Big\} \le 5\, 2^{2N-1} \frac{\sigma + x^{-1}}{x} \exp\Big(-\frac{x^2}{2(\sigma + x^{-1})^2}\Big), \tag{3.5}$$

which, together with (3.3), yields (3.1) and thus Lemma 3.1. □

For the next lemma, we need the following notation. For every $t \in \mathbb{R}^N$, $x \in \mathbb{R}$ and $\ell = 1, \ldots, N$, we denote by $(\widehat{t_\ell}, x)$ the vector in $\mathbb{R}^N$ obtained from t by replacing its ℓth coordinate t_ℓ by x. For example, $(\widehat{t_N}, x) = (t_1, \ldots, t_{N-1}, x)$.

Lemma 3.2 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}$ and satisfy the upper bound in Condition (A1) with $I = [0,1]^N$. Then for any $\varepsilon > 0$ there exist positive constants $u_0 = u_0(\varepsilon, c_{2,2})$ and $c_{3,2} = c_{3,2}(\varepsilon, c_{2,2}, H, N)$ such that for all $x \in [0,1]$, $0 < y \le z \le 1$ that satisfy $[x, x+y] \subset [0,1]$, all $u \ge u_0$ and $1 \le \ell \le N$*

$$\mathbb{P}\bigg\{ \sup_{t \in [0,1]^N} |X(\widehat{t_\ell}, x+y) - X(\widehat{t_\ell}, x)| \ge u z^{H_\ell} \bigg\} \le c_{3,2}\, z^{-\frac{(N-1)H_\ell}{\min\{H_i\}}} u^{\frac{N-1}{\min\{H_i\}}} e^{-\frac{u^2}{(2+\varepsilon)c_{2,2}}}. \tag{3.6}$$

Here and in the sequel the minimum of H_i is taken over all $1 \le i \le N$.

Proof. For simplicity of notation, we only prove (3.6) for $\ell = N$ and write $(\widehat{t_N}, x)$ as (t, x), where $t \in [0,1]^{N-1}$ or more generally $t \in \mathbb{R}^{N-1}$. For any $x \in [0,1]$ and $0 < y \le z$, we consider the Gaussian process $Y = \{Y(t), t \in \mathbb{R}^{N-1}\}$ defined by

$$Y(t) = \frac{X(t, x+y) - X(t,x)}{z^{H_N}}, \qquad \forall t \in \mathbb{R}^{N-1}.$$

We now show (3.6) by applying Lemma 3.1 to Y with $S = [0,1]^{N-1}$.

By the Minkowski inequality and Condition (A1), we have

$$\begin{aligned} d(s,t) &\le \frac{1}{z^{H_N}} \Big[\mathbb{E}(X(t,x+y) - X(s,x+y))^2 + \mathbb{E}(X(t,x) - X(s,x))^2 \Big]^{1/2} \\ &\le \frac{2c_{2,2}^{1/2}}{z^{H_N}} \sum_{j=1}^{N-1} |s_j - t_j|^{H_j} \end{aligned}$$

for all $s, t \in S$. By Jensen's inequality we derive

$$\begin{aligned} \sum_{j=1}^{N-1} |t_j - s_j|^{2H_j} &\le (N-1)^{1-\min\{H_i\}} \Bigg(\sum_{j=1}^{N-1} |t_j - s_j|^{2H_j/\min\{H_i\}} \Bigg)^{\min\{H_i\}} \\ &\le (N-1)^{1-\min\{H_i\}} \Bigg(\sum_{j=1}^{N-1} |t_j - s_j|^2 \Bigg)^{\min\{H_i\}} \\ &= (N-1)^{1-\min\{H_i\}} \|t-s\|^{2\min\{H_i\}}. \end{aligned}$$

Thus

$$d(s,t) \le \frac{2c_{2,2}^{1/2}}{z^{H_N}} (N-1)^{(1-\min\{H_i\})/2} \|t-s\|^{\min\{H_i\}}.$$

It follows that

$$\gamma(\varepsilon) = \sup_{s,t \in S, \|s-t\| \le \varepsilon} d(s,t) \le \frac{2c_{2,2}^{1/2}}{z^{H_N}} (N-1)^{(1-\min\{H_i\})/2} \varepsilon^{\min\{H_i\}}.$$

This implies that

$$Q(h) \le c\, h^{\min\{H_i\}} / z^{H_N}$$

and the inverse function of Q satisfies

$$Q^{-1}(r) \geq c\, r^{1/\min\{H_i\}} z^{H_N/\min\{H_i\}}.$$

Since $\sigma^2 = \sup_{t \in S} \mathbb{E}(Y(t)^2) \leq c_{2,2}$ by Condition (A1), we use (3.2) to derive that for any $\varepsilon > 0$, there exists a positive constant $u_0 = u_0(\varepsilon,\, c_{2,2})$ such that for all $u \geq u_0$,

$$\begin{aligned} \mathbb{P}\Big\{ \sup_{t \in [0,1]^{N-1}} |X(t, x+y) - X(t,x)| \geq u z^{H_N} \Big\} \\ \leq c\, u^{(N-1)/\min\{H_i\}} z^{-(N-1)H_N/\min\{H_i\}}\, e^{-\frac{u^2}{(2+\varepsilon)c_{2,2}}}. \end{aligned}$$

This yields (3.6) for $\ell = N$. $\square$

The following is the main result of this section.

Proposition 3.3 *Let $\{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field in $\mathbb{R}$ satisfying the upper bound in Condition (A1) for $I = [0,1]^N$. For any $\tau > 0$ and any $\varepsilon > 0$ there exist positive constants $u^* = u^*(\varepsilon, c_{2,2})$ and $c_{3,3} = c(\tau, \varepsilon, c_{2,2}, H, N)$ such that*

$$\begin{aligned} &\mathbb{P}\Bigg\{ \sup_{\langle x_i \rangle \leq t \leq \langle x_i + T_i \rangle} \sup_{\langle 0 \rangle \leq s \leq \langle a_i \rangle} |X(t+s) - X(t)| \geq ((1+\tau)u + \tau) \sum_{j=1}^N a_j^{H_j} \Bigg\} \\ &\leq c_{3,3} \left[\sum_{j=1}^N \Big(\frac{T_j}{a_j} + 1 \Big) \cdot \Big(\frac{1}{a_j} \Big)^{\frac{(N-1)H_j}{\min\{H_i\}}} \right] \Big(u^{\frac{N-1}{\min\{H_i\}}} + 1 \Big) e^{-\frac{u^2}{(2+\varepsilon)c_{2,2}}} \end{aligned} \tag{3.7}$$

for all $u \geq u^*$, x_i, $T_i \in [0,1]$ and $a_i \in (0,1]$ $(i = 1, ..., N)$ which satisfy $[\langle x_i \rangle, \langle x_i + T_i + a_i \rangle] \subset [0,1]^N$.

Proof. We prove (3.7) by using induction on N. If $N = 1$, (3.7) is an immediate consequence of Lemma 2.1 of Csáki *et al.* (1992).

Now we consider the case $N \geq 2$. It is easy to see that for any $s, t \in \mathbb{R}^N$,

$$\begin{aligned} X(t+s) - X(t) \;&= \sum_{j=1}^N \Big(X(t_1, ..., t_{j-1}, t_j + s_j, t_{j+1} + s_{j+1}, ..., t_N + s_N) \\ &\qquad\qquad - X(t_1, ..., t_{j-1}, t_j, t_{j+1} + s_{j+1}, ..., t_N + s_N) \Big) \end{aligned} \tag{3.8}$$

with the convention that $X(t_1, ..., t_{j-1}, t_j + s_j, t_{j+1} + s_{j+1}, ..., t_N + s_N) = X(t+s)$ if $j = 1$ and $X(t_1, ..., t_{j-1}, t_j, t_{j+1} + s_{j+1}, ..., t_N + s_N) = X(t)$ if $j = N$. By (3.8), we have for each $N \geq 2$

$$\begin{aligned} &\sup_{\langle x_i \rangle \leq t \leq \langle x_i + T_i \rangle} \sup_{\langle 0 \rangle \leq s \leq \langle a_i \rangle} |X(t+s) - X(t)| \\ &\quad \leq \sum_{j=1}^N \sup_{t \in [0,1]^{N-1}} \sup_{x_j \leq t_j \leq x_j + T_j} \sup_{0 \leq s_j \leq a_j} |X(t, t_j + s_j) - X(t, t_j)|. \end{aligned}$$

Here and in the rest of the proof, for $t \in \mathbb{R}^{N-1}$, we write $(t, t_j) \in \mathbb{R}^N$ for the point whose jth coordinate is t_j and so $(t, t_j) = (t_1, ..., t_N)$.

Thus for any $\tau > 0$ and $u > 0$

$$
\begin{aligned}
&\mathbb{P}\left\{ \sup_{\langle x_i\rangle \le t \le \langle x_i + T_i\rangle} \sup_{\langle 0\rangle \le s \le \langle a_i\rangle} |X(t+s) - X(t)| \ge \big((1+\tau)u + \tau\big) \sum_{j=1}^N a_j^{H_j} \right\} \\
&\le \sum_{j=1}^N \mathbb{P}\Big\{ \sup_{t\in[0,1]^{N-1}} \sup_{x_j \le t_j \le x_j + T_j} \sup_{0\le s_j \le a_j} \big|X(t, t_j + s_j) - X(t, t_j)\big| \ge \big((1+\tau)u + \tau\big) a_j^{H_j} \Big\} \qquad (3.9) \\
&=: \sum_{j=1}^N P_j.
\end{aligned}
$$

Let $1 \le j \le N$ be fixed. We follow the proof of Lemma 2.1 of Csáki *et al.* (1992) to estimate P_j $(1 \le j \le N)$ by using a chaining argument. For any positive real number r and $n \ge 3$, denote $(r)_n = a_j \lfloor r \frac{2^n}{a_j} \rfloor / 2^n$, where $\lfloor x \rfloor$ is the largest integer $\le x$. Clearly $(r)_n \to r$ as $n \to \infty$.

Let $k \ge 3$ be a fixed integer whose value will be specified later. By the triangle inequality, we have

$$
\begin{aligned}
&\big|X(t, t_j + s_j) - X(t, t_j)\big| \\
&\le \big|X(t, (t_j + s_j)_k) - X(t, (t_j)_k)\big| + \big|X(t, t_j + s_j) - X(t, (t_j + s_j)_k)\big| \\
&\qquad + \big|X(t, t_j) - X(t, (t_j)_k)\big| \\
&\le \big|X(t, (t_j + s_j)_k) - X(t, (t_j)_k)\big| + \sum_{l=0}^\infty \big|X(t, (t_j + s_j)_{k+l+1}) - X(t, (t_j + s_j)_{k+l})\big| \qquad (3.10) \\
&\qquad + \sum_{l=0}^\infty \big|X(t, (t_j)_{k+l+1}) - X(t, (t_j)_{k+l})\big|.
\end{aligned}
$$

In order to use the above inequality to estimate P_j, we make some preparation first. For $l \ge 0$, put

$$
u_l^2 = u^2 + c_{2,2}\left(2 + \frac{2(N-1)H_j}{\min\{H_i\}} + \frac{N-1}{\min\{H_i\}}\right)(k+l+1).
$$

Then

$$
u_l \le u + c_{2,2}^{1/2}\left(2 + \frac{2(N-1)H_j}{\min\{H_i\}} + \frac{N-1}{\min\{H_i\}}\right)^{1/2} (k+l+1)^{1/2}.
$$

It follows that

$$
\begin{aligned}
&u a_j^{H_j} + u\Big(\frac{2a_j}{2^k}\Big)^{H_j} + 2\sum_{l=0}^\infty u_l \Big(\frac{a_j}{2^{k+l}}\Big)^{H_j} \\
&\le u a_j^{H_j} + u\left[\Big(\frac{2a_j}{2^k}\Big)^{H_j} + \sum_{l=0}^\infty \Big(\frac{a_j}{2^{k+l}}\Big)^{H_j}\right] \\
&\quad + c_{2,2}^{1/2}\left(2 + \tfrac{2(N-1)H_j}{\min\{H_i\}} + \tfrac{N-1}{\min\{H_i\}}\right)^{1/2} \sum_{l=0}^\infty (k+l+1)^{1/2} \Big(\frac{a_j}{2^{k+l}}\Big)^{H_j} \qquad (3.11) \\
&= u a_j^{H_j} + u a_j^{H_j} \frac{2^{H_j} + (1 - 2^{-H_j})^{-1}}{2^{kH_j}} \\
&\quad + c_{2,2}^{1/2} a_j^{H_j}\left(2 + \tfrac{2(N-1)H_j}{\min\{H_i\}} + \tfrac{N-1}{\min\{H_i\}}\right)^{1/2} \sum_{l=0}^\infty \frac{(k+l+1)^{1/2}}{2^{H_j(k+l)}} \\
&\le u a_j^{H_j} + \tau u a_j^{H_j} + \tau a_j^{H_j} = ((1+\tau)u + \tau) a_j^{H_j}.
\end{aligned}
$$

In deriving the last inequality we have chosen k large enough such that

$$\frac{2^{H_j} + (1-2^{-H_j})^{-1}}{2^{kH_j}} \le \tau$$

and

$$c_{2,2}^{1/2}\left(2 + \frac{2(N-1)H_j}{\min\{H_i\}} + \frac{N-1}{\min\{H_i\}}\right)^{1/2} \sum_{l=k}^{\infty} \frac{(l+1)^{1/2}}{2^{H_j l}} \le \tau.$$

It follows from (3.10) and (3.11) that

$$\begin{aligned}
P_j \le\ & \mathbb{P}\Big\{ \sup_{t\in[0,1]^{N-1}} \sup_{x_j\le t_j\le x_j+T_j} \sup_{0\le s_j\le a_j} \big|X(t,(t_j+s_j)_k) - X(t,(t_j)_k)\big| \ge (1+2^{-k+1}) u a_j^{H_j} \Big\} \\
&+ \sum_{l=0}^{\infty} \mathbb{P}\Big\{ \sup_{t\in[0,1]^{N-1}} \sup_{x_j\le t_j\le x_j+T_j} \sup_{0\le s_j\le a_j} \big|X(t,(t_j+s_j)_{k+l+1}) - X(t,(t_j+s_j)_{k+l})\big| \ge u_l \Big(\frac{a_j}{2^{k+l}}\Big)^{H_j} \Big\} \\
&+ \sum_{l=0}^{\infty} \mathbb{P}\Big\{ \sup_{t\in[0,1]^{N-1}} \sup_{x_j\le t_j\le x_j+T_j} \big|X(t,(t_j)_{k+l+1}) - X(t,(t_j)_{k+l})\big| \ge u_l \Big(\frac{a_j}{2^{k+l}}\Big)^{H_j} \Big\} \\
&=: P_{j,1} + P_{j,2} + P_{j,3}.
\end{aligned}$$

We estimate the terms $P_{j,1}$, $P_{j,2}$ and $P_{j,3}$ separately. In order to estimate $P_{j,1}$, we use the triangle inequality again to write

$$\begin{aligned}
&\sup_{t\in[0,1]^{N-1}} \sup_{x_j\le t_j\le x_j+T_j} \sup_{0\le s_j\le a_j} \big|X(t,(t_j+s_j)_k) - X(t,(t_j)_k)\big| \\
&\quad\le \sup_{t\in[0,1]^{N-1}} \sup_{x_j\le t_j\le x_j+T_j} \sup_{0\le s_j\le a_j(1-2^{-k})} \big|X(t,(t_j+s_j)_k) - X(t,(t_j)_k)\big| \\
&\quad\quad + \sup_{t\in[0,1]^{N-1}} \sup_{x_j\le t_j\le x_j+T_j} \sup_{a_j(1-2^{-k})\le s_j\le a_j} \big|X(t,(t_j+s_j)_k) - X(t,(t_j+a_j(1-2^{-k}))_k)\big|.
\end{aligned} \tag{3.12}$$

Since

$$\sup_{0\le t_j\le T_j} \sup_{0\le s_j\le a_j(1-2^{-k})} \big|(t_j+s_j)_k - (t_j)_k\big| \le a_j,$$

and there are at most $2^k\big(\frac{T_j}{a_j}+1\big)$ different points $(t_j)_k$ and at most 2^k different $(t_j+s_j)_k$, we derive from Lemma 3.2 with $z = a_j$ that for each $u \ge u^*$,

$$\begin{aligned}
&\mathbb{P}\left\{ \sup_{t\in[0,1]^{N-1}} \sup_{x_j\le t_j\le x_j+T_j} \sup_{0\le s_j\le a_j(1-2^{-k})} |X(t,(t_j+s_j)_k) - X(t,(t_j)_k)| \ge u a_j^{H_j} \right\} \\
&\quad\le c_{3,2}\, 2^{2k}\left(\frac{T_j}{a_j}+1\right)\cdot\left(\frac{1}{a_j}\right)^{\frac{(N-1)H_j}{\min\{H_i\}}} u^{\frac{N-1}{\min\{H_i\}}} \exp\left(-\frac{u^2}{(2+\varepsilon)c_{2,2}}\right).
\end{aligned} \tag{3.13}$$

Similarly, because

$$\sup_{x_j\le t_j\le x_j+T_j} \sup_{a_j(1-2^{-k})\le s_j\le a_j} \big|(t_j+s_j)_k - (t_j+a_j(1-2^{-k}))_k\big| \le 2a_j\cdot 2^{-k},$$

we derive from Lemma 3.2 that for each $u \ge u^*$,

$$\begin{aligned}
&\mathbb{P}\Big\{ \sup_{t\in[0,1]^{N-1}} \sup_{x_j\le t_j\le x_j+T_j} \sup_{a_j(1-2^{-k})\le s_j\le a_j} \big|X(t,(t_j+s_j)_k) - X(t,(t_j+a_j(1-2^{-k}))_k)\big| \\
&\qquad\qquad \ge u(2^{-k+1}a_j)^{H_j} \Big\}
\end{aligned} \tag{3.14}$$

$$\le c_{3,2}\, 2^{k}\left(\frac{T_j}{a_j}+1\right)\cdot\left(\frac{2^{k+1}}{a_j}\right)^{\frac{(N-1)H_j}{\min\{H_i\}}} u^{\frac{N-1}{\min\{H_i\}}} \exp\left(-\frac{u^2}{(2+\varepsilon)c_{2,2}}\right).$$

Combining (3.12)–(3.14), we obtain

$$P_{j,1} \le c_{3,4} \left(\frac{T_j}{a_j} + 1\right) \cdot \left(\frac{1}{a_j}\right)^{\frac{(N-1)H_j}{\min\{H_i\}}} u^{\frac{N-1}{\min\{H_i\}}} \exp\left(- \frac{u^2}{(2+\varepsilon)c_{2,2}}\right). \tag{3.15}$$

In the same way, note that

$$\sup_{x_j \le t_j \le x_j + T_j} \sup_{0 \le s_j \le a_j} \left|(t_j + s_j)_{k+l+1} - (t_j+s_j)_{k+l}\right| \le a_j \cdot 2^{-(k+l)},$$

we apply Lemma 3.2 to derive

$$\begin{aligned}
&\mathbb{P}\left\{ \sup_{t \in [0,1]^{N-1}} \sup_{x_j \le t_j \le x_j + T_j} \sup_{0 \le s_j \le a_j} \left|X(t,(t_j+s_j)_{k+l+1}) - X(t,(t_j+s_j)_{k+l})\right| \ge u_l (a_j 2^{-(k+l)})^{H_j}\right\}\\
&\le c_{3,2}\, 2^{k+l+1} \left(\frac{T_j}{a_j} + 1\right) \cdot \left(\frac{2^{k+l}}{a_j}\right)^{\frac{(N-1)H_j}{\min\{H_i\}}} u_l^{\frac{N-1}{\min\{H_i\}}} \exp\left(- \frac{u_l^2}{(2+\varepsilon)c_{2,2}}\right)
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\mathbb{P}\left\{ \sup_{t \in [0,1]^{N-1}} \sup_{x_j \le t_j \le x_j + T_j} \left|X(t,(t_j)_{k+l+1}) - X(t,(t_j)_{k+l})\right| \ge u_l (a_j 2^{-(k+l)})^{H_j}\right\}\\
&\le c_{3,2}\, 2^{k+l+1} \left(\frac{T_j}{a_j} + 1\right) \cdot \left(\frac{2^{k+l}}{a_j}\right)^{\frac{(N-1)H_j}{\min\{H_i\}}} u_l^{\frac{N-1}{\min\{H_i\}}} \exp\left(- \frac{u_l^2}{(2+\varepsilon)c_{2,2}}\right).
\end{aligned} \tag{3.17}$$

By the definition of u_l, we have

$$u_l^{\frac{N-1}{\min\{H_i\}}} \le 2^{\frac{N-1}{\min\{H_i\}}-1}\left[u^{\frac{N-1}{\min\{H_i\}}} c_{2,2}^{\frac{N-1}{2\min\{H_i\}}} \left(2 + \frac{(N-1)(2H_j+1)}{\min\{H_i\}}\right)^{\frac{N-1}{2\min\{H_i\}}} (k+l+1)^{\frac{N-1}{2\min\{H_i\}}}\right]$$

and

$$\begin{aligned}
&\sum_{l=0}^{\infty} 2^{\left(1+\frac{(N-1)H_j}{\min\{H_i\}}\right)(k+l+1)} (k+l+1)^{\frac{N-1}{2\min\{H_i\}}} \exp\left(- \frac{u_l^2}{(2+\varepsilon)c_{2,2}}\right)\\
&\le \sum_{l=0}^{\infty} 2^{\left(1+\frac{(N-1)H_j}{\min\{H_i\}}+\frac{N-1}{2\min\{H_i\}}\right)(k+l+1)} \exp\left(- \frac{u_l^2}{(2+\varepsilon)c_{2,2}}\right)\\
&= \sum_{l=0}^{\infty} 2^{\left(1+\frac{(N-1)H_j}{\min\{H_i\}}+\frac{N-1}{2\min\{H_i\}}\right)(k+l+1)} \exp\left(- \left(1 + \frac{(N-1)H_j}{\min\{H_i\}} + \frac{N-1}{2\min\{H_i\}}\right)(k+l+1)\right)\\
&\qquad\qquad \times \exp\left(- \frac{u^2}{(2+\varepsilon)c_{2,2}}\right)\\
&\le \exp\left(- \frac{u^2}{(2+\varepsilon)c_{2,2}}\right).
\end{aligned}$$

Hence, (3.16) and (3.17) yield

$$P_{j,2} + P_{j,3} \le c_{3,5} \left(\frac{T_j}{a_j} + 1\right) \cdot \left(\frac{1}{a_j}\right)^{\frac{(N-1)H_j}{\min\{H_i\}}} \left(u^{\frac{N-1}{\min\{H_i\}}} + 1\right) \exp\left(- \frac{u^2}{(2+\varepsilon)c_{2,2}}\right).$$

Combining the above inequality with (3.15) shows that for each $1 \le j \le N$

$$P_j \le c_{3,5} \left(\frac{T_j}{a_j} + 1\right) \cdot \left(\frac{1}{a_j}\right)^{\frac{(N-1)H_j}{\min\{H_i\}}} \left(u^{\frac{N-1}{\min\{H_i\}}} + 1\right) \exp\left(- \frac{u^2}{(2+\varepsilon)c_{2,2}}\right).$$

This, together with (3.9), implies (3.7). The proof is now completed. □

4 Uniform modulus of continuity

In this section we establish exact uniform moduli of continuity for a large class of anisotropic Gaussian random fields. The main result is Theorem 4.1. As a consequence, we will derive an exact uniform modulus of continuity of fractional Brownian sheets, which extends and strengthens Theorem 1 of Ayache and Xiao (2005) and Theorem 3.2 of Wang (2007) (see Section 6 below).

Theorem 4.1 *Let $\{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}$ and satisfy Conditions (A1) and (A2). Put*

$$\beta(s,t) = \sigma(s,t)\sqrt{\log(1+\sigma(s,t)^{-1})}, \qquad s,t \in I. \tag{4.1}$$

Then

$$\lim_{\varepsilon\to 0+} \sup_{s,t\in I, \sigma(s,t)\le\varepsilon} \frac{|X(t)-X(s)|}{\beta(s,t)} = \kappa_1 \quad a.s., \tag{4.2}$$

where κ_1 is a positive constant satisfying

$$\sqrt{\frac{2c_{2,3}}{c_{2,2}\min\{H_i\}}} \le \kappa_1 \le \sqrt{\frac{2N^3 c_{2,2}}{c_{2,1}\min\{H_i\}}}. \tag{4.3}$$

Proof. For simplicity of notation, we assume $I = [a,1]^N$, where $a \in [0,1)$ is a constant. For any $\varepsilon > 0$, put

$$J(\varepsilon) = \sup_{s,t\in I, \sigma(s,t)\le\varepsilon} \frac{|X(t)-X(s)|}{\beta(s,t)},$$

then $\varepsilon \mapsto J(\varepsilon)$ is non-decreasing. Hence the limit in the left hand side of (4.2) exists almost surely. By Lemma 2.1 it only remains to show

$$\lim_{\varepsilon\to 0+} J(\varepsilon) \le c_{4,2} \qquad \text{a.s.} \tag{4.4}$$

and

$$\lim_{\varepsilon\to 0+} J(\varepsilon) \ge c_{4,1} \qquad \text{a.s.,} \tag{4.5}$$

where

$$c_{4,1} = \sqrt{\frac{2c_{2,3}}{c_{2,2}\min\{H_i\}}} \qquad \text{and} \qquad c_{4,2} = \sqrt{\frac{2N^3 c_{2,2}}{c_{2,1}\min\{H_i\}}}.$$

We first show (4.4). Proofs of (4.4) with a generic constant by using general Gaussian principles such as majorizing measure or isoperimetric inequality are available [see, e.g., Marcus and Rosen (2006, Chapter 7) or Xiao (2009)]. Here we apply Proposition 3.3 to provide a more careful deviation to connect this constant with the constants in (A1). Since the function $x \mapsto x\sqrt{\log(1+1/x)}$ is increasing for $x \in (0,1)$ small and

$$\sigma(s,t)^2 \ge c_{2,1}\max\{|t_j - s_j|^{2H_j}\},$$

we have

$$\beta(s,t) \ge c_{2,1}^{1/2}|t_j - s_j|^{H_j}\sqrt{\log(1 + c_{2,1}^{-1/2}|t_j - s_j|^{-H_j})}$$

for every $1 \le j \le N$. Thus by (3.8) we have

$$
\begin{aligned}
J(\varepsilon) &\le \sum_{j=1}^N \sup_{\substack{t\in[a,1]^{N-1}, s_j, t_j\in[a,1]\\ \sigma(s,t)\le\varepsilon}} \frac{|X(t,t_j)-X(t,s_j)|}{\beta(s,t)}\\
&\le \sum_{j=1}^N \sup_{\substack{t\in[a,1]^{N-1}, s_j, t_j\in[a,1]\\ c_{2,1}^{1/2}|t_j-s_j|^{H_j}\le\varepsilon}} \frac{|X(t,t_j)-X(t,s_j)|}{\beta(s,t)}\\
&\le \sum_{j=1}^N \sup_{\substack{t\in[a,1]^{N-1}, s_j, t_j\in[a,1]\\ c_{2,1}^{1/2}|t_j-s_j|^{H_j}\le\varepsilon}} \frac{|X(t,t_j)-X(t,s_j)|}{c_{2,1}^{1/2}|t_j-s_j|^{H_j}\sqrt{\log(1+c_{2,1}^{-1/2}|t_j-s_j|^{-H_j})}}\\
&=: \sum_{j=1}^N J_j(\varepsilon).
\end{aligned}
\tag{4.6}
$$

We now show that for each $1 \le j \le N$,

$$
\limsup_{\varepsilon\to 0+} J_j(\varepsilon) \le \frac{c_{4,2}}{N} \qquad \text{a.s.} \tag{4.7}
$$

For every $1 \le j \le N$ and $\mu > 0$, define the event

$$
E_j(l,k,n) = \left\{ \sup \frac{|X(t,t_j)-X(t,s_j)|}{c_{2,1}^{1/2}|t_j-s_j|^{H_j}\sqrt{\log(1+c_{2,1}^{-1/2}|t_j-s_j|^{-H_j})}} \ge \frac{(1+\mu)c_{4,2}}{N} \right\},
$$

where the supremum is taken over $t \in [a,1]^{N-1}$ and all s_j, t_j which satisfy

$$
\frac{l-1}{2^n} \le s_j < \frac{l}{2^n}, \qquad \frac{l+k}{2^n} \le t_j < \frac{l+k+1}{2^n}. \tag{4.8}
$$

Let

$$
A = c_{2,1}^{1/2} k^{H_j}\, 2^{-nH_j} \qquad \text{and} \qquad B = c_{2,1}^{1/2}(k+2)^{H_j}\, 2^{-nH_j}.
$$

Then A is the infimum of $c_{2,1}^{1/2}|t_j - s_j|^{H_j}$ taken over s_j, t_j satisfy (4.8), and B is the corresponding supremum. The parameters k and l will be restricted to the following ranges:

$$
a2^n + 1 \le l \le 2^n, \quad \frac{1}{4}n \le k \le n \tag{4.9}
$$

for $n = 3, 4, \ldots$. It is easy to check that

$$
0 \le 1 - AB^{-1} \le c\, n^{-1} \to 0.
$$

By taking $x_j = \frac{l-1}{2^n}$, $T_j = a_j = \frac{k+2}{2^n}$ and $u = \frac{(1+\mu)c_{4,2}c_{2,1}^{1/2}}{N}\sqrt{\log(1+B^{-1})}$ in (3.7), we obtain that for n large enough

$$
\begin{aligned}
\mathbb{P}\Big(E_j(l,k,n)\Big) &\le c\left(\frac{2^n}{k+2}\right)^{\frac{(N-1)H_j}{\min\{H_i\}}} \exp\left(-\frac{(1+\mu/2)^2 c_{4,2}^2 c_{2,1}\log(1+B^{-1})}{2N^2 c_{2,2}}\right)\\
&\le c\left(\frac{n}{2^n}\right)^{\frac{(1+\mu/2)^2 c_{4,2}^2 c_{2,1} H_j}{2N^2 c_{2,2}} - \frac{(N-1)H_j}{\min\{H_i\}}}.
\end{aligned}
$$

Thus

$$\sum_{n=1}^{\infty}\sum_{l,k}\mathbb{P}(E_j(l,k,n)) \le c\sum_{n=1}^{\infty} n2^n\left(\frac{n}{2^n}\right)^{\frac{(1+\mu/2)^2c_{4,2}^2c_{2,1}H_j}{2N^2c_{2,2}}-\frac{(N-1)H_j}{\min\{H_i\}}} < \infty,$$

where the summation $\sum_{l,k}$ is taken over integers l,k that satisfy (4.9) and, for obtaining the last inequality, we have used the definition of $c_{4,2}$. Thus, by the Borel-Cantelli lemma, there exists a.s. an integer $n_0 = n_0(\omega)$ such that none of the events $E_j(l,k,n)$ occur for $n \ge n_0$ and l,k satisfying (4.9).

Let $[s_j,t_j]$ be an interval with $t_j - s_j \le n_0 2^{-n_0}$. Now we first choose $n \ge n_0$ so that

$$(n+1)2^{-n-1} < t_j - s_j \le n2^{-n}$$

and then l and k so that

$$(l-1)2^{-n} \le s_j < l2^{-n}, \qquad (l+k)2^{-n} \le t_j < (l+k+1)2^{-n}.$$

It is now easy to check that if $[s_j,t_j] \subseteq [a,1]$ then the indices l,k satisfy (4.9) and $[s_j,t_j]$ is one of the intervals in the event $E_j(l,k,n)$. Hence $\limsup_{\varepsilon\to 0+} J_j(\varepsilon) \le (1+\mu)c_{4,2}/N$ a.s. Letting $\mu \downarrow 0$ yields (4.7) for every $1 \le j \le N$. Combining (4.6) and (4.7) we obtain (4.4) immediately.

Now we show (4.5). Let $1 < \theta < 2$ be a constant which will be specified later and, for all $n \ge 1$, let

$$\varepsilon_n = \sqrt{c_{2,2}\sum_{j=1}^{N}\theta^{-2H_jn}}.$$

Since for any $0 < \varepsilon < 1$ there is an integer $n \ge 2$ such that $\varepsilon_n < \varepsilon \le \varepsilon_{n-1}$, by the monotonicity of $J(\varepsilon)$ and Condition (A1), we have

$$\begin{aligned}\lim_{\varepsilon\to 0+} J(\varepsilon) &= \lim_{n\to\infty}\sup_{s,t\in I,\sigma(s,t)\le\varepsilon_n}\frac{|X(t)-X(s)|}{\beta(s,t)}\\ &\ge \liminf_{n\to\infty}\max_{a\theta^n/2\le i\le(\theta^n-1)/2}\frac{|X(\langle(2i+1)\theta^{-n}\rangle)-X(\langle 2i\theta^{-n}\rangle)|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}}\\ &=: \liminf_{n\to\infty} J_n.\end{aligned} \tag{4.10}$$

Recall that $\langle c(i)\rangle$ means the N-dimensional vector $(c(i),...,c(i))$.

For any $\mu \in (0,1)$, we have

$$\begin{aligned}&\mathbb{P}\Big(J_n \le (1-\mu)c_{4,1}\Big)\\ &\le \mathbb{P}\left(\left\{\frac{|X(\langle 1\rangle)-X(\langle 1-\theta^{-n}\rangle)|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}} \le (1-\mu)c_{4,1}\right\}\right.\\ &\qquad\left.\bigcap\left\{\max_{a\theta^n/2\le i\le(\theta^n-1)/2-1}\frac{|X(\langle(2i+1)\theta^{-n}\rangle)-X(\langle 2i\theta^{-n}\rangle)|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}} \le (1-\mu)c_{4,1}\right\}\right)\\ &= P_1(n)\cdot P_2(n),\end{aligned}$$

where

$$P_1(n) = \mathbb{P}\left(\max_{a\theta^n/2\le i\le(\theta^n-1)/2-1}\frac{|X(\langle(2i+1)\theta^{-n}\rangle)-X(\langle 2i\theta^{-n}\rangle)|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}} \le (1-\mu)c_{4,1}\right)$$

and

$$P_2(n) \quad = \mathbb{P}\Bigg(\frac{|X(\langle 1\rangle) - X(\langle 1-\theta^{-n}\rangle)|}{\varepsilon_n \sqrt{\log(1+\varepsilon_n^{-1})}} \le (1-\mu)c_{4,1} \,\Big|\, X(\langle 1-\theta^{-n}\rangle); X(\langle (2i+1)\theta^{-n}\rangle), \\ X(\langle 2i\theta^{-n}\rangle),\ a\theta^n/2 \le i \le (\theta^n-1)/2-1 \Bigg).$$

By Condition (A2), we have

$$\mathrm{Var}\Big(X(\langle 1\rangle) - X(\langle 1-\theta^{-n}\rangle) \,\Big|\, X(\langle 1-\theta^{-n}\rangle); X(\langle (2i+1)\theta^{-n}\rangle), \\ X(\langle 2i\theta^{-n}\rangle),\ a\theta^n/2 \le i \le (\theta^n-1)/2-1 \Big) \\ \ge c_{2,3} \sum_{j=1}^{N} \theta^{-2H_j n} = \frac{c_{2,3}}{c_{2,2}}\, \varepsilon_n^2.$$

Thus, by the fact that the conditional distributions of the Gaussian process is still Gaussian and Anderson's inequality (see Anderson (1955)), we derive

$$P_2(n) \le \mathbb{P}\Big(N(0,1) \le (1-\mu)c_{4,1} \sqrt{(c_{2,2}/c_{2,3})\log(1+\varepsilon_n^{-1})} \Big),$$

where $N(0,1)$ denotes a standard normal random variable. By using the following well known inequality

$$(2\pi)^{-\frac{1}{2}}(1-x^{-2})x^{-1}e^{-\frac{x^2}{2}} \le \mathbb{P}(N(0,1) > x) \le (2\pi)^{-\frac{1}{2}}x^{-1}e^{-\frac{x^2}{2}}, \qquad \forall\, x > 0, \tag{4.11}$$

we derive that for all n large enough

$$\begin{aligned} P_2(n) \quad &= 1 - \mathbb{P}\Big(N(0,1) > (1-\mu)c_{4,1} \sqrt{(c_{2,2}/c_{2,3})\log(1+\varepsilon_n^{-1})} \Big) \\ &\le 1 - \varepsilon_n^{\frac{(1-\mu/2)^2 c_{2,2} 2c_{4,1}^2}{2c_{2,3}}} \\ &\le \exp\bigg(-\varepsilon_n^{\frac{(1-\mu/2)^2 c_{2,2} c_{4,1}^2}{2c_{2,3}}} \bigg), \end{aligned}$$

where for obtaining the last inequality we have used the elementary inequality $\forall x,\ 1-x \le e^{-x}$. Hence

$$\mathbb{P}\Big(J_n \le (1-\mu)c_{4,1} \Big) \le \exp\bigg(-\varepsilon_n^{\frac{(1-\mu/2)^2 c_{2,2} c_{4,1}^2}{2c_{2,3}}} \bigg) \cdot P_1(n).$$

By repeating the above argument, we derive that

$$\begin{aligned} \mathbb{P}\Big(J_n \le (1-\mu)c_{4,1} \Big) \quad &\le \exp\bigg(-\frac{(1-a)\theta^n - 1}{2}\, \varepsilon_n^{\frac{(1-\mu/2)^2 c_{2,2} c_{4,1}^2}{2c_{2,3}}} \bigg) \\ &\le \exp\bigg(-c\theta^n \theta^{-\frac{\min\{H_i\}(1-\mu/2)^2 c_{2,2} c_{4,1}^2 n}{2c_{2,3}}} \bigg), \end{aligned}$$

where the last inequality follows from the estimate: $\varepsilon_n^2 \ge c_{2,2}\theta^{-2\min\{H_i\}n}$. Thus, by the definition of $c_{4,1}$, we get

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(J_n \le (1-\mu)c_{4,1} \Big) < \infty.$$

Thus, the Borel-Cantelli lemma implies

$$\liminf_{n\to\infty} J_n \geq (1-\mu)c_{4,1} \quad \text{a.s.} \tag{4.12}$$

Letting $\mu \downarrow 0$ along a sequence, by (4.10) and (4.12) we obtain (4.5). The proof of Theorem 4.1 is completed. □

5 Laws of the iterated logarithm

In this section we investigate the exact local moduli of continuity for anisotropic random fields with stationary increments. Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with $X(\langle 0\rangle) = 0$. We assume that the covariance function $R(s,t) = \mathbb{E}[X(s)X(t)]$ is continuous and X has stationary increments. The latter means that for any $h \in \mathbb{R}^N$,

$$\{X(t+h) - X(h),\, t \in \mathbb{R}^N\} \stackrel{d}{=} \{X(t),\, t \in \mathbb{R}^N\},$$

where $\stackrel{d}{=}$ means equality in finite dimensional distributions. According to Yaglom (1957), $R(s,t)$ can be represented as

$$R(s,t) = \int_{\mathbb{R}^N} (e^{i\langle s,\xi\rangle} - 1)(e^{i\langle t,\xi\rangle} - 1)\Delta(d\xi) + \langle s, Qt\rangle, \tag{5.1}$$

where $Q = (q_{ij})$ is an $N \times N$ non-negative definite matrix and $\Delta(d\xi)$ is a nonnegative symmetric measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^N} \frac{|\xi|^2}{1+|\xi|^2}\Delta(d\xi) < \infty. \tag{5.2}$$

The measure Δ and its density $f(\xi)$ (if it exists) are called the spectral measure and spectral density of X, respectively.

It follows from (5.1) that X has the following stochastic integral representation:

$$X(t) = \int_{\mathbb{R}^N} (e^{i\langle t,\xi\rangle} - 1)\mathcal{M}(d\xi) + \langle \mathbf{Y}, t\rangle, \tag{5.3}$$

where $\mathbf{Y}$ is an N-dimensional Gaussian random vector with mean zero and $\mathcal{M}(d\xi)$ is a centered complex-valued Gaussian random measure which is independent of $\mathbf{Y}$ and satisfies

$$\mathbb{E}\Big[\mathcal{M}(A)\overline{\mathcal{M}(B)}\Big] = \Delta(A \cap B) \quad \text{and} \quad \mathcal{M}(-A) = \overline{\mathcal{M}(A)}$$

for all Borel sets $A, B \subset \mathbb{R}^N$ with finite Δ-measure. The spectral measure Δ is called the control measure of $\mathcal{M}$. Since the linear term $\langle \mathbf{Y}, t\rangle$ in (5.3) will not have any effect on the problems considered in this paper, we will from now on assume $\mathbf{Y} = 0$. This is equivalent to assuming $Q = 0$ in (5.1). Consequently, we have

$$\sigma^2(h) = \mathbb{E}\big[(X(t+h) - X(t))^2\big] = 2\int_{\mathbb{R}^N} (1 - \cos\langle h, \xi\rangle)\Delta(d\xi). \tag{5.4}$$

For $t_0 \in \mathbb{R}^N$ and a family of neighborhoods $\{O(\delta), \delta > 0\}$ of $\langle 0\rangle \in \mathbb{R}^N$ whose diameters go to 0 as $\delta \to 0$, we consider the corresponding local moduli of continuity of X at t_0

$$\omega(t_0, \delta) = \sup_{s \in O(\delta)} |X(t_0 + s) - X(t_0)|.$$

It can be seen that, if X is anisotropic, then the rate at which $\omega(t_0, \delta)$ goes to 0 as $\delta \to 0$ depends on the shape of $O(\delta)$.

In the following, we consider two kinds of local moduli of continuity for X. Theorem 5.1 is concerned with the local modulus of continuity measured in the most general way. Theorem 5.6 provides the exact local modulus of continuity in the metric σ. It should be noticed that the logarithmic factors in these two theorems are quite different, since (A1) implies $\sigma(s) \asymp \sum_{j=1}^N |s_j|^{H_j}$ as $s \to 0$ (ratio remains bounded away from zero and infinity) in Theorem 5.6, and the corresponding term $\prod_{j=1}^N |s_j|^{H_j}$ in Theorem 5.1 is much smaller.

Theorem 5.1 *Let $\{X(t), t \in \mathbb{R}^N\}$ be a real valued, centered Gaussian random field with stationary increments and $X(\langle 0 \rangle) = 0$. If X satisfies Condition (A1) for $I = [0, 1]^N$, then there is a positive constant κ_2 such that for every $t_0 \in \mathbb{R}^N$ we have*

$$\limsup_{\|\varepsilon\| \to 0+} \sup_{\langle |s_j| \rangle \le \langle \varepsilon_j \rangle} \frac{|X(t_0 + s) - X(t_0)|}{\gamma(s)} = \kappa_2 \quad a.s., \tag{5.5}$$

where

$$\gamma(s) = \sigma(s) \left[\log\log \left(1 + \frac{1}{\prod_{j=1}^N |s_j|^{H_j}} \right) \right]^{\frac{1}{2}}, \qquad \forall\, s \in \mathbb{R}^N. \tag{5.6}$$

Remark 5.2 Eq. (5.5) means that, for any $\eta > 0$, there exists a.s. $\delta_0 = \delta_0(\omega) > 0$ such that for all $\varepsilon \in (0, 1)^N$ which satisfies $\|\varepsilon\| \le \delta_0$, we have

$$\sup_{\langle |s_j| \rangle \le \langle \varepsilon_j \rangle} \frac{|X(t_0 + s) - X(t_0)|}{\gamma(s)} < \kappa_2 + \eta.$$

Moreover for any $\eta > 0$ there exists a sequence $\varepsilon(n) \in (0, 1)^N$ such that $\|\varepsilon(n)\| \to 0$ and

$$\sup_{\langle |s_j| \rangle \le \varepsilon(n)} \frac{|X(t_0 + s) - X(t_0)|}{\gamma(s)} > \kappa_2 - \eta$$

for n large enough.

In order to show Theorem 5.1, we will make use of the following lemmas. The first one (Lemma 5.3) is taken from Talagrand (1995). Let $\{Z(t),\, t \in S\}$ be a centered Gaussian process with values in $\mathbb{R}$. The index set S is equipped with the pseudo-metric $d(s, t) = \left[\mathbb{E}(Z(t) - Z(s))^2\right]^{1/2}$. We denote by $N_d(S, \varepsilon)$ the smallest number of (open) d-balls of radius ε needed to cover S and we denote by D the diameter of S, that is, $D = \sup\{d(s, t) : s, t \in S\}$.

Lemma 5.3. *Given $x > 0$, we have*

$$\mathbb{P}\left(\sup_{s,t \in S} |Z(t) - Z(s)| \ge c_{5,1} \left(x + \int_0^D \sqrt{\log N_d(S, \varepsilon)}\, d\varepsilon \right) \right) \le \exp\left(- \frac{x^2}{D^2} \right), \tag{5.7}$$

where $c_{5,1}$ is a positive and finite constant.

Lemma 5.4. *Let $\{X(t), t \in \mathbb{R}^N\}$ be a real valued, centered Gaussian random field satisfying the upper bound in Condition (A1). Then there exist positive and finite constants u_0 and $c_{5,2}$ such that for all*

$t_0 \in I$ and $u \geq u_0$

$$\mathbb{P}\left(\sup_{\langle |s_j| \rangle \leq \langle a_j \rangle} |X(t_0+s) - X(t_0)| \geq u \sum_{j=1}^N a_j^{H_j} \right) \leq e^{-c_{5,2}\, u^2} \tag{5.8}$$

for all $\langle a_j \rangle \in (0,1]^N$ such that $t_0 - \langle a_j \rangle \in I$ and $t_0 + \langle a_j \rangle \in I$.

Proof. We will use Lemma 5.3 to prove this lemma. Consider the Gaussian random field $\{Z(t), t \in S\}$ defined by $Z(t) = X(t_0+t) - X(t_0)$, where $S = \{t \in \mathbb{R}^N : |t_j| \leq a_j\}$. By Condition (A1), we have

$$d(s,t) = d_Z(s,t) \leq c_{2,2} \sum_{j=1}^N |s_j - t_j|^{H_j}, \qquad \forall\, s,t \in S.$$

Thus

$$N_d(S,\varepsilon) \leq c \left(\prod_{j=1}^N a_j \right) \varepsilon^{-\sum_{j=1}^N \frac{1}{H_j}}$$

and the diameter D of S is at most $c\sum_{j=1}^N a_j^{H_j}$. Let j_0 be the index such that $a_{j_0}^{H_{j_0}} = \max\{a_j^{H_j}, 1 \leq j \leq N\}$. It is elementary to verify that

$$\begin{aligned}
\int_0^D \sqrt{\log N_d(S,\varepsilon)}d\varepsilon &\leq c \int_0^D \sqrt{\log\left(\prod_{j=1}^N \frac{a_j}{\varepsilon^{1/H_j}} \right)}\, d\varepsilon \\
&\leq c \int_0^{cNa_{j_0}^{H_{j_0}}} \sqrt{\log \left(\frac{a_{j_0}^{H_{j_0}}}{\varepsilon} \right)^{\sum_{j=1}^N \frac{1}{H_j}}}\, d\varepsilon \\
&= c\, a_{j_0}^{H_{j_0}} \int_0^{cN} \sqrt{\log\left(\frac{1}{\eta}\right)}\, d\eta \\
&\leq c \sum_{j=1}^N a_j^{H_j}.
\end{aligned}$$

Hence the conclusion of the lemma follows from (5.7). □

The following truncation inequalities are extensions of those in Loéve (1977, p. 209) for $N=1$ and (3.4) and (3.5) in Xiao (1996) for $N>1$ and ρ being replaced by the Euclidean metric. In the current form, they are proved in Luan and Xiao (2010).

Lemma 5.5. *Let Δ be a nonnegative symmetric Borel measure on $\mathbb{R}^N \backslash \{0\}$ which satisfies (5.2). Then for any $u > 0$ and any $t \in \mathbb{R}^N$ with $\rho(0,t)u \leq 1/N$ we have*

$$\int_{\{\xi:\rho(0,\xi)<u\}} \langle t,\xi\rangle^2 \Delta(d\xi) \leq c \int_{\mathbb{R}^N} (1-\cos\langle t,\xi\rangle)\Delta(d\xi) \tag{5.9}$$

and for all $u>0$

$$\int_{\{\xi:\rho(0,\xi)\geq u\}} \Delta(d\xi) \leq c\, u^Q \int_{\{v:\rho(0,v)\leq 1/u\}} dv \int_{\mathbb{R}^N} (1-\cos\langle v,\xi\rangle)\Delta(d\xi), \tag{5.10}$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$.

We are in position to prove Theorem 5.1. Due to the stationarity of increments of X, it is sufficient to consider $t_0 = \langle 0 \rangle$. However, we will keep writing t_0 because the method for proving Theorem 5.1

remains valid as long as X has an appropriate stochastic integral representation. In particular, we will see in Section 6 that the proof below can be modified to obtain local moduli of continuity for fractional Brownian sheets, which do not have stationary increments in the usual sense.

Proof of Theorem 5.1. For any $\varepsilon = \langle \varepsilon_j \rangle \in (0,1)^N$, put

$$M(\varepsilon) = \sup_{\langle |s_j| \rangle \le \langle \varepsilon_j \rangle} \frac{|X(t_0+s) - X(t_0)|}{\gamma(s)}.$$

Note that, even though $M(\varepsilon)$ is a nondecreasing function of $\varepsilon \in (0,1)^N$ in the partial order $\le$, it is in general not monotone in $\|\varepsilon\|$. We claim that

$$\limsup_{\|\varepsilon\| \to 0+} M(\varepsilon) \le c_{5,3} \qquad \text{a.s.} \tag{5.11}$$

for some constant $c_{5,3} > 0$ and

$$\limsup_{\|\varepsilon\| \to 0+} M(\varepsilon) \ge \sqrt{2} \qquad \text{a.s.} \tag{5.12}$$

Before proving (5.11) and (5.12), let us notice that, (5.11) and the proof of Lemma 7.1.1 in Marcus and Rosen (2006) imply (5.5) and the constant $\kappa_2 \in [\sqrt{2}, c_{5,3}]$.

Hence, it only remains to verify (5.11) and (5.12). We show (5.11) first. For any $\mathbf{n} = (n_1, ..., n_N) \in \mathbb{N}^N$, let $h_{\mathbf{n}} = \langle 2^{-n_j} \rangle$. Let $\delta > 0$ be a constant whose value will be determined later. Define the event

$$F_{\mathbf{n}} = \left\{ \sup_{h_{\mathbf{n}} \le \langle |s_j| \rangle \le h_{\mathbf{n}-\langle 1 \rangle}} \gamma(s)^{-1} |X(t_0+s) - X(t_0)| \ge \delta \right\}.$$

By Condition (A1), we see that for any $s \in \mathbb{R}^N$ that satisfies $h_{\mathbf{n}} \le \langle |s_j| \rangle \le h_{\mathbf{n}-\langle 1 \rangle}$ we have

$$\gamma(s) \ge c_{2,1} \left(\sum_{j=1}^N 2^{-n_j H_j} \right) \sqrt{\log\log \left(1 + \prod_{j=1}^N 2^{(n_j-1)H_j} \right)}.$$

This and Lemma 5.4 imply

$$\begin{aligned} \mathbb{P}(F_{\mathbf{n}}) \quad &\le \exp\left(-c_{5,4}\delta^2 \log\log\left(1 + \prod_{j=1}^N 2^{(n_j-1)H_j}\right)\right) \\ &\le c \left(\sum_{j=1}^N n_j \right)^{-c_{5,4}\delta^2}, \end{aligned}$$

where $c_{5,4} = c_{5,2} c_{2,1}^2$. By taking δ large enough such that $c_{5,4}\delta^2 > N$, we see that

$$\sum_{\mathbf{n} \in \mathbb{N}^N} \mathbb{P}(F_{\mathbf{n}}) \le c \sum_{\mathbf{n} \in \mathbb{N}^N} \|\mathbf{n}\|^{-c_{5,4}\delta^2} < \infty.$$

Thus, by the Borel-Cantelli lemma, a.s. only finitely many of the events $F_{\mathbf{n}}$ occur. This implies

$$\limsup_{\|\mathbf{n}\| \to \infty} \sup_{h_{\mathbf{n}} \le \langle |s_j| \rangle \le h_{\mathbf{n}-\langle 1 \rangle}} \frac{|X(t_0+s) - X(t_0)|}{\gamma(s)} \le \delta \qquad \text{a.s.} \tag{5.13}$$

Let $s \in \mathbb{R}^N$ be a point such that, for some $\mathbf{n}^0 \in \mathbb{N}^N$ we have $|s_j| \le 2^{-n_j^0}$ for every $1 \le j \le N$. Now we choose $\mathbf{n} \in \mathbb{N}^N$ so that

$$2^{-n_j} < |s_j| \le 2^{-n_j+1}$$

for every $1 \le j \le N$. This and (5.13) yield (5.11).

Now we show that (5.12) holds. For this purpose, it is sufficient to provide a sequence $\langle \varepsilon_j^{(n)} \rangle \in (0,1)^N$ such that $\|\langle \varepsilon_j^{(n)} \rangle\| \to 0$ and

$$\limsup_{n\to\infty} \frac{\big|X(t_0 + \langle \varepsilon_j^{(n)} \rangle) - X(t_0)\big|}{\gamma(\langle \varepsilon_j^{(n)} \rangle)} \ge \sqrt{2} \qquad \text{a.s.} \tag{5.14}$$

To this end we will use the spectral representation (5.3) of X to create independence among the random variables. This argument is a modification of those in Monrad and Rootzen (1995), Talagrand (1995) or Li and Shao (2001) so that it adapts to the anisotropy of X.

For any $0 < \mu < 1$ and $n \ge 1$, we define $\langle \varepsilon_j^{(n)} \rangle = (\varepsilon_1^{(n)}, \ldots, \varepsilon_N^{(n)})$ by

$$\varepsilon_j^{(n)} = \exp\big(- H_j^{-1}\, n^{1+\mu}\big) \quad (j = 1, \ldots, N).$$

Then $\rho(0, \langle \varepsilon_j^{(n)} \rangle) = N \exp\big(- n^{1+\mu}\big)$.

For every integer $n \ge 1$, we denote

$$d_n = \exp(n^{1+\mu} + n^{\mu})$$

and define the following Gaussian random fields

$$\widetilde{X}_n(t) = \int_{\{\rho(0,\xi) \notin (d_{n-1}, d_n]\}} \big(e^{i\langle t, \xi\rangle} - 1\big)\, \mathcal{M}(d\xi), \qquad \forall t \in \mathbb{R}^N \tag{5.15}$$

and

$$X_n(t) = \int_{\{\rho(0,\xi) \in (d_{n-1}, d_n]\}} \big(e^{i\langle t, \xi\rangle} - 1\big)\, \mathcal{M}(d\xi), \qquad \forall t \in \mathbb{R}^N. \tag{5.16}$$

Then $\{\widetilde{X}_n(t), t \in \mathbb{R}^N\}$ and $\{X_n(t), t \in \mathbb{R}^N\}$ are independent and $X(t) = X_n(t) + \widetilde{X}_n(t)$ for all $t \in \mathbb{R}^N$. Moreover, the random fields $\{X_n(t), t \in \mathbb{R}^N\}$, $n = 1, 2, ...$ are independent. Notice that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{|X(t_0 + \langle \varepsilon_j^{(n)} \rangle) - X(t_0)|}{\gamma(\langle \varepsilon_j^{(n)} \rangle)} \\
&\ge \limsup_{n\to\infty} \frac{|X_n(t_0 + \langle \varepsilon_j^{(n)} \rangle) - X_n(t_0)|}{\gamma(\langle \varepsilon_j^{(n)} \rangle)} - \limsup_{n\to\infty} \frac{|\widetilde{X}_n(t_0 + \langle \varepsilon_j^{(n)} \rangle) - \widetilde{X}_n(t_0)|}{\gamma(\langle \varepsilon_j^{(n)} \rangle)} \\
&=: \limsup_{n\to\infty} I_1(n) - \limsup_{n\to\infty} I_2(n).
\end{aligned} \tag{5.17}$$

By the definition of $\widetilde{X}_n(t)$ we have

$$\begin{aligned}
&\mathbb{E}\big(\widetilde{X}_n(t_0 + \langle \varepsilon_j^{(n)} \rangle) - \widetilde{X}_n(t_0)\big)^2 \\
&= 2\left(\int_{\rho(0,\xi) \le d_{n-1}} + \int_{\rho(0,\xi) > d_n} \right) \big(1 - \cos\langle \langle \varepsilon_j^{(n)} \rangle, \xi\rangle\big)\, \Delta(d\xi) \\
&\le \int_{\rho(0,\xi) \le d_{n-1}} \langle \varepsilon_j^{(n)}, \xi \rangle^2\, \Delta(d\xi) + 4 \int_{\rho(0,\xi) > d_n} \Delta(d\xi).
\end{aligned}$$

To derive the above inequality, we bound $1 - \cos\langle t, x\rangle$ by $\langle t, x\rangle^2/2$ and by 2, respectively.

Now we estimate the last two integrals separately. Denote $U = \exp\big(\mu(n-1)^{\mu}\big)$. Notice that

$$\begin{aligned}\rho(0, \langle\varepsilon_j^{(n)}U\rangle)d_{n-1} &\le N\, U^{\max\{H_i\}} \exp\big(-n^{1+\mu} + (n-1)^{1+\mu} + (n-1)^{\mu}\big)\\ &\le N\, \exp\big(-\mu(1-\max\{H_i\})(n-1)^{\mu}\big),\end{aligned}$$

which is smaller than $1/N$ for n large. It follows from (5.9) that

$$\begin{aligned}\int_{\rho(0,\xi)\le d_{n-1}} \langle\langle\varepsilon_j^{(n)}\rangle, \xi\rangle^2\,\Delta(d\xi) &= U^{-2}\int_{\rho(0,\xi)\le d_{n-1}} \langle\langle\varepsilon_j^{(n)}U\rangle, \xi\rangle^2\,\Delta(d\xi)\\ &\le c\,U^{-2}\sigma^2(\langle\varepsilon_j^{(n)}U\rangle)\\ &\le c\,U^{-2(1-\max\{H_i\})}\rho(0, \langle\varepsilon_j^{(n)}\rangle)^2,\end{aligned} \tag{5.18}$$

where the last inequality follows from condition (A1).

On the other hand, (5.10) and condition (A1) imply that

$$\begin{aligned}\int_{\rho(0,\xi)>d_n} \Delta(d\xi) &\le c\,d_n^{Q}\int_{v:\rho(0,v)\le d_n^{-1}} \sigma^2(v)\,dv\\ &\le c\,d_n^{-2} = c\,\rho(0, \langle\varepsilon_j^{(n)}\rangle)^2 \exp\big(-2n^{\mu}\big).\end{aligned} \tag{5.19}$$

Combining (5.18) and (5.19) we obtain

$$\begin{aligned}\mathbb{E}\big(\widetilde{X}_n(t_0 + \langle\varepsilon_j^{(n)}\rangle) - \widetilde{X}_n(t_0)\big)^2 &\le c\big(U^{-2(1-\max\{H_i\})}\rho(0, \langle\varepsilon_j^{(n)}\rangle)^2 + d_n^{-2}\big)\\ &\le c\,\rho(0, \langle\varepsilon_j^{(n)}\rangle)^2\,\exp\big(-2(1-\max\{H_i\})\mu n^{\mu}\big).\end{aligned} \tag{5.20}$$

Hence for any $\eta > 0$ we have

$$\begin{aligned}\mathbb{P}\big(I_2(n) \ge \eta\big) &\le \mathbb{P}\Big(\big|\widetilde{X}_n(t_0 + \langle\varepsilon_j^{(n)}\rangle) - \widetilde{X}_n(t_0)\big| \ge \eta\gamma(\langle\varepsilon_j^{(n)}\rangle)\Big)\\ &\le \mathbb{P}\Big(|N(0,1)| \ge \eta\sqrt{\log n}\,\exp\big((1-\max\{H_i\})\mu\, n^{\mu}\big)\Big)\\ &\le n^{-2}\end{aligned}$$

for all n large enough. Thus $\sum_{n=1}^{\infty} \mathbb{P}\big(I_2(n) \ge \eta\big) < \infty$. By the Borel-Cantelli lemma and the arbitrariness of η, we obtain

$$\limsup_{n\to\infty} I_2(n) = 0 \qquad \text{a.s.} \tag{5.21}$$

In order to estimate $\limsup_{n\to\infty} I_1(n)$, notice that

$$\mathbb{E}\Big(X_n(t_0 + \langle\varepsilon_j^{(n)}\rangle) - X_n(t_0)\Big)^2 \le \sigma^2\big(0, \langle\varepsilon_j^{(n)}\rangle\big).$$

It follows from this and (4.11) that for any $0 < \eta < 1$,

$$\begin{aligned}&\mathbb{P}\Big(|X_n(t_0 + \langle\varepsilon_j^{(n)}\rangle) - X_n(t_0)| \ge (1-\eta)\sqrt{2}\,\gamma(\langle\varepsilon_j^{(n)}\rangle)\Big)\\ &\ge \mathbb{P}\Big(|N(0,1)| \ge (1-\eta)\sqrt{2\log\log(1+\exp(N\,n^{1+\mu})}\Big)\\ &\ge c\,n^{-(1-\eta)^2(1+\mu)}.\end{aligned}$$

Now we choose $\mu > 0$ small such that $(1-\eta)^2(1+\mu) < 1$ and consequently $\sum_{n=1}^{\infty} \mathbb{P}\big(I_1(n) \ge (1-\eta)\sqrt{2}\big) = \infty$. Since the events $\{I_1(n) \ge (1-\eta)\sqrt{2}\}$ $(n = 1, 2, \ldots)$ are independent, the Borel-Cantelli lemma and the arbitrariness of η yield

$$\limsup_{n\to\infty} I_1(n) \ge \sqrt{2} \qquad \text{a.s.} \tag{5.22}$$

Hence (5.12) follows from (5.17), (5.21) and (5.22). The proof of Theorem 5.1 is now completed. □

Combining Theorem 5.1 and Lemma 2.1 we derive the following exact local modulus of continuity.

Theorem 5.6 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with stationary increments and $X(\langle 0\rangle) = 0$. If X satisfies Condition (A1) for $I = [0, 1]^N$, then there is a positive and finite constant κ_3 such that for every $t_0 \in \mathbb{R}^N$ we have*

$$\lim_{\varepsilon\to 0+} \sup_{s:\sigma(s)\le\varepsilon} \frac{|X(t_0 + s) - X(t_0)|}{\sigma(s)\sqrt{\log\log(1 + \sigma(s)^{-1})}} = \kappa_3 \quad a.s. \tag{5.23}$$

Proof. By Lemma 2.1, it suffices to prove that there exists a finite constant c such that

$$\lim_{\varepsilon\to 0+} \sup_{s:\sigma(s)\le\varepsilon} \frac{|X(t_0 + s) - X(t_0)|}{\sigma(s)\sqrt{\log\log(1 + \sigma(s)^{-1})}} \le c \quad \text{a.s.} \tag{5.24}$$

and

$$\lim_{\varepsilon\to 0+} \sup_{s:\sigma(s)\le\varepsilon} \frac{|X(t_0 + s) - X(t_0)|}{\sigma(s)\sqrt{\log\log(1 + \sigma(s)^{-1})}} \ge \sqrt{2}. \tag{5.25}$$

Eq. (5.24) can be proved by using Lemma 5.4 and the Borel-Cantelli lemma which is simpler than the proof of (5.11). We omit the details.

On the other hand, we notice that for $\langle\varepsilon_j^{(n)}\rangle$ as in the proof of Theorem 5.1, we have

$$\sigma(\langle\varepsilon_j^{(n)}\rangle)\sqrt{\log\log(1 + \sigma(\langle\varepsilon_j^{(n)}\rangle)^{-1})} \sim \gamma(\langle\varepsilon^{(n)}\rangle)$$

as $n \to \infty$. It follows from (5.14) that

$$\lim_{n\to\infty} \frac{|X(t_0 + \langle\varepsilon_j^{(n)}\rangle) - X(t_0)|}{\sigma(\langle\varepsilon_j^{(n)}\rangle)\sqrt{\log\log(1 + \sigma(\langle\varepsilon_j^{(n)}\rangle)^{-1})}} \ge \sqrt{2},$$

which implies (5.25). The proof is finished.

6 Applications

In this section we apply Theorems 4.1, 5.1 and 5.6 to a class of Gaussian random fields with stationary increments discussed in Biermé, Meerschaert and Scheffler (2007), Xiao (2009) and Xue and Xiao (2009) and to fractional Brownian sheets.

6.1 Gaussian random fields with stationary increments

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments and $X(\langle 0\rangle) = 0$ a.s. We further assume that its spectral measure is absolutely continuous with density function $f(\xi)$ which satisfies the following property: There exist positive finite constants $c_{6,1}$, $c_{6,2}$ and a vector $(H_1, \ldots, H_N) \in (0, 1)^N$ such that

$$\frac{c_{6,1}}{\left(\sum_{j=1}^N |\xi_j|^{H_j}\right)^{2+Q}} \le f(\xi) \le \frac{c_{6,2}}{\left(\sum_{j=1}^N |\xi_j|^{H_j}\right)^{2+Q}} \tag{6.1}$$

for all $\xi \in \mathbb{R}^N$, with $|\xi| > 1$. Here $Q = \sum_{j=1}^N \frac{1}{H_j}$.

As an interesting example, we mention that Robeva and Pitt (2004, Proposition 10) showed that the Gaussian random field

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{e^{i(\xi_1 t + \xi_2 x)} - 1}{i\xi_1 + \xi_2^2} \, W(d\xi_1, d\xi_2), \qquad \forall\, t \in \mathbb{R}_+,\, x \in \mathbb{R} \tag{6.2}$$

is a solution to the stochastic heat equation $\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + \dot{W}$, where $\dot{W}(t,x)$ is an $\mathbb{R}$-valued space-time white noise and $u(0,0) = 0$. The Gaussian random field $\{u(t,x), t \in \mathbb{R}_+,\, x \in \mathbb{R}\}$ in (6.2) has stationary increments with spectral density

$$f(\xi) = \frac{1}{\xi_1^2 + \xi_2^4}.$$

It satisfies (6.1) with $H_1 = 1/4$ and $H_2 = 1/2$.

Another example comes from Biermé, Meerschaert and Scheffler (2007). Use the stochastic integral representation (5.3) with $\mathbf{Y} = 0$, and $\mathcal{M}(d\xi) = \psi(\xi)^{-1-Q/2}\widetilde{W}(d\xi)$, where $\widetilde{W}(d\xi)$ is a centered complex-valued Gaussian random measure with Lebesgue control measure, and the filter $\psi(\xi)$ satisfies $\psi(\xi) \neq 0$ for all $\xi \neq 0$ as well as the scaling property $\psi(c^{E'}\xi) = c\psi(\xi)$ for all $c > 0$ and all $\xi \in \mathbb{R}^N$. Here $Q = \mathrm{trace}(E)$, E' is the transpose of E, $c^E = \exp(E \ln c)$ where $\exp(A) = \sum_{n=0}^\infty A^n/n!$ is the usual matrix exponential, and we assume that every eigenvalue of the matrix E has real part greater than one. Then Theorem 4.1 in Biermé, Meerschaert and Scheffler (2007) shows that the Gaussian random field

$$X(t) = \int_{\mathbb{R}^N} (e^{i\langle t, \xi\rangle} - 1)\psi(\xi)^{-1-Q/2}\widetilde{W}(d\xi) \tag{6.3}$$

exists and is stochastically continuous. Corollary 4.2 in Biermé, Meerschaert and Scheffler (2007) shows that $X(t)$ is anisotropic, satisfies the following operator scaling property

$$\{X(c^E t),\, t \in \mathbb{R}^N\} \stackrel{d}{=} \{cX(t),\, t \in \mathbb{R}^N\}, \qquad \forall\, c > 0. \tag{6.4}$$

and has stationary increments. For example, we can take $E = \mathrm{diag}(1/H_1, \ldots, 1/H_N)$ where $0 < H_j < 1$. Then, (6.4), along with stationary increments, implies that for every $t_0 \in \mathbb{R}^N$,

$$\big\{X(t_0 + cre_i) - X(t_0),\, r \in \mathbb{R}\big\} \stackrel{d}{=} \big\{c^{H_i}\big(X(t_0 + re_i) - X(t_0)\big),\, r \in \mathbb{R}\big\},$$

where $e_1, \ldots, e_N$ are the standard basis vectors for $\mathbb{R}^N$. Hence the ith coordinate process started at any point $t_0 \in \mathbb{R}^N$ is a fractional Brownian motion with Hurst index H_i.

To be explicit, we could use the symmetric filter

$$\psi(\xi) = \sum_{j=1}^N C_j |\xi_j|^{H_j}, \tag{6.5}$$

where $C_j > 0$ are arbitrary constants. Then the spectral density of $X(t)$ is

$$f(\xi) = \frac{1}{\Big(\sum_{j=1}^N C_j |\xi_j|^{H_j}\Big)^{2+Q}}$$

so that (6.1) holds.

The anisotropic Gaussian random fields (6.3) are useful for modeling physical properties in underground aquifers, see for example Benson *et al.* (2006), Hu *et al.* (2009), and references cited therein. The anisotropy in these applications reflects the layering and self-organization of the porous medium under the effects of gravity and ground water flow, which causes a different Hurst index in each coordinate. A local Whittle method for estimating the Hurst indices $H_1, \ldots, H_N$ in this model from data was provided by Guo, Lim and Meerschaert (2009). A least squares estimator for $H_1, \ldots, H_N$ appeared in Beran, Ghosh and Schell (2009).

Some regularity and fractal properties of Gaussian random fields with stationary increments whose spectral density satisfies (6.1) have been studied by Xiao (2009), Xue and Xiao (2009). In particular, the following lemma is a consequence of Theorem 3.1 and Lemma 3.2 in Xue and Xiao (2009).

Lemma 6.1 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments and spectral density $f(\xi)$ which satisfies (6.1). Then, for any given constant $T > 0$, there exist constants $c_{6,3} > 0$ and $c_{6,4} > 0$ such that for $s,\, t \in [-T, T]^N$,*

$$c_{6,3} \sum_{j=1}^N |s_j - t_j|^{2H_j} \le \mathbb{E}\big(X(s) - X(t)\big)^2 \le c_{6,4} \sum_{j=1}^N |s_j - t_j|^{2H_j}. \tag{6.6}$$

Moreover there exists a constant $c_{6,5} > 0$ such that for all integers $n \ge 1$ and all $u,\, t^1, \cdots, t^n \in [-T, T]^N$,

$$\mathrm{Var}\big(X(u)|X(t^1), \cdots, X(t^n)\big) \ge c_{6,5} \min_{0 \le k \le n} \sum_{j=1}^N |u_j - t_j^k|^{2H_j}, \tag{6.7}$$

where $t^0 = 0$.

It follows from Lemma 6.1 that X satisfies conditions (A1) and (A3) on all compact intervals $I \subset \mathbb{R}^N$. Hence we can derive from Theorems 4.1 and 5.6 the following exact uniform and local moduli of continuity for X.

Theorem 6.2 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments. We assume its spectral density function satisfies (6.1). Then*

$$\lim_{\varepsilon \to 0+} \sup_{s,t \in [0,1]^N : \sigma(t-s) \le \varepsilon} \frac{|X(t) - X(s)|}{\sigma(t-s)\sqrt{\log(1 + \sigma(t-s)^{-1})}} = \kappa_4 \qquad a.s. \tag{6.8}$$

Moreover, for every $t_0 \in \mathbb{R}^N$,

$$\lim_{\|\varepsilon\| \to 0+} \sup_{\langle |s_j| \rangle \le \langle \varepsilon_j \rangle} \frac{|X(t_0 + s) - X(t_0)|}{\gamma(s)} = \kappa_5 \quad a.s. \tag{6.9}$$

where $\gamma(s)$ is defined in (5.6), and

$$\lim_{\varepsilon \to 0+} \sup_{s : \sigma(s) \le \varepsilon} \frac{|X(t_0 + s) - X(t_0)|}{\sigma(s)\sqrt{\log\log(1 + \sigma(s)^{-1})}} = \kappa_6 \quad a.s. \tag{6.10}$$

In the above, κ_4, κ_5 and κ_6 are positive and finite constants.

6.2 Fractional Brownian sheets

Fractional Brownian sheets were introduced by Kamont (1996) who also studied some of their regularity properties. Recall that, for a given vector $H = (H_1, ..., H_N) \in (0,1)^N$, a real-valued fractional Brownian sheet $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ with Hurst index H is a centered Gaussian random field with covariance function given by

$$\mathbb{E}\Big[B^H(s)B^H(t)\Big] = \prod_{j=1}^{N} \frac{1}{2}\Big(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\Big), \quad s, t \in \mathbb{R}^N. \tag{6.11}$$

It follows from (6.11) that B^H is an anisotropic Gaussian random field and $B^H = 0$ a.s. for every $t \in \mathbb{R}^N$ with at least one zero coordinate. Moreover, B^H satisfies a stronger scaling property: For all constants $c_1, \ldots, c_N > 0$,

$$\big\{B_H(c_1^{1/H_1}t_1, \ldots, c_N^{1/H_N}t_N),\, t \in \mathbb{R}^N\big\} \stackrel{d}{=} \big\{c_1 \cdots c_N B_H(t_1, \ldots, t_N),\, t \in \mathbb{R}^N\big\}. \tag{6.12}$$

Namely, B^H satisfies the *multi-self-similarity* as called by Genton, Perrin and Taqqu (2007). In particular (6.12) implies that for all $c > 0$,

$$\big\{B^H(c^E t),\, t \in \mathbb{R}^N\big\} \stackrel{d}{=} \Big\{c^N\, B^H(t),\ t \in \mathbb{R}^N\Big\}, \tag{6.13}$$

where $E = \mathrm{diag}(1/H_1, \ldots, 1/H_N)$. This is clearly different from (6.4).

Fractional Brownian sheets and the operator-scaling, stationary increment field $\{X(t),\, t \in \mathbb{R}^N\}$ defined by (6.3) and (6.5) are two important representatives of anisotropic Gaussian random fields. They share many sample path properties. For example, the random fractals such as the range, graph, inverse images of these two types of Gaussian random fields have the same Hausdorff dimensions; see Ayache and Xiao (2005), Biermé, Lacaux and Xiao (2009) and Xiao (2009). However, there are subtle and sometimes significant differences in other aspects such as small ball probabilities [Mason and Shi (2001), Belinsky and Linde (2002), Xiao (2009)], sharp Hölder conditions for the local times [Ayache, Wu and Xiao (2008), Wu and Xiao (2009)], Chung-type laws of the iterated logarithm and exact Hausdorff measure functions [Luan and Xiao (2010)]. Many questions regarding these later aspects remain open and seem to be challenging [e.g., the small probability problem]. Even though the properties of strong local nondeterminism have been useful for studying local times and exact Hausdorff measure functions in the aforementioned references as well as in this paper, new methods have to be developed for attacking these open problems regarding anisotropic Gaussian random fields.

The rest of this section is concerned with sample path regularity properties of fractional Brownian sheets. When $H = \langle 1/2 \rangle$, B^H is the ordinary Brownian sheet whose uniform and local moduli of continuity have been established by Orey and Pruitt (1973). Several authors have studied asymptotic properties of B^H for general $H \in (0,1)^N$. For example, Ayache and Xiao (2005) derived a sharp upper bound for the uniform modulus of continuity of fractional Brownian sheets by using the wavelet method. Wang (2007) established the exact uniform and local moduli of continuity for the increments of B^H over intervals and also obtained upper and lower bounds for the uniform modulus of continuity of fractional Brownian sheets. Blath and Martin (2008) provided an analysis of the propagation of singularities of a class of fractional Brownian sheets. In the following we study the exact uniform and local moduli of continuity for B^H in the sense of (1.1) and (1.2).

In order to apply our results in Sections 4 and 5 to fractional Brownian sheets, we notice that Ayache and Xiao (2005) and Wu and Xiao (2007) have shown that for every fixed $H = (H_1, \ldots, H_N) \in (0,1)$ and $\varepsilon \in (0,1)$, B^H satisfies Conditions (A1) and (A2) for all compact intervals $I \subset [\varepsilon, \infty)^N$. Hence the following theorem is a consequence of Theorem 4.1. It extends Theorem 2.4 of Orey and Pruitt (1973) and strengthens Theorem 1 in Ayache and Xiao (2005) and Theorem 3.2 in Wang (2007).

Theorem 6.3 *Let $\{B^H(t), t \in \mathbb{R}^N\}$ be a fractional Brownian sheet with index $H = (H_1, ..., H_N) \in (0,1)^N$ and let $I = [a,1]^N$, where $a \in (0,1)$ is a constant. Then*

$$\lim_{\varepsilon \to 0+} \sup_{s,t \in I, \sigma(s,t) \le \varepsilon} \frac{|B^H(t) - B^H(s)|}{\beta(s,t)} = \kappa_7 \quad a.s., \tag{6.14}$$

where $\beta(s,t)$ is defined as in (4.1) and κ_7 is a positive and finite constant.

Even though Theorems 5.1 and 5.6 can not be applied directly to B^H because it does not have stationary increments in the ordinary sense, one can apply the harmonizable representation of B^H and modify the proof of Theorem 5.6 to prove the following Theorem 6.4, which is complementary to Theorem 2 and Proposition 1 in Ayache and Xiao (2005). It should also be mentioned that, as shown by Theorems 4.1 and 4.2 of Wang (2007), the local moduli of continuity of B^H at $t = \langle 0 \rangle$ are rather different from (6.15) and (6.16) below. This difference is caused by the asymptotic behavior of B^H near the boundary $\partial \mathbb{R}^N_+$ (e.g., $\mathbb{E}\big(B^H(t)^2\big) = \prod_{j=1}^N |t_j|^{2H_j}$ goes to 0 faster than $\rho(0,t)^2$ as $\|t\| \to 0$) and the lack of stationarity for the increments of B^H.

Theorem 6.4 *Let $\{B^H(t), t \in \mathbb{R}^N\}$ be a fractional Brownian sheet with index $H = (H_1, \ldots, H_N) \in (0,1)^N$ and let $a \in (0,1)$ be a constant. Then for every $t_0 \in [a,1]^N$ there exist positive and finite constants κ_8 and κ_9 such that*

$$\limsup_{\|\varepsilon\| \to 0+} \sup_{\langle |s_j| \rangle \le \langle \varepsilon_j \rangle} \frac{|B^H(t_0 + s) - B^H(t_0)|}{\rho(0,s)\sqrt{\log\log(1 + \prod_{j=1}^N |s_j|^{-H_j})}} = \kappa_8 \quad a.s. \tag{6.15}$$

and

$$\lim_{\varepsilon \to 0+} \sup_{s: \sigma(s) \le \varepsilon} \frac{|B^H(t_0 + s) - B^H(t_0)|}{\rho(0,s)\sqrt{\log\log(1 + \rho(0,s)^{-1})}} = \kappa_9 \quad a.s. \tag{6.16}$$

Remark 6.5 The constants κ_8 and κ_9 may depend on t_0, but they are bounded from above and below by positive constants which only depend on H, a and N.

Proof of Theorem 6.4. The upper bounds in (6.15) and (6.16) follows respectively from the proofs of the upper bounds in Theorems 5.1 and 5.6, which only rely on condition (A1). For proving the lower bounds in (6.15) and (6.16), we need to modify the proofs of the lower bound in Theorems 5.1. Instead of using (5.3), we will make use of the following harmonizable representation for B^H:

$$B^H(t) = K_H^{-1} \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{e^{it_j \xi_j} - 1}{|\xi_j|^{H_j + \frac{1}{2}}} \widetilde{W}(d\xi), \tag{6.17}$$

where $K_H > 0$ is a normalizing constant and $\widetilde{W}$ is a centered complex-valued Gaussian random measure in $\mathbb{R}^N$ with Lebesgue control measure. Or one may use the stochastic integral representation given by (2.6) in Wang (2007).

Let $\langle \varepsilon_j^{(n)} \rangle$ and d_n be defined as in the proof of Theorem 5.1. Similarly to (5.15) and (5.16), we define

$$\widetilde{B}_n^H(t) = \int_{\rho(0,\xi)\notin(d_{n-1},d_n]} \prod_{j=1}^N \frac{e^{it_j\xi_j}-1}{|\xi_j|^{H_j+\frac{1}{2}}}\, \widetilde{W}(d\xi), \quad \forall t \in \mathbb{R}^N$$

and

$$B_n^H(t) = \int_{\rho(0,\xi)\in(d_{n-1},d_n]} \prod_{j=1}^N \frac{e^{it_j\xi_j}-1}{|\xi_j|^{H_j+\frac{1}{2}}}\, \widetilde{W}(d\xi), \quad \forall t \in \mathbb{R}^N.$$

Then the random fields $\widetilde{B}_n^H$ and B_n^H are independent. Moreover

$$\begin{aligned}
&\mathbb{E}\big(\widetilde{B}_n^H(t_0 + \langle \varepsilon_j^{(n)} \rangle) - \widetilde{B}_n^H(t_0)\big)^2 \\
&\le \int_{\rho(0,\xi)\le d_{n-1}} \bigg| \prod_{j=1}^N \big(e^{i(t_j+\varepsilon_j^{(n)})\xi_j} - 1\big) - \prod_{j=1}^N \big(e^{it_j\xi_j} - 1\big)\bigg| \frac{d\xi}{\prod_{j=1}^N |\xi_j|^{2H_j+1}} \\
&\quad + \int_{\rho(0,\xi)> d_n} \bigg| \prod_{j=1}^N \big(e^{i(t_j+\varepsilon_j^{(n)})\xi_j} - 1\big) - \prod_{j=1}^N \big(e^{it_j\xi_j} - 1\big)\bigg| \frac{d\xi}{\prod_{j=1}^N |\xi_j|^{2H_j+1}} \\
&= J_1 + J_2.
\end{aligned} \tag{6.18}$$

By using the triangle inequality and the fact that $t_0 \in [a,1]^N$, we derive directly that

$$\begin{aligned}
J_1 &\le c \sum_{j=1}^N \int_{|\xi_j|^{H_j}\le d_{n-1}} \big(1-\cos(\varepsilon_j^{(n)}\xi_j)\big) \frac{d\xi_j}{|\xi_j|^{2H_j+1}} \\
&\le c\, U^{-2(1-H_N)} \rho\big(0, \langle \varepsilon_j^{(n)} \rangle\big)^2.
\end{aligned} \tag{6.19}$$

To bound J_2, notice that $\rho(0,\xi) > d_n$ implies $|\xi_{j_0}|^{H_{j_0}} > d_n/N$ for some $j_0 \in \{1,\ldots,N\}$. For simplicity of notation, we assume $j_0 = 1$. Then

$$J_2 \le c \int_{|\xi_1|^{H_1}>d_n/N} \frac{d\xi_1}{|\xi_1|^{2H_1+1}} = c\, d_n^{-2}. \tag{6.20}$$

Combining (6.18), (6.19) and (6.20) we obtain

$$\mathbb{E}\big(\widetilde{B}_n^H(t_0 + \langle \varepsilon_j^{(n)} \rangle) - \widetilde{B}_n^H(t_0)\big)^2 \le c\, \rho(0, \langle \varepsilon_j^{(n)} \rangle)^2 \exp\big(-2(1-H_N)\mu n^\mu\big),$$

which is the same as (5.20). Now the same proof as that of Theorem 5.1 shows that for every $t \in [a,1]^N$

$$\limsup_{n\to\infty} \frac{\big|B^H(t_0 + \langle \varepsilon_j^{(n)} \rangle) - B^H(t_0)\big|}{\rho(0, \langle \varepsilon_j^{(n)} \rangle)\sqrt{\log\log(1 + \prod_{j=1}^N |\varepsilon_j^{(n)}|^{-H_j})}} \ge c_{6,6} \qquad \text{a.s.}$$

This proves the lower bounds in (6.15) and (6.16). Finally, by using Lemma 7.1.1 of Marcus and Rosen (2006) [one may also apply the wavelet expansion for B^H in Ayache and Xiao (2005) and Kolmorogov's 0-1 law], we derive from the above that (6.15) and (6.16) hold. □

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