A Class of Fractional Brownian Fields from Branching Systems and Their Regularity Properties

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Abstract

In this paper, we study the smoothness and exact modulus of continuity of a class of fractional Brownian fields arisen from fluctuation limits of branching systems. These Gaussian random fields satisfy a kind of operator scaling property and, depending on the choice of their parameters, may share similar fractal properties as those of fractional Brownian sheets or may be smooth in some (or all) directions. It is proved that these Gaussian random fields satisfy the property of sectorial local nondeterminism which is useful for further studying their sample path properties.

Keywords: Pseudo-fractional Brownian sheet, Directional differentiability, Exact modulus of continuity, Sectorial local nondeterminism, Branching particle system.

AMS 2010 Subject Classification: 60G15; 60G17; 60G18; 60G60; 60F17; 60J80

Running Title: A class of fractional Brownian fields and their regularity

*Research of Y. Li is partially supported by NSFC (No: 10901054).

†Research of Y. Xiao is partially supported by the NSF grant DMS-1006903.

1. Introduction

Consider the following branching particle system in $\mathbb{R}^N$. Particles start off at time $t=0$ from a Poisson random field with N-dimensional Lebesgue λ as its intensity measure, and they evolve independently. The spatial motion consists of an $\mathbb{R}^N$-valued stochastic process

$$\vec{\xi}=\{\vec{\xi}(t),t\geq 0\}=\{(\xi_1(t),\xi_2(t),\cdots,\xi_N(t)),t>0\},$$

where for every $0<k\leq N$, $\xi_k=\{\xi_k(t),t\geq 0\}$ is a symmetric α_k-stable Lévy process ($0<\alpha_k\leq 2$) and $\xi_1,\cdots,\xi_N$ are independent of each other. In addition, they split at a rate γ and the branching law at age t has the following generating function

$$g(s,t)=1-\frac{\mathrm{e}^{-\delta t}}{2}+\frac{\mathrm{e}^{-\delta t}}{2}s^2,\qquad 0\leq s\leq 1,\ t\geq 0.$$

Intuitively, in this model, the particles' motion in different direction is controlled by different mechanism and their ability of splitting new particles declines as their ages increase. For simplicity of notation, the vector $(\alpha_1,\cdots,\alpha_N)$ is denoted by $\vec{\alpha}$, and this model is called a $(N,\vec{\alpha},\delta,\gamma)$-degenerate branching particle system.

Let $N(s)$ denote the empirical measure of the particle system at time s, i.e. $N(s)(A)$ is the number of particles in the set $A\subset\mathbb{R}^N$ at time s. Li and Xiao [16] studied the limit of a sequence of scaled occupation time fluctuations

$$X_n(t)=\frac{1}{F_n}\int_0^{nt}(N_n(s)-f_n(s)\lambda)\mathrm{d}s \tag{1.1}$$

of $(N,\vec{\alpha},\delta_n,\gamma)$-degenerate branching particle systems, where F_n is a scaling constant and

$$f_n(s):=\bar{f}_n(s)\mathrm{e}^{-\delta_n s}:=\Big[1+\frac{\delta_n}{\gamma-\delta_n}(1-\mathrm{e}^{-(\gamma-\delta_n)s})\Big]\mathrm{e}^{-\delta_n s}. \tag{1.2}$$

Under the condition $n\delta_n\to\theta\in[0,\infty)$ as $n\to\infty$, which is referred to as weak degeneration, they proved that when $\bar{\alpha}:=\sum_{k=1}^N\alpha_k^{-1}>2$, the limit process of X_n gives rise to two anisotropic centered Gaussian random fields $Y_1=\{Y_1(x),x\in\mathbb{R}^N\}$ and $Y_2=\{Y_2(x),x\in\mathbb{R}^N\}$ which, up to a constant, have the following covariance functions

$$\mathbb{E}(Y_j(x)Y_j(y))=\int_{\mathbb{R}^N}\prod_{k=1}^N(\mathrm{e}^{ix_kz_k}-1)(\mathrm{e}^{-iy_kz_k}-1)\frac{\mathrm{d}z}{\psi_j^2(z)} \tag{1.3}$$

for all $x,y\in\mathbb{R}^N$ and $j=1,2$, where

$$\psi_1(z)=\sqrt{\sum_{k=1}^N|z_k|^{\alpha_k}\prod_{k=1}^N z_k},\qquad\text{and}\qquad \psi_2(z)=\sum_{k=1}^N|z_k|^{\alpha_k}\prod_{k=1}^N z_k, \tag{1.4}$$

respectively. Therefore, by Samorodnitsky and Taqqu [22, P.325-326], one can readily check that Y_j $(j=1,2)$ has the following harmonizable stochastic integral representation

$$Y_j(x)=\int_{\mathbb{R}^N}\prod_{k=1}^N(\mathrm{e}^{ix_kz_k}-1)\frac{\widetilde{\mathcal{M}}(\mathrm{d}z)}{\psi_j(z)}, \tag{1.5}$$

where $\widetilde{\mathcal{M}}$ is a centered complex-valued Gaussian random measure in $\mathbb{R}^N$ with Lebesgue control measure. It follows from (1.3) that Y_j is operator-scaling with exponent E which is the $N \times N$ diagonal matrix $E = \mathrm{diag}(1/\alpha_k)_{1\le k\le N}$. That is, for all constants $c > 0$

$$\big\{Y_j(c^E x),\, x \in \mathbb{R}^N\big\} \overset{f.d}{=} \big\{c^{\beta_j}\, Y_j(x),\, x \in \mathbb{R}^N\big\},$$

where $\overset{f.d}{=}$ means equality of all finite dimensional distributions and $\beta_j = \frac{1}{2}\big(j + \bar{\alpha}\big)$. We refer to Biermé, Meerschaert and Scheffler [5], Xiao [25] and Li and Xiao [15] for further information on operator-scaling random fields. Note that the Gaussian field Y_j defined by (1.5) does not have stationary increments in the ordinary sense, and it reminicent to a fractional Brownian sheet (fBs).

Recall that, for a fixed vector $\vec{h} = (h_k)_{1\le k\le N} \in (0,1)^N$, the fraction Brownian sheet $B_0^{\vec{h}} = \{B_0^{\vec{h}}(x), x \in \mathbb{R}^N\}$ has the harmonizable representation

$$B_0^{\vec{h}}(x) = C \int_{\mathbb{R}^N} \prod_{k=1}^{N} (\mathrm{e}^{i x_k z_k} - 1) \frac{\widetilde{\mathcal{M}}(\mathrm{d}z)}{\psi_0(z)}, \tag{1.6}$$

where $C > 0$ is a constant and $\psi_0(z) = \prod_{k=1}^N |z_k|^{h_k+1/2}$. Fractional Brownian sheets were introduced by Kamont [13] who also studied some of their regularity properties. As an important example of anisotropic Gaussian fields, fractional Brownian sheets have also been studied by several other authors. See, for example, Mason and Shi [18], Ayache, et al. [3], Ayache and Xiao [4], Herbin [12], Wu and Xiao [24]. Recently, Meerschaert, et al. [19] established the exact moduli of continuity for a large class of Gaussian random fields, including fraction Brownian sheets.

Motivated by the aforementioned articles, we introduce in this paper a class of fractional Brownian fields, which we call pseudo-fractional Brownian sheets (or pseudo-fBs, for brevity) and include Y_j $(j = 1, 2)$ in (1.5) as special cases; see Section 2 below. These Gaussian random fields have rich analytic and geometric properties. In this paper, we focus on the directional differentiability and the exact modulus of continuity of the sample paths. To study the exact modulus of continuity, we determine when pseudo-fractional Brownian sheets have the property of sectorial local nondeterminism.

From the results in Section 2, we know that for $j = 1, 2$, if $\alpha_k > 1/j$ for some k, then the Gaussian random field $Y_j(t)$ has a continuous version $\widetilde{Y}_j$ such that its partial derivative in the k-th direction is almost surely continuous and, if $\alpha_k \in (0, 1/j)$ for all $k = 1, 2, \cdots, N$, then Y_j does not have mean-square partial derivative in any direction. In the latter case, we show that Y_j satisfies the sectorial local non-determinism and the exact uniform modulus of continuity of Y_j on $[0,1]^N$ is $\rho_j(x,y)\sqrt{\ln(1+\rho_j(x,y)^{-1})}$ and, for every fixed $x \in (0,\infty)^N$, the exact local modulus of continuity of Y_j at x is $\rho_j(x,y)\sqrt{\ln\ln(1+\rho_j(x,y)^{-1})}$, where ρ_j is a metric defined by

$$\rho_j(x,y) = \sum_{k=1}^{N} |x_k - y_k|^{(1+j\alpha_k)/2}$$

for any $x = (x_k), y = (y_k) \in \mathbb{R}^N$.

The rest of this paper is organized as follows. In Section 2, we first define a general pseudo-fractional Brownian sheet and study the (directional) differentiability of its sample functions. In Section 3, we study the sectorial local nondeterminism of pseudo-fractional Brownian sheets. In Section 4, we prove the exact modulus of continuity of a nondifferentiable pseudo-fBs. In the last section, we reproduce the pseudo-fBs Y_1 and Y_2 via the functional fluctuation limits of a sequence of $(N, \vec{\alpha}, \delta_n, \gamma)$-degenerate branching particle systems under more a general condition than that in Li and Xiao [16]. Moreover, our Theorem 5.1 also strengthens Theorem 2.1 in [16] by proving weak convergence of X_n in the function space $C([\varepsilon, 1], \mathcal{S}'(\mathbb{R}^N))$.

Throughout the paper, we use C to denote an unspecified positive finite constant which may not necessarily be the same in each occurrence. More specific constants are denoted by $C_1, \cdots, C_7$.

2. Pseudo-fBs: definition and differentiability of sample paths

Given a constant $\beta > 0$ and a vector $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N) \in (0, 2]^N$ which satisfy the following condition:

$$\beta < \bar{\alpha} = \sum_{k=1}^{N} \frac{1}{\alpha_k}, \tag{2.1}$$

the Gaussian random field $Y = \{Y(x), x \in \mathbb{R}^N\}$ defined by

$$Y(x) = \int_{\mathbb{R}^N} \prod_{k=1}^{N} (\mathrm{e}^{i x_k z_k} - 1) \frac{\widetilde{\mathcal{M}}(\mathrm{d}z)}{\psi(z)}, \tag{2.2}$$

is called a pseudo-fractional Brownian sheet with parameters β and $\vec{\alpha}$. In the above, $\widetilde{\mathcal{M}}$ is a centered complex-valued Gaussian random measure in $\mathbb{R}^N$ with Lebesgue control measure and the function $\psi = \psi_{\beta,\vec{\alpha}}$ is defined by

$$\psi(z) = \left(\prod_{k=1}^{N} |z_k| \right) \left(\sum_{k=1}^{N} |z_k|^{\alpha_k} \right)^{\beta/2}.$$

It is easy to see that, in (1.5), Y_1 corresponds to $\beta = 1$ and Y_2 corresponds to $\beta = 2$. When $N = 1$ and $\beta\alpha_1 < 1$, the Gaussian process Y in (2.2) becomes fractional Brownian motion with index $H = (1 + \beta\alpha_1)/2$.

Proposition 2.1 *The Gaussian random field Y in (2.2) is well-defined if and only if (2.1) holds. Moreover, Y has the following operator-scaling property: For all constants $c > 0$,*

$$\big\{Y(c^E x),\, x \in \mathbb{R}^N\big\} \overset{f.d}{=} \big\{c^{(\bar{\alpha}-\beta)/2}\, Y(x),\, x \in \mathbb{R}^N\big\},$$

where E is the $N \times N$ diagonal matrix $E = \mathrm{diag}(1/\alpha_k)_{1 \le k \le N}$ and $\bar{\alpha} = \sum_{k=1}^{N} \alpha_k^{-1}$.

Proof. It is well-known that Y is a well-defined Gaussian field if and only if

$$\int_{\mathbb{R}^N} \prod_{k=1}^{N} \big(1 - \cos(x_k z_k)\big) \frac{\mathrm{d}z}{\psi^2(z)} = \int_{\mathbb{R}^N} \prod_{k=1}^{N} \frac{1 - \cos(x_k z_k)}{z_k^2} \frac{\mathrm{d}z}{(\sum_{k=1}^{N} |z_k|^{\alpha_k})^{\beta}} < \infty \tag{2.3}$$

for all $x = (x_k)_k \in \mathbb{R}^N$. We split the domain of integration into $\mathbb{R}^N \backslash [-1,1]^N$ and $[-1,1]^N$, and note that the integral over $\mathbb{R}^N \backslash [-1,1]^N$ is always convergent. Hence, (2.3) holds if and only if

$$\int_{[-1,1]^N} \frac{\mathrm{d}z}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta} < \infty. \tag{2.4}$$

It is elementary to show (see Lemma 8.6 in Xiao [25]) that (2.4) holds if and only if $\bar{\alpha} > \beta$. Finally, the operator-scaling property of Y can be verified by checking the covariance functions. This is straightforward and is omitted. □

Now we study the (directional) differentiability of pseudo-fBs. Let us recall some useful concepts. Suppose $u \in \mathbb{R}^N$ is a unit vector. A second order random field $\{X(x), x \in \mathbb{R}^N\}$ has mean square directional derivative $X_u'(x)$ at $x \in \mathbb{R}^N$ in the direction u if, as $h \to 0$, the random variables

$$X_{u,h}(x) = \frac{X(x+hu) - X(x)}{h}$$

converge to $X_u'(x)$ in the $L^2(\Omega, \mathbb{P})$ sense. In this case, we write $X_u'(x) = \mathrm{l.i.m.}_{h\to 0} X_{u,h}(x)$ and say that $X(x)$ is mean-square differentiable in the direction u. Especially, if

$$u = e_k = (\underbrace{0, \cdots, 0, 1}_{k}, 0, \cdots, 0),$$

then we say that $X(x)$ is mean-square differentiable in the k-th direction.

The existence of mean square directional derivative can be given in terms of the covariance function of X (see, e.g., Adler [1, Theorem 2.2.2]). However, for many theoretical and applied purposes, one often needs to work with random fields which have smooth sample functions. We refer to Adler and Taylor [2] and the reference therein for more information.

The following theorem provides an explicit criterion for Y to have a version whose sample functions have continuous partial derivatives. Higher order partial derivatives can also be considered similarly.

Theorem 2.1 *Suppose the Gaussian random field $Y = \{Y(x), x \in \mathbb{R}^N\}$ in (2.2) is well-defined, i.e. (2.1) holds.*

(i). If $\alpha_k > 1/\beta$ for some $k \in \{1, \ldots, N\}$, then Y has a version $\widetilde{Y}$ with continuous sample functions such that its k-th partial derivative $\widetilde{Y}_k(x)$ is continuous almost surely.

(ii). If $\alpha_k > 1/\beta$ for every $k \in \{1, \ldots, N\}$, then Y has a version $\widetilde{Y}$ whose sample functions are continuously differentiable almost surely.

Proof. The method of proof is similar to that of Theorem 4.8 in Xue and Xiao [26]. See also Potthoff [21] for related results. For simplicity, we only prove (i) and suppose, without loss of generality, that $k = N$.

The proof is divided into three parts, namely, (a) for every $x \in \mathbb{R}^N$, the mean square partial derivative $Y_N'(x)$ exists; (b) for any $T > 0$, there exist constants $C > 0$ and $\eta > 0$ such that

$$\mathbb{E}\big[Y_N'(x) - Y_N'(y)\big]^2 \le C\,|x-y|^\eta, \quad \forall\, x, y \in [-T,T]^N \tag{2.5}$$

and (c) construction of a version $\widetilde{Y}$ with the desired property.

(a). For any fixed $x \in \mathbb{R}^N$, to prove $Y_{N,h}(x) = \frac{Y(x+he_N)-Y(x)}{h}$ converges in $L^2(\Omega, \mathbb{P})$ as $h \to 0$ is equivalent to verify that $\{Y_{N,h}(x)\}_h$ is a Cauchy sequence. For simplicity, we show that for any $0 < r < h$

$$\mathbb{E}(|Y_{N,h}(x) - Y_{N,r}(x)|^2) \to 0, \tag{2.6}$$

as $h \to 0$. By (2.2), we have

$$\begin{aligned} & Y_{N,h}(x) - Y_{N,r}(x) \\ &= \int_{\mathbb{R}^N} \Big(\prod_{k=1}^{N-1} \frac{\mathrm{e}^{ix_k z_k} - 1}{z_k} \Big) \Big(\frac{\mathrm{e}^{ihz_N} - 1}{hz_N} - \frac{\mathrm{e}^{irz_N} - 1}{rz_N} \Big) \frac{\mathrm{e}^{ix_N z_N} \widetilde{\mathcal{M}}(\mathrm{d}z)}{(\sum_{k=1}^N |z_k|^{\alpha_k})^{\beta/2}}. \end{aligned}$$

Therefore

$$\mathbb{E}\left(|Y_{N,h}(x) - Y_{N,r}(x)|^2\right) = \int_{\mathbb{R}^N} g(x, z^{(N-1)}) f(h, r, z_N) \frac{\mathrm{d}z}{(\sum_{k=1}^N |z_k|^{\alpha_k})^{\beta}}, \tag{2.7}$$

where $z^{(N-1)} = (z_1, \cdots, z_{N-1})$,

$$g(x, z^{(N-1)}) := 2^{N-1} \prod_{k=1}^{N-1} \frac{1 - \cos(x_k z_k)}{z_k^2} \le C(x), \tag{2.8}$$

where $C(x)$ denotes a constant depending on x, and

$$f(h, r, z_N) := \left| \frac{\mathrm{e}^{ihz_N} - 1}{hz_N} - \frac{\mathrm{e}^{irz_N} - 1}{rz_N} \right|^2 \le 2 \tag{2.9}$$

for all positive r, h, z_N. Furthermore

$$\begin{aligned} f(h, r, z_N) &= \frac{1}{r^2 h^2 z_N^2} \Big(\big|r\cos(hz_N) - r - h\cos(rz_N) + h\big|^2 \\ &\qquad\qquad + \big|r\sin(hz_N) - h\sin(rz_N)\big|^2 \Big) \\ &\le \frac{1}{r^2 h^2 z_N^2} \Big(r^2 h^4 z_N^4 + h^2 r^4 z_N^4 + h^2 r^2 z_N^6 (h^2 + r^2)^2 \Big) \\ &\le 2h^2 z_N^2 + 4h^4 z_N^4, \end{aligned} \tag{2.10}$$

where we have used the elementary inequalities $|1 - \cos u| \le u^2/2$ and $|u \sin v - v \sin xu| \le uv^3 + vu^3$ for all $u, v > 0$. (The second inequality follows from the inequalities $\sin v \le v$ and $\sin u \ge u - \frac{u^3}{3!}$ for all u, $v > 0$.)

Now we break the right hand side of (2.7) into five parts.

$$\text{r.h.s. of (2.7)} = I_1 + I_2 + I_3 + I_4 + I_5, \tag{2.11}$$

where

$$I_1 = \int_{|z_N|\le 1} f(h,r,z_N)\mathrm{d}z_N \int_{[-1,1]^{N-1}} \frac{g(x,z^{(N-1)})\,\mathrm{d}z^{(N-1)}}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta}, \tag{2.12}$$

$$I_2 = \int_{1\le|z_N|\le 1/\sqrt{h}} f(h,r,z_N)\mathrm{d}z_N \int_{[-1,1]^{N-1}} \frac{g(x,z^{(N-1)})\,\mathrm{d}z^{(N-1)}}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta}, \tag{2.13}$$

$$I_3 = \int_{|z_N|\ge 1/\sqrt{h}} f(h,r,z_N)\mathrm{d}z_N \int_{[-1,1]^{N-1}} \frac{g(x,z^{(N-1)})\,\mathrm{d}z^{(N-1)}}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta}, \tag{2.14}$$

$$I_4 = \int_{|z_N|\le 1/\sqrt{h}} f(h,r,z_N)\mathrm{d}z_N \int_{\mathbb{R}^{N-1}\setminus[-1,1]^{N-1}} \frac{g(x,z^{(N-1)})\,\mathrm{d}z^{(N-1)}}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta}, \tag{2.15}$$

$$I_5 = \int_{|z_N|\ge 1/\sqrt{h}} f(h,r,z_N)\mathrm{d}z_N \int_{\mathbb{R}^{N-1}\setminus[-1,1]^{N-1}} \frac{g(x,z^{(N-1)})\mathrm{d}z^{(N-1)}}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta}. \tag{2.16}$$

Substituting (2.8), (2.10) into (2.12) and (2.13) yields that

$$I_1 \le C(h^2+h^4)\int_{[-1,1]^N} \frac{\mathrm{d}z}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta} \le C(h^2+h^4), \tag{2.17}$$

since $\int_{[-1,1]^N} \frac{\mathrm{d}z}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta} < \infty$ for $\bar{\alpha} > \beta$, and that

$$I_2 \le C(h+h^2)\int_1^{1/\sqrt{h}} \mathrm{d}z_N \int_{[-1,1]^{N-1}} \frac{\mathrm{d}z^{(N-1)}}{|z_N|^{\alpha_N\beta}} \le C(h+h^2), \tag{2.18}$$

since $\alpha_N\beta > 1$. Furthermore, substituting (2.8), (2.9) into (2.14) leads to

$$I_3 \le C\int_{1/\sqrt{h}}^\infty \mathrm{d}z_N \int_{[-1,1]^{N-1}} \frac{\mathrm{d}z^{(N-1)}}{|z_N|^{\alpha_N\beta}} \le Ch^{(\beta\alpha_N-1)/2}. \tag{2.19}$$

Notice that the second integral in (2.15) and (2.16) are convergent and, for $z^{(N-1)} \in \mathbb{R}^{N-1}\setminus[-1,1]^{N-1}$, we have $(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta \ge (1+|z_N|^{\alpha_N})^\beta$. It follows that

$$I_4 \le C(h+h^2)\int_0^{1/\sqrt{h}} \frac{\mathrm{d}z_N}{(1+|z_N|^{\alpha_N})^\beta} \le C(h+h^2) \tag{2.20}$$

and

$$I_5 \le C\int_{1/\sqrt{h}}^\infty \frac{\mathrm{d}z_N}{(1+|z_N|^{\alpha_N})^\beta} \le Ch^{(\beta\alpha_N-1)/2}, \tag{2.21}$$

since $\alpha_N\beta > 1$. Combining (2.7) with (2.10)-(2.21), we get that as $h \to 0$

$$\mathbb{E}(|Y_{N,h}(x) - Y_{N,s}(x)|^2) \le Ch^{(\beta\alpha_N-1)/2} \to 0.$$

This proves (2.6).

(b). For any fixed constant $T > 0$ and any $x, y \in [-T, T]^N$, we have

$$
\begin{aligned}
&\mathbb{E}\big(Y_N'(x) - Y_N'(y)\big)^2 \\
&= \int_{\mathbb{R}^N} \Big| \prod_{k=1}^{N-1} \Big(\frac{e^{ix_k z_k} - 1}{z_k}\Big) e^{ix_N z_N} - \prod_{k=1}^{N-1} \Big(\frac{e^{iy_k z_k} - 1}{z_k}\Big) e^{iy_N z_N} \Big|^2 \frac{\mathrm{d}z}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta} \\
&\le \int_{\mathbb{R}^N} \Big| \prod_{k=1}^{N-1} \frac{e^{ix_k z_k} - 1}{z_k} \Big|^2 \Big| e^{ix_N z_N} - e^{iy_N z_N} \Big|^2 \frac{\mathrm{d}z}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta} \\
&\qquad + \int_{\mathbb{R}^N} \Big| \prod_{k=1}^{N-1} \Big(\frac{e^{ix_k z_k} - 1}{z_k}\Big) - \prod_{k=1}^{N-1} \Big(\frac{e^{iy_k z_k} - 1}{z_k}\Big) \Big|^2 \frac{\mathrm{d}z}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta} \\
&:= J_1 + J_2.
\end{aligned} \tag{2.22}
$$

In order to estimate J_1, we take $\theta \in (0, 1)$ such that $\alpha_N \beta - 2\theta > 1$. By using the elementary inequality $|e^{iu} - 1| \le 2^{1-\theta} |u|^\theta$ for all $u \in \mathbb{R}$ and a change of variable when integrating $[\mathrm{d}z_N]$, we obtain that

$$
\begin{aligned}
J_1 &\le C|x_N - y_N|^{2\theta} \int_{\mathbb{R}^N} \Big| \prod_{k=1}^{N-1} \frac{e^{ix_k z_k} - 1}{z_k} \Big|^2 \frac{|z_N|^{2\theta}\, \mathrm{d}z}{(\sum_{k=1}^N |z_k|^{\alpha_k})^\beta} \\
&= C|x_N - y_N|^{2\theta} \int_{\mathbb{R}^{N-1}} \Big| \prod_{k=1}^{N-1} \frac{e^{ix_k z_k} - 1}{z_k} \Big|^2 \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_{N-1}}{\big(\sum_{k=1}^{N-1} |z_k|^{\alpha_k}\big)^{\beta - \frac{1+2\theta}{\alpha_N}}} \\
&\le C\, |x_N - y_N|^{2\theta},
\end{aligned} \tag{2.23}
$$

where the last inequality follows from Proposition 2.1 and our choice of θ.

To estimate J_2, we first integrate $[\mathrm{d}z_N]$ and then apply Proposition 3.1 from Section 3 to derive

$$
\begin{aligned}
J_2 &= \int_{\mathbb{R}^{N-1}} \Big| \prod_{k=1}^{N-1} \Big(\frac{e^{ix_k z_k} - 1}{z_k}\Big) - \prod_{k=1}^{N-1} \Big(\frac{e^{iy_k z_k} - 1}{z_k}\Big) \Big|^2 \frac{\mathrm{d}z}{\big(\sum_{k=1}^{N-1} |z_k|^{\alpha_k}\big)^{\beta - \frac{1}{\alpha_N}}} \\
&\le C \sum_{k=1}^{N-1} |x_k - y_k|^{1 + (\beta - \frac{1}{\alpha_N})\alpha_k},
\end{aligned} \tag{2.24}
$$

where $C > 0$ is a finite constant which depends on T. Combining (2.22)–(2.24) yields (2.5) with $\eta = \min\{2\theta, 1 + (\beta - \frac{1}{\alpha_N})\alpha_k, 1 \le k \le N-1\}$.

(c). The construction of a version $\widetilde{Y}$ of Y with a.s. continuous partial derivative $\widetilde{Y}_N'$ is the same as in the proof of Theorem 4.8 in Xue and Xiao [26] and will be omitted. Hence the proof of Theorem 2.1 is completed. □

3. Sectorial local nondeterminism

Theorem 2.1 shows that the sample functions of a pseudo-fBs Y may either be smooth in all directions or have fractal properties, depending on the choices of the parameters $\alpha_1, \ldots, \alpha_N$ and β. Different tools are needed for studying fine structures of Y. In this and the next sections, we focus on the fractal case and always assume that $\alpha_k \in (0, 1/\beta \wedge 2)$ for all $k = 1, 2, \cdots, N$. We will study the uniform and local moduli of continuity of the sample function of Y.

It is well-known that many properties of the sample functions of a Gaussian process $\{G(t),\, t \in \mathbb{R}^N\}$ are determined by the pseudo-metric $d_G(x, y)$ defined by

$$d_G(x,y) = \left[\mathbb{E}\big(G(x) - G(y)\big)^2\right]^{1/2}$$

for all $x, y \in \mathbb{R}^N$. In order to study the uniform and local moduli of continuity of the sample function of pseudo-fBs Y, we define a metric $\rho_{\vec{\alpha},\beta}$ on $\mathbb{R}^N$ by

$$\rho_{\vec{\alpha},\beta}(x,y) = \sum_{k=1}^{N} |x_k - y_k|^{(1+\beta\alpha_k)/2}. \tag{3.1}$$

Due to the assumption that $\alpha_k \in (0, 1/\beta \wedge 2]$ for all $1 \le k \le N$, it is easy to verify that $\rho_{\vec{\alpha},\beta}(x, y)$ defined by (3.1) is indeed a metric on $\mathbb{R}^N$. In this section we investigate the relation between the pseudo-metric d_Y and the metric $\rho_{\vec{\alpha},\beta}$ and prove that Y satisfies the property of sectorial local nondeterminism. This latter property will play an important role not only in this paper, but also in studying the local times, self-intersections and other sample path properties of Y. See Xiao [25] for further information.

To simplify the notation, in the rest of this paper, we will suppress the notation $\vec{\alpha}$ and β from $\rho_{\vec{\alpha},\beta}$. The main results of this section read as follows.

Theorem 3.1 *Given constants $0 < a < T$, there exists a positive constant C_1 such that for all $n \ge 1$, and all $x^{(1)}, \ldots, x^{(n)} \in [a, T]^N$,*

$$\mathrm{Var}\Big(\, Y(x^{(n)}) \,\Big|\, Y(x^{(1)}), \ldots, Y(x^{(n-1)})\Big) \ge C_1 \sum_{j=1}^{N} \min_{0\le k\le n-1} \big|x_j^{(n)} - x_j^{(k)}\big|^{1+\beta\alpha_j}, \tag{3.2}$$

where $x^{(0)} = 0$.

Proof. For any $1 \le m \le N$, let $r_m = \min\limits_{0\le k\le n-1} |x_m^{(n)} - x_m^{(k)}|$, where $x^{(0)} = 0$. Since

$$\mathrm{Var}\Big(Y(x^{(n)})\big|Y(x^{(1)}), \cdots, Y(x^{(n-1)})\Big) = \inf_{u_1,\cdots,u_{n-1}\in\mathbb{R}} \mathbb{E}\bigg(Y(x^{(n)}) - \sum_{k=1}^{n-1} u_k Y(x^{(k)})\bigg)^2,$$

it suffices to prove that for some positive constant $C > 0$

$$\mathbb{E}\bigg(Y(x^{(n)}) - \sum_{k=1}^{n-1} u_k Y(x^{(k)})\bigg)^2 \ge C\, r_m^{1+\beta\alpha_m} \tag{3.3}$$

for all $u_k \in \mathbb{R}$, $k = 1, 2, \cdots, n-1$. The proof is similar to that of Theorem 1 in Wu and Xiao [24] and is included for completeness.

From (2.2), we know

$$\begin{aligned}
&\mathbb{E}\bigg(Y(x^{(n)}) - \sum_{k=1}^{n-1} u_k Y(x^{(k)})\bigg)^2 \\
&\quad = \int_{\mathbb{R}^N} \bigg| \prod_{j=1}^{N} (\mathrm{e}^{ix_j^{(n)} z_j} - 1) - \sum_{k=1}^{n-1} u_k \prod_{j=1}^{N} (\mathrm{e}^{ix_j^{(k)} z_j} - 1)\bigg|^2 \frac{\mathrm{d}z}{\psi^2(z)}.
\end{aligned}$$

For every $j = 1, 2, \cdots, N$, let $\delta_j(\cdot) : \mathbb{R} \mapsto [0, 1]$ be a function in $C^\infty(\mathbb{R})$ such that $\delta_j(0) = 1$,

$$\delta_j(z) < 1/2 \ \text{ for } \ |z| > a, \tag{3.4}$$

and it vanishes outside the open set $B = \{z : |z| < 1\}$. Denote by $\widehat{\delta}_j$ the Fourier transform of δ_j. Then $\widehat{\delta}_j \in C^\infty(\mathbb{R})$ as well and $\widehat{\delta}_j(z)$ decays rapidly as $z \to \infty$, i.e., for any $k > 0$, $|z|^k |\widehat{\delta}_j(z)| \to 0$ as $z \to \infty$. Furthermore, let $\delta_m^{r_m}(t) = r_m^{-1} \delta_m(r_m^{-1} t)$. Then

$$\delta_m^{r_m}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-i\langle t, z\rangle} \widehat{\delta}_m(r_m z) \mathrm{d}z.$$

Since $\min\{|x_m^{(n)} - x_m^{(k)}|,\ 0 \le k \le n-1\} = r_m$, we have

$$\delta_m^{r_m}(x_m^{(n)} - x_m^{(k)}) = 0, \tag{3.5}$$

for all $k = 0, 1, \cdots, n-1$. Therefore

$$\begin{aligned} J := \int_{\mathbb{R}^N} \Bigg[\Big(\prod_{j=1}^N (\mathrm{e}^{i x_j^{(n)} z_j} - 1) - \sum_{k=1}^{n-1} u_k \prod_{j=1}^N (\mathrm{e}^{i x_j^{(k)} z_j} - 1) \Big) \\ \times \widehat{\delta}_m(r_m z_m) \prod_{j \ne m} \widehat{\delta}_j(z_j) \prod_{j=1}^N \mathrm{e}^{-i x_j^{(n)} z_j} \Bigg] \mathrm{d}z \end{aligned}$$

equals

$$\begin{aligned} &(2\pi)^N \Big(\prod_{j \ne m} (\delta_j(0) - \delta_j(x_j^{(n)})) \Big) \Big(\delta_m^{r_m}(0) - \delta_m^{r_m}(x_m^{(n)}) \Big) \\ &- (2\pi)^N \sum_{k=1}^{n-1} u_k \Big(\prod_{j \ne m} (\delta_j(x_j^{(n)} - x_j^{(k)}) - \delta_j(x_j^{(n)})) \Big) \Big(\delta_m^{r_m}(x_m^{(n)} - x_m^{(k)}) - \delta_m^{r_m}(x_m^{(n)}) \Big). \end{aligned}$$

This and (3.4),(3.5) together imply that

$$J \ge 2\pi^N / r_m. \tag{3.6}$$

Furthermore, by Hölder inequality,

$$\begin{aligned} J^2 \le& \int_{\mathbb{R}^N} \Big| \prod_{j=1}^N (\mathrm{e}^{i x_j^{(n)} z_j} - 1) - \sum_{k=1}^{n-1} u_k \prod_{j=1}^N (\mathrm{e}^{i x_j^{(k)} z_j} - 1) \Big|^2 \frac{\mathrm{d}z}{\psi^2(z)} \\ &\times \int_{\mathbb{R}^N} \psi^2(z) \Big| \widehat{\delta}_m(r_m z_m) \prod_{j \ne m} \widehat{\delta}_j(z_j) \Big|^2 \mathrm{d}z \\ =& \mathbb{E}\Big(\Big| Y(x^{(n)}) - \sum_{k=1}^{n-1} u_k Y(x^{(k)}) \Big|^2 \Big) \int_{\mathbb{R}^N} \psi^2(z) \Big| \widehat{\delta}_m(r_m z_m) \prod_{j \ne m} \widehat{\delta}_j(z_j) \Big|^2 \mathrm{d}z. \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \psi^2(z)\Big|\widehat{\delta}_m(r_m z_m)\prod_{j\neq m}\widehat{\delta}_j(z_j)\Big|^2 \mathrm{d}z \\
&= r_m^{-3-\beta\alpha_m}\int_{\mathbb{R}^N}\Big(\prod_{j=1}^N z_j^2\Big)\Big(|z_m|^{\alpha_m}+r_m^{\alpha_m}\sum_{j\neq m}^N |z_j|^{\alpha_j}\Big)^\beta \Big|\prod_{j=1}^N \widehat{\delta}_j(z_j)\Big|^2\mathrm{d}z \\
&\le r_m^{-3-\beta\alpha_m}\int_{\mathbb{R}^N}\prod_{j=1}^N z_j^2\Big(|z_m|^{\alpha_m}+T^{\alpha_m}\sum_{j\neq m}^N |z_j|^{\alpha_j}\Big)^\beta \Big|\prod_{j=1}^N \widehat{\delta}_j(z_j)\Big|^2\mathrm{d}z \le C r_m^{-3-\beta\alpha_m}
\end{aligned}$$

for some constant $C>0$. Then from (3.7), we have that

$$J^2 \le C\,\mathbb{E}\bigg(\Big|Y(x^{(n)})-\sum_{i=1}^{n-1}u_i Y(x^{(i)})\Big|^2\bigg) r_m^{-3-\beta\alpha_m}.$$

This and (3.6) together yield (3.3) for an appropriate constant C. □

From Theorem 3.1, we can easily get the following corollary.

Corollary 3.1 *Given constants $a\in(0,1)$ and $T>a$, there exists a positive constant C_2 such that $d_Y^2(x,y)\ge C_2\,\rho^2(x,y)$ for all $x,y\in[a,T]^N$.*

Proof. Let $n=2$ and $x^{(2)}=x$, $x^{(1)}=y$. Since $x,y\in[a,T]^N$, we have

$$|x_j-y_j|\le \frac{T}{a}\,|x_j| \qquad \forall\ 1\le j\le N.$$

It follows from Theorem 3.1 and Jensen's inequality that

$$\begin{aligned}
d_Y^2(x,y)\ge \mathrm{Var}(Y(x)|Y(y)) &\ge C_1\sum_{j=1}^N \min_{0\le k\le 1}|x_j-x_j^{(k)}|^{1+\beta\alpha_j} \\
&\ge \frac{C_1 a}{T}\sum_{j=1}^N |x_j-y_j|^{1+\beta\alpha_j}\ge \frac{C_1 a}{NT}\rho^2(x,y),
\end{aligned}$$

which completes the proof. □

Proposition 3.1 *For any bounded set $D\subset\mathbb{R}^N$, there exists a positive constant C_3 such that for all $x,y\in D$*

$$d_Y^2(x,y)\le C_3\rho^2(x,y). \tag{3.8}$$

Proof. We use induction to prove (3.8). For any $n=1,2,\cdots,N$, let $Y_n=\{Y_n(x),x\in\mathbb{R}^n\}$ be the Gaussian random field defined as in (2.2) with ψ being replaced by $\psi_n(z)=\big(\prod_{k=1}^n|z_k|\big)\big(\sum_{k=1}^n|z_k|^{\alpha_k}\big)^{\beta/2}$. Let

$$d_n^2(x,y)=\int_{\mathbb{R}^n}\Big|\prod_{k=1}^n(\mathrm{e}^{ix_kz_k}-1)-\prod_{k=1}^n(\mathrm{e}^{iy_kz_k}-1)\Big|^2\frac{\mathrm{d}z_1\cdots\mathrm{d}z_n}{\psi_n^2(z)}.$$

It is clear that, for $n=1$, we have $d_{Y_1}^2(x,y)=C\,|x-y|^{1+\alpha_1\beta}$ for all $x,y\in\mathbb{R}$, where $C>0$ is a finite constant. Hence (3.8) holds for Y_n with $n=1$.

Assume that (3.8) holds for random fields Y_n with $n = 1, \ldots, N-1$. Next we consider $Y = Y_N$. By (1.5), we have that

$$d_Y^2(x,y) = \mathbb{E}\big(Y(x) - Y(y)\big)^2 = d_N^2(x,y), \quad \forall\, x, y \in \mathbb{R}^N. \tag{3.9}$$

Note that

$$\begin{aligned}\Big|\prod_{k=1}^{N}(\mathrm{e}^{ix_kz_k}-1)-\prod_{k=1}^{N}(\mathrm{e}^{iy_kz_k}-1)\Big|^2 &\le \Big|\prod_{k=1}^{N-1}(\mathrm{e}^{iy_kz_k}-1)\Big|^2\Big|\mathrm{e}^{i(x_N-y_N)z_N}-1\Big|^2\\ &\quad+\Big|\prod_{k=1}^{N-1}(\mathrm{e}^{ix_kz_k}-1)-\prod_{k=1}^{N-1}(\mathrm{e}^{ix_kz_k}-1)\Big|^2\Big|\mathrm{e}^{iy_Nz_N}-1\Big|^2\end{aligned}$$

and

$$\frac{1}{\psi^2(z)} \le \left(\frac{1}{z_N^2\big(\sum_{k=1}^{N-1}|z_k|^{\alpha_k}\big)^\beta} \wedge \frac{1}{|z_N|^{2+\beta\alpha_N}}\right)\prod_{k=1}^{N-1}\frac{1}{z_k^2}. \tag{3.10}$$

From (3.9) and (3.10) we have that

$$\begin{aligned}d_N^2(x,y) \le d_{N-1}^2(x,y)\int_{\mathbb{R}}\frac{|\mathrm{e}^{ix_Nz_N}-1|^2}{z_N^2}\mathrm{d}z_N + \int_{\mathbb{R}}\frac{|\mathrm{e}^{i(x_N-y_N)z_N}-1|^2}{z_N^{2+\beta\alpha_N}}\mathrm{d}z_N\\ \times\int_{\mathbb{R}^{N-1}}\Big|\prod_{k=1}^{N-1}(\mathrm{e}^{iy_kz_k}-1)\Big|^2\frac{dz_1\cdots \mathrm{d}z_{N-1}}{\prod_{k=1}^{N-1}z_k^2}.\end{aligned} \tag{3.11}$$

Since $x, y \in D$ which is a bounded set, by direct verification we see that the integrals

$$\int_{\mathbb{R}}\frac{|\mathrm{e}^{ix_Nz_N}-1|^2}{z_N^2}\mathrm{d}z_N \quad\text{and}\quad \int_{\mathbb{R}^{N-1}}\Big|\prod_{k=1}^{N-1}(\mathrm{e}^{iy_kz_k}-1)\Big|^2\frac{dz_1\cdots \mathrm{d}z_{N-1}}{\prod_{k=1}^{N-1}z_k^2}$$

are convergent (due to $\alpha_N < 1/\beta$) and bounded from above by a finite constant and

$$\int_{\mathbb{R}}\frac{|\mathrm{e}^{i(x_N-y_N)z_N}-1|^2}{z_N^{2+\beta\alpha_N}}\mathrm{d}z_N = C|x_N-y_N|^{1+\beta\alpha_N}$$

for some positive constant $C > 0$. Thus, it follows from (3.11) there exists a constant $C > 0$, which may depend on N, $\vec{\alpha}$ and β, such that

$$d_Y^2(x,y) = d_N^2(x,y) \le C\left(d_{N-1}^2(x,y) + |x_N-y_N|^{1+\beta\alpha_N}\right).$$

This proves Proposition 3.1. □

Remark 3.1 Because of Theorem 3.1 and Proposition 3.1, many sample path properties such as fractal dimensions, existence and continuity of local times of Y can be investigated by using the methods in Xiao (2009).

4. Exact uniform and local moduli of continuity of pseudo-fBs

In this section, we determine the exact uniform and local moduli of continuity of Y (see Theorems 4.1 and 4.2). Even though the main methods are reminiscent to those in Meerschaert, Wang and Xiao [19], both Theorems 4.1 and 4.2 can not be derived directly from those in [19].

We recall from Marcus and Rosen [17] the definition of exact modulus of continuity. Suppose (S, τ) is a compact metric or pseudo-metric space and $X = \{X(t),\, t \in S\}$ is a centered Gaussian random field with values in $\mathbb{R}$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ is called an exact uniform modulus of continuity for X on (S, τ) if

$$\lim_{\varepsilon\to 0} \sup_{s;t\in S;\tau(s,t)\le\varepsilon} \frac{|X(s)-X(t)|}{\varphi(\tau(s,t))} = C \quad \text{a.s.}$$

for some constant $C \in (0,\infty)$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ is called an exact local modulus of continuity for X at $t_0 \in T$ if

$$\lim_{\varepsilon\to 0} \sup_{s\in T;\tau(s;t_0)\le\varepsilon} \frac{|X(s)-X(t)|}{\varphi(\tau(s,t))} = C \quad \text{a.s.}$$

for some constant $C \in (0,\infty)$.

The following is the main result of this section.

Theorem 4.1 *Let $I = [0,1]^N$. There exists a constant $0 < C_4 < \infty$ such that*

$$\lim_{\varepsilon\to 0} \sup_{\substack{x,y\in I\\ \rho(x,y)\le\varepsilon}} \frac{|Y(x)-Y(y)|}{\rho(x,y)\sqrt{\log(1+\rho(x,y)^{-1})}} = C_4 \qquad a.s. \tag{4.1}$$

Proof. Due to monotonicity the limit in the left hand side of (4.1) exists almost surely. The key point of the theorem is that this limit is non-random, positive and finite.

We first show that the limit in (4.1) is non-random and finite. Our proof is simpler than that of (4.4) in Meerschaert, Wang and Xiao [19].

Consider an auxiliary Gaussian random field:

$$W = \big\{W(x,y), x \in I, y \in B_\rho(r)\big\}$$

defined by $W(x,y) = Y(x+y) - Y(x)$, where $B_\rho(r) := \{x \in \mathbb{R}^N, \rho(x,0) \le r\}$. Then the metric d_W on $S := I \times B_\rho(r)$ associated with W satisfies the following inequality:

$$\begin{aligned} &d_W((x,y),(x',y')) \\ &\quad \le C \min\{d_Y(x+y,x) + d_Y(x'+y',x'), d_Y(x+y,x'+y') + d_Y(x,x')\}. \end{aligned}$$

Therefore, by (3.8), there exists a constant $C > 0$ such that

$$d_W((x,y),(x',y')) \le C \min\{\rho(y,0) + \rho(y',0), \rho(x,x') + \rho(y,y')\} \tag{4.2}$$

for all (x,y), $(x',y') \in S$. Denote the diameter of S in the metric d_W by D. Then (4.2) implies that $D \le 2Cr$. Furthermore, for small $\varepsilon > 0$, if for every $j = 1, 2, \cdots, N$,

$$|x_j - x'_j| < \Big(\frac{\varepsilon}{2CN}\Big)^{2/(1+\beta\alpha_j)} \text{ and } |y_j - y'_j| < \Big(\frac{\varepsilon}{2CN}\Big)^{2/(1+\beta\alpha_j)},$$

then by (4.2),

$$(x', y') \in O_{d_W}((x, y), \varepsilon) = \Big\{(x', y') : d_W((x', y'), (x, y)) < \varepsilon\Big\}.$$

Hence, $N_d(S, \varepsilon)$, the smallest number of open d_W-balls of radius ε needed to cover S, satisfies

$$N_d(S, \varepsilon) \leq C\varepsilon^{-4\sum_{j=1}^{N}\frac{1}{1+\beta\alpha_j}}$$

for some constant $C > 0$. Then one can verify that for some constant $C > 0$

$$\int_0^D \sqrt{\ln N_d(S, \varepsilon)}\, d\varepsilon \leq Cr\sqrt{\log(1 + r^{-1})}.$$

It follows from Lemma 2.1 in Talagrand [23] that for all $u \geq Cr\sqrt{\log(1 + r^{-1})}$,

$$\mathbb{P}\bigg\{\sup_{(x,y)\in S}\big|Y(x + y) - Y(x)\big| \geq u\bigg\} \leq \exp\bigg(-\frac{u^2}{D^2}\bigg).$$

By a standard Borel-Cantelli argument, we have that for some positive constant $C < \infty$

$$\limsup_{r\to 0}\sup_{\substack{x\in I\\ \rho(x,y)\leq r}}\frac{|Y(x) - Y(y)|}{r\sqrt{\log(1 + r^{-1})}} \leq C \qquad \text{a.s.}$$

The monotonicity of the functions $r \mapsto r\sqrt{\log(1 + r^{-1})}$ implies that

$$\limsup_{r\to 0}\sup_{\substack{x,y\in I\\ \rho(x,y)\leq r}}\frac{|Y(x) - Y(y)|}{\rho(x, y)\sqrt{\log(1 + \rho(x, y)^{-1})}} \leq C \qquad \text{a.s.} \tag{4.3}$$

It follows from the proof of Lemma 7.1.1 in Marcus and Rosen [17] and (4.3) that the limit in (4.1) is almost surely a finite constant C_4.

It remains to prove that the limit C_4 is positive. It is sufficient to show that, for a constant $a \in (0, 1)$ and $I_a = [a, 1]^N$, we have

$$\limsup_{\varepsilon\to 0}\sup_{\substack{x,y\in I_a\\ \rho(x,y)\leq\varepsilon}}\frac{|Y(x) - Y(y)|}{\rho(x, y)\sqrt{\log(1 + \rho(x, y)^{-1})}} \geq C_5, \qquad \text{a.s.} \tag{4.4}$$

where $C_5 > 0$ is a constant. Thanks to Theorem 3.1, the proof of (4.4) is the same as the proof of (4.5) in Meerschaert, Wang and Xiao [19, Theorem 4.1]. This finishes the proof of Theorem 4.1. □

Our next theorem is concerned with local oscillation of Y.

Theorem 4.2 *For every fixed $x \in (0, \infty)^N$, there exists a constant $0 < C_6 = C_6(x) < \infty$, such that*

$$\lim_{\varepsilon\to 0}\sup_{y-x\in B_\rho(\varepsilon)}\frac{|Y(y) - Y(x)|}{\rho(y, x)\sqrt{\log\log(1 + \rho(y, x)^{-1})}} = C_6 \qquad a.s. \tag{4.5}$$

For proving Theorem 4.2, we will make use of the following lemma, which is a consequence of Proposition 3.1 and the proof of Lemma 5.4 in [19] with $H_j = (1 + \beta\alpha_k j)/2$ $(1 \leq j \leq N)$.

Lemma 4.1 *Let $T > 0$ be a constant. There exist positive and finite constants u_0 and C_7 (depending on T) such that for every $x \in [-T, T]^N$, $u \geq u_0$ and sufficiently small $r > 0$,*

$$\mathbb{P}\left(\sup_{y \in B_\rho(r)} |Y(y+x) - Y(x)| \geq ur \right) \leq e^{-C_7 u^2}.$$

Proof of Theorem 4.2. Without loss of generality, we assume that $x \in [a, 1]^N$ for some $a \in (0, 1)$. The proof is in essence similar to that of Theorem 6.4 in [19] for fractional Brownian sheets. Some modifications are needed due to different stochastic integral representations.

For any $n \geq 1$, let $\varepsilon_n = \mathrm{e}^{-n}$. For any fixed constant $u > \frac{1}{\sqrt{C_7}}$, we consider the event

$$E_n = \left\{ \sup_{y-x \in B_\rho(\varepsilon_n)} \frac{|Y(y) - Y(x)|}{\varepsilon_n \sqrt{\log\log(1+\varepsilon_n^{-1})}} > u \right\}.$$

By Lemma 4.1, for sufficiently large n,

$$\mathbb{P}(E_n) \leq \mathrm{e}^{-C_7 u^2 \log n}.$$

Hence, by the Borel-Cantelli lemma and a standard monotonicity argument, we get

$$\limsup_{\varepsilon \to 0} \sup_{y-x \in B_\rho(\varepsilon)} \frac{|Y(y) - Y(x)|}{\rho(s,t)\sqrt{\log\log(1+\rho(y,t)^{-1})}} \leq u.$$

This and Lemma 7.1.1 in [17] imply that there is a constant $C_6 \in [0, u]$ such that

$$\lim_{\varepsilon \to 0} \sup_{y-x \in B_\rho(\varepsilon)} \frac{|Y(y) - Y(x)|}{\rho(y,x)\sqrt{\log\log(1+\rho(y,x)^{-1})}} = C_6 \quad \text{a.s.}$$

Below we prove $C_6 > 0$. In fact we will prove that

$$\limsup_{n \to \infty} \frac{|Y(x+y^{(n)}) - Y(x)|}{\rho(y^{(n)},0)\sqrt{\log\log(1+\rho(y^{(n)},0)^{-1})}} \geq \sqrt{C_2} \tag{4.6}$$

for $y^{(n)} = \left(y_1^{(n)}, \cdots, y_N^{(n)}\right) = \left(\mathrm{e}^{-2n^{1+\mu}/(1+\beta\alpha_1)}, \cdots, \mathrm{e}^{-2n^{1+\mu}/(1+\beta\alpha_N)}\right)$, where μ is a positive constant whose value will be specified later.

Obviously, $\rho(y^{(n)}, 0) = N\mathrm{e}^{-n^{1+\mu}} := \eta_n \to 0$ as $n \to \infty$. Let $d_n = \exp(n^{1+\mu} + n^\mu)$ for any $n \geq 1$. We decompose Y into two independent parts as follows. Let

$$\begin{aligned} \Omega_n &= \{z \in \mathbb{R}^N : \rho(z,0) \leq d_{n-1}\} \cup \{z \in \mathbb{R}^N : \rho(z,0) > d_n]\} \\ &:= \Delta_{n-1} \cup \bar{\Delta}_n \end{aligned}$$

and

$$\overline{\Omega}_n = \{z \in \mathbb{R}^N : \rho(z,0) \in (d_{n-1}, d_n]\}.$$

Define

$$\widetilde{Y}_n(t) = \int_{\Omega_n} \prod_{k=1}^N \left(e^{it_k z_k} - 1\right) \frac{\widetilde{\mathcal{M}}(\mathrm{d}z)}{\psi(z)}$$

and

$$Y_n(t) = \int_{\overline{\Omega}_n} \prod_{k=1}^N \left(e^{it_k z_k} - 1\right) \frac{\widetilde{\mathcal{M}}(\mathrm{d}z)}{\psi(z)}.$$

Then, for each fixed $n \geq 1$, $\{\widetilde{Y}_n(t),\, t \in \mathbb{R}^N\}$ and $\{Y_n(t),\, t \in \mathbb{R}^N\}$ are independent. Moreover, the random fields $\{Y_n(t),\, t \in \mathbb{R}^N\}$ $(n = 1, 2, \cdots)$ are independent. Note that

$$\begin{aligned}\frac{|Y(x+y^{(n)}) - Y(x)|}{\rho(y^{(n)},0)\sqrt{\log\log(1+\rho(y^{(n)},0)^{-1})}} &\geq \frac{|Y_n(x+y^{(n)}) - Y_n(x)|}{\rho(y^{(n)},0)\sqrt{\log\log(1+\rho(y^{(n)},0)^{-1})}} \\ &\quad - \frac{|\widetilde{Y}_n(x+y^{(n)}) - \widetilde{Y}_n(x)|}{\rho(y^{(n)},0)\sqrt{\log\log(1+\rho(y^{(n)},0)^{-1})}} \\ &=: J_1(n) - J_2(n). \end{aligned} \tag{4.7}$$

We will show that, as $n \to \infty$, $J_1(n)$ is the main term and $J_2(n)$ is negligible.

By the definition of $\widetilde{Y}_n$ we have

$$\begin{aligned}\mathbb{E}\left(\widetilde{Y}_n(x+y^{(n)}) - \widetilde{Y}_n(x)\right)^2 &= \int_{\Delta_{n-1}} \Big| \prod_{k=1}^N (\mathrm{e}^{i(x_k+y_k^{(n)})z_k} - 1) - \prod_{k=1}^N (\mathrm{e}^{ix_kz_k} - 1)\Big|^2 \frac{dz}{\psi^2(z)} \\ &\quad + \int_{\overline{\Delta}_n} \Big| \prod_{k=1}^N (\mathrm{e}^{i(x_k+y_k^{(n)})z_k} - 1) - \prod_{k=1}^N (\mathrm{e}^{ix_kz_k} - 1)\Big|^2 \frac{dz}{\psi^2(z)} \\ &= J_{2,1}(n) + J_{2,2}(n). \end{aligned} \tag{4.8}$$

For convenience of notation, we let

$$\Pi_j(y,x,z) = \prod_{k=1}^{j-1} \Big|\mathrm{e}^{ix_kz_k} - 1\Big|^2 \Big|\mathrm{e}^{iy_j^{(n)}z_j} - 1\Big|^2 \prod_{k=j+1}^N \Big|\mathrm{e}^{i(x_k+y_k^{(n)})z_k} - 1\Big|^2$$

and $\psi_j^2(z) := |z_j|^{\alpha_j\beta} \prod_{k=1}^N z_k^2 \leq \psi^2(z)$. By the triangle inequality, we have

$$\Big| \prod_{k=1}^N (\mathrm{e}^{i(x_k+y_k^{(n)})z_k} - 1) - \prod_{k=1}^N (\mathrm{e}^{ix_kz_k} - 1)\Big|^2 \frac{1}{\psi^2(z)} \leq \sum_{j=1}^N \frac{\Pi_j(y,x,z)}{\psi_j^2(z)}. \tag{4.9}$$

Furthermore, for $j \geq 1$,

$$\begin{aligned}\int_{\Delta_{n-1}} \frac{\Pi_j(y,x,z)}{\psi_j^2(z)} \mathrm{d}z &\leq \int_{|z_j| \leq (N\mathrm{e}^{-\mu(n-1)^\mu}\eta_n^{-1})^{2/(1+\beta\alpha_j)}} \frac{|\mathrm{e}^{iy_j^{(n)}z_j} - 1|^2}{|z_j|^{2+\beta\alpha_j}} \mathrm{d}z_j \\ &\quad \times \prod_{k=j+1}^N \int_{\mathbb{R}} \Big|\mathrm{e}^{ix_kz_k} - 1\Big|^2 \frac{dz_k}{z_k^2} \prod_{k=1}^{j-1} \int_{\mathbb{R}} \Big|\mathrm{e}^{i(x_k+y_n^{(n)})z_k} - 1\Big|^2 \frac{dz_k}{z_k^2},\end{aligned}$$

since $\Delta_{n-1} \subset \{z : |z_j| \leq (N\mathrm{e}^{-\mu(n-1)^\mu}\eta_n^{-1})^{2/(1+\beta\alpha_j)}\}$ which follows from the fact that

$$|z_j|^{(1+\beta\alpha_j)/2} \geq \mathrm{e}^{-\mu(n-1)^\mu}\mathrm{e}^{n^{1+\mu}} \geq \mathrm{e}^{(n-1)^{1+\mu}+(n-1)^\mu} = d_{n-1},$$

for all $|z_j| > (N\mathrm{e}^{-\mu(n-1)^\mu}\eta_n^{-1})^{2/(1+\beta\alpha_j)}$. Note that

$$\prod_{k=j+1}^{N}\int_{\mathbb{R}}\left|\mathrm{e}^{ix_kz_k}-1\right|^2\frac{dz_k}{z_k^2}\prod_{k=1}^{j-1}\int_{\mathbb{R}}\left|\mathrm{e}^{i(x_k+y_n^{(n)})z_k}-1\right|^2\frac{dz_k}{z_k^2}$$

is uniformly bounded for all $x\in[0,1]^N$ and $y\in B_\rho(1)$. There exists a constant C such that

$$\begin{aligned}\int_{\Delta_{n-1}}\frac{\Pi_j(y,x,z)}{\psi_j^2(z)}\mathrm{d}z &\le C\int_{|z_j|\le(N\mathrm{e}^{-\mu(n-1)^\mu}\eta_n^{-1})^{2/(1+\beta\alpha_j)}}\frac{|\mathrm{e}^{iy_j^{(n)}z_j}-1|^2}{|z_j|^{2+\beta\alpha_j}}\mathrm{d}z_j\\ &\le C(y_j^{(n)})^2\int_0^{(N\mathrm{e}^{-\mu(n-1)^\mu}\eta_n^{-1})^{2/(1+\beta\alpha_j)}}\frac{1}{z_j^{\beta\alpha_j}}\mathrm{d}z_j\\ &\le C\mathrm{e}^{-2\frac{1-\beta\alpha_j}{1+\beta\alpha_j}\mu(n-1)^\mu}\,\mathrm{e}^{2\frac{1-\beta\alpha_j}{1+\beta\alpha_j}n^{1+\mu}}(y_j^{(n)})^2,\end{aligned}\tag{4.10}$$

where we have used the fact $1-\cos z\le z^2/2$. Recall that $y_j^{(n)}=\mathrm{e}^{-2n^{1+\mu}/(1+\beta\alpha_j)}$ we have

$$\mathrm{e}^{2\frac{1-\beta\alpha_j}{1+\beta\alpha_j}n^{1+\mu}}(y_j^{(n)})^2=\frac{\eta_n^2}{N^2}.$$

Let $\hat{\alpha}=\min\left\{\frac{1-\beta\alpha_j}{1+\beta\alpha_j}, j=1,2,\cdots,N\right\}>0$. Applying (4.9) and (4.10) to (4.8) we arrive at

$$J_{2,1}(n)\le C\mathrm{e}^{-2\hat{\alpha}\mu(n-1)^\mu}\eta_n^2.\tag{4.11}$$

To estimate $J_{2,2}(n)$, noting that $\rho(z,0)>d_n$ implies $|z_j|^{(1+\beta\alpha_j)/2}>d_n/N$ for some $j\in\{1,2,\cdots,N\}$. For simplicity of notation, we assume $j=N$. Then from (3.11), we know there is a constant C which may depend on x such that

$$J_{2,2}(n)\le C\int_{|z_N|^{(1+\beta\alpha_N)/2}>d_n/N}\frac{\mathrm{d}z_N}{|z_N|^{2+\beta\alpha_N}}=Cd_n^{-2}.\tag{4.12}$$

Since $d_n^{-2}=\eta_n^2\frac{\mathrm{e}^{-2n^\mu}}{N^2}$, we combine (4.8), (4.11) and (4.12) to see that there exists a positive finite constant C such that

$$\mathbb{E}\big(\widetilde{Y}_n(x+y^{(n)})-\widetilde{Y}_n(x)\big)^2\le C\eta_n^2\mathrm{e}^{-2\hat{\alpha}\mu(n-1)^\mu}\tag{4.13}$$

for n large enough. Hence for any given $\varepsilon>0$, when n is large enough,

$$\begin{aligned}\mathbb{P}\big(J_2(n)\ge\varepsilon\big) &=\mathbb{P}\Big(|\widetilde{Y}_n(x+y^{(n)})-\widetilde{Y}_n(s)|\ge\varepsilon\eta_n\sqrt{\log\log(1+\eta_n^{-1})}\Big)\\ &\le\mathbb{P}\Big(|N(0,1)|\ge C\varepsilon\sqrt{\log n}\,\mathrm{e}^{\hat{\alpha}\mu(n-1)^\mu}\Big).\end{aligned}\tag{4.14}$$

On the other hand, by using the independence of Y_n and $\widetilde{Y}_n$, Corollary 3.1 and (4.13), we get that

$$\begin{aligned}\mathbb{E}\big(Y_n(x+y^{(n)})-Y_n(x)\big)^2 &=\mathbb{E}\big(Y(x+y^{(n)})-Y(x)\big)^2-\mathbb{E}\big(\widetilde{Y}_n(x+y^{(n)})-\widetilde{Y}_n(x)\big)^2\\ &\ge d_Y^2(x+y^{(n)},x)-C\rho^2(y^{(n)},0)\mathrm{e}^{-2n^\mu}\\ &\ge\rho^2(y^{(n)},0)(C_2-C\mathrm{e}^{-2n^\mu})\\ &\ge\frac{C_2}{2}\rho^2(y^{(n)},0)=\frac{C_2}{2}\eta_n^2,\end{aligned}\tag{4.15}$$

for sufficiently large n. Therefore, from (4.15), we obtain that

$$\begin{aligned}
&\mathbb{P}\big(J_1(n) \geq (1-\varepsilon)\sqrt{C_2}\big) \\
&\quad = \mathbb{P}\Big(|Y_n(t+y^{(n)}) - Y_n(t)| \geq \eta_n(1-\varepsilon)\sqrt{C_2 \log\log(1+\eta_n^{-1})}\Big) \\
&\quad \geq \mathbb{P}\Big(|N(0,1)| \geq (1-\varepsilon)\sqrt{2\log\log(1+\eta_n^{-1})}\Big). \qquad (4.16)
\end{aligned}$$

Recall that for standard Gaussian random variable, we have

$$(2\pi)^{-\frac{1}{2}}(1-x^{-2})x^{-1}\mathrm{e}^{-\frac{x^2}{2}} \leq \mathbb{P}(N(0,1) > x) \leq (2\pi)^{-\frac{1}{2}}x^{-1}\mathrm{e}^{-\frac{x^2}{2}}.$$

Therefore, (4.14) and (4.16) yield that for sufficiently large n,

$$\mathbb{P}\big(J_2(n) \geq \varepsilon\big) \leq \frac{1}{\varepsilon} n^{-\frac{K^2}{2}\varepsilon^2 \mathrm{e}^{2\hat{\alpha}\mu(n-1)^{\mu}}} \leq n^{-2} \qquad (4.17)$$

and

$$\mathbb{P}\big(J_1(n) \geq (1-\varepsilon)\sqrt{C_2}\big) \geq C n^{-(1-\varepsilon)^2(1+\mu)} \qquad (4.18)$$

for some constant $C > 0$, respectively. Thus, $\sum_{n=1}^{\infty}\mathbb{P}\big(J_2(n) \geq \varepsilon\big) < \infty$. By the Borel-Cantelli lemma and the arbitrariness of ε, we obtain

$$\lim_{n\to\infty} J_2(n) = 0, \quad \text{a.s.} \qquad (4.19)$$

Now we choose $\mu > 0$ small such that $(1-\varepsilon)^2(1+\mu) < 1$ and consequently

$$\sum_{n=1}^{\infty}\mathbb{P}\big(J_1(n) \geq (1-\varepsilon)\sqrt{C_2}\big) = \infty.$$

Since the events $\{J_1(n) \geq (1-\varepsilon)\sqrt{C_2}\}_{n=1,2,\cdots}$ are independent, the Borel-Cantelli lemma and the arbitrariness of η yield

$$\limsup_{n\to\infty} J_1(n) \geq \sqrt{C_2} \qquad \text{a.s.} \qquad (4.20)$$

Hence (4.6) follows from (4.7), (4.19) and (4.20). The proof of Theorem 4.2 is now completed. □

Remark 4.1 The constant C_6 in (4.5) may depend on x. However, for any $0 < a < T$ and all $x \in [a,T]^N$, $C_6(x)$ is bounded from above and below by positive constants which only depend on the constants $(\alpha_k)_{1\leq k\leq N}$, β, a and T.

5. A functional limit theorem for branching systems

This section is a continuation of Li and Xiao [16]. Consider a sequence of $(N, \vec{\alpha}, \delta_n, \gamma)$-branching particle systems. We assume that δ_n satisfies the condition

$$n^{\kappa}\delta_n \to \theta \in (0,\infty) \ \text{ for some constant } \ \kappa \in (0,1), \qquad (5.1)$$

which is referred to as strong degeneration. The case of $\kappa = 1$ (weak degeneration) has been considered by Li and Xiao [16].

The objective of this section is to prove that in the case of $\bar{\alpha} := \sum_{k=1}^{N} \frac{1}{\alpha_k} > 2$, the temporal structure of the functional limit of the scaled occupation time fluctuations (1.1) is degenerate, that is, the limit process in any positive time interval is a time-independent Gaussian process valued in $\mathcal{S}'(\mathbb{R}^N)$, where $\mathcal{S}'(\mathbb{R}^N)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^N)$, the space of smooth rapidly decreasing functions. At the same time, we show that the functional convergence can be strengthened from the integral sense (in Theorem 2.1 of Li and Xiao [16] for $\kappa = 1$) to the weak convergence in $C([\varepsilon, 1], \mathcal{S}'(\mathbb{R}^N))$ for every $\varepsilon \in (0, 1)$.

Let $N_n = \{N_n(t), t \geq 0\}$ be the empirical measure of the $(N, \vec{\alpha}, \delta_n, \gamma)$-branching particle system such that (5.1) holds. The occupation time fluctuation of N_n is defined as

$$\langle X_n(t), \phi \rangle = \frac{1}{F_n} \int_0^{nt} \langle N_n(s) - f_n(s)\lambda, \phi \rangle \mathrm{d}s \tag{5.2}$$

for every $\phi \in \mathcal{S}(\mathbb{R}^N)$, where $\{F_n\}$ is a suitable scaling sequence of normalizing constants and $f_n(s)$ is defined as in (1.2).

The following is the main result of this section.

Theorem 5.1 *Assume $\bar{\alpha} = \sum_{k=1}^{N} \alpha_k^{-1} > 2$ and (5.1) holds. Let $\{X_n(t), t \geq 0\}$ be defined as in (5.2) and let $F_n = n^{\kappa/2}$. For any $\varepsilon \in (0, 1)$, $X_n \Rightarrow X$ in $C([\varepsilon, 1], \mathcal{S}'(\mathbb{R}^N))$ as $n \to \infty$, where $X = \{X(t), t \in [\varepsilon, 1]\}$ is a centered (constant in t) Gaussian process valued in $\mathcal{S}'(\mathbb{R}^N)$, with covariance function*

$$\begin{aligned} &\mathrm{Cov}(\langle X(s), \phi_1 \rangle, \langle X(t), \phi_2 \rangle) \\ &\quad = \frac{1}{\theta(2\pi)^N} \int_{\mathbb{R}^N} \left[\frac{2}{\sum_{k=1}^{N} |z_k|^{\alpha_k}} + \frac{\gamma}{(\sum_{k=1}^{N} |z_k|^{\alpha_k})^2} \right] \widehat{\phi}_1(z) \overline{\widehat{\phi}_2(z)} \, \mathrm{d}z \end{aligned} \tag{5.3}$$

for all ϕ_1 and ϕ_2 in $\mathcal{S}(\mathbb{R}^N)$. Here and in the sequel, $\widehat{f}(z), (z \in \mathbb{R}^N)$ denotes the Fourier transform of f.

Remark 5.1 Here are some remarks on comparing Theorem 5.1 with the corresponding result in Li and Xiao [16].

(a) The sense of convergence is stronger than that in Theorem 2.1 in [16]. It was noted in Remark 2.1 in [16] that, in the case of weak degeneration, the method to strengthen the sense of convergence is tedious and calculation-intensive. The condition of strong degeneration (5.1) in Theorem 5.1 assures a relatively simpler treatment. The cost is that X_n can not fulfil the weak convergence in $C([0, 1], \mathcal{S}'(\mathbb{R}^N))$ but in $C([\varepsilon, 1], \mathcal{S}'(\mathbb{R}^N))$. The reason lies in that $X_n(\cdot)$ is continuous but $X(\cdot)$ is impossibly continuous on $[0, 1]$.

(b) Unlike Theorem 2.1 in Li and Xiao [16], the covariance function of $X(t)$ in (5.3) is independent of the time t. That is, the temporal structure of the limiting process is degenerate.

(c) As in Theorem 2.1 in Li and Xiao [16], we obtain from the limit process X in Theorem 5.1 two independent $\mathcal{S}'(\mathbb{R}^N)$-valued Gaussian variables, say $\bar{X}_1$ and $\bar{X}_2$, whose covariance functions are given by

$$\mathrm{Cov}\big(\langle \bar{X}_1, \phi_1\rangle, \langle \bar{X}_1, \phi_2\rangle\big) = \frac{1}{\theta(2\pi)^N}\int_{\mathbb{R}^N} \frac{2\widehat{\phi}_1(z)\overline{\widehat{\phi}_2(z)}}{\sum_{k=1}^N |z_k|^{\alpha_k}}\,\mathrm{d}z,$$

$$\mathrm{Cov}\big(\langle \bar{X}_2, \phi_1\rangle, \langle \bar{X}_2, \phi_2\rangle\big) = \frac{1}{\theta(2\pi)^N}\int_{\mathbb{R}^N} \frac{\gamma\widehat{\phi}_1(z)\overline{\widehat{\phi}_2(z)}}{(\sum_{k=1}^N |z_k|^{\alpha_k})^2}\,\mathrm{d}z,$$

respectively. Define $Y_1(x) = \langle \bar{X}_1, \mathbf{1}_{D(x)}\rangle$ and $Y_2(x) = \langle \bar{X}_2, \mathbf{1}_{D(x)}\rangle$ for all $x = (x_k)_{1\le k\le N} \in \mathbb{R}^N$, where

$$D(x) = \{(y_1, \cdots, y_N), 0 \le y_k \le x_k, 1 \le k \le N\}$$

if $x \in \mathbb{R}^N_+$ and is defined similarly for $x \in \mathbb{R}^N \backslash \mathbb{R}^N_+$. Then, from (4.11) in Li and Xiao [16] and the discussion in Introduction, $Y_1 = \{Y_1(x), x \in \mathbb{R}^N\}$ and $Y_2 = \{Y_2(x), x \in \mathbb{R}^N\}$, up to a constant, are pseudo-fractional Brownian sheets. Here, in contrast with Li and Xiao [16], we have constructed Y_1 and Y_2 without requiring the spatial behavior of the limit process at a fixed time.

As in Bojedcki *et al* [7, 8, 9], Li [14] and Li and Xiao [16], the method for proving Theorems 5.1 relies on the Laplace functionals of the occupation time fluctuations. Before proving Theorem 5.1, we define a sequence of random variables $\widetilde{X}_n$ in $\mathcal{S}'(\mathbb{R}^{N+1})$ as follows: For any $n \ge 0$ and $\psi \in \mathcal{S}(\mathbb{R}^{N+1})$, let

$$\langle \widetilde{X}_n, \psi\rangle = \int_0^1 \langle X_n(t), \psi(\cdot, t)\rangle\,\mathrm{d}t. \tag{5.4}$$

From (3.15) in Li and Xiao [16], for $\psi = \psi(x,t) = \phi(x)h(t)$, where $\phi \in \mathcal{S}(\mathbb{R}^N)$ and $h \in \mathcal{S}(\mathbb{R})$ are nonnegative, we have

$$\mathbb{E}(\mathrm{e}^{-\langle \widetilde{X}_n, \psi\rangle}) = \exp\Big(I_1(n,\psi_n) + I_2(n,\psi_n) + I_3(n,\psi_n)\Big), \tag{5.5}$$

where $I_1(n,\psi_n), I_2(n,\psi_n)$ and $I_3(n,\psi_n)$ are given respectively by

$$I_1(n,\psi_n) = \frac{\gamma}{2}\int_{\mathbb{R}^N}\mathrm{d}x\int_0^n \mathrm{e}^{-\delta_n s} V^2_{n,\psi_n}(x, n-s, s)\mathrm{d}s, \tag{5.6}$$

$$I_2(n,\psi_n) = \int_{\mathbb{R}^N}\mathrm{d}x\int_0^n \mathrm{e}^{-\delta_n s}\psi_n(x,s) V_{n,\psi_n}(x, n-s, s)\mathrm{d}s, \tag{5.7}$$

$$I_3(n,\psi_n) = \delta_n\int_{\mathbb{R}^N}\mathrm{d}x\int_0^n \mathrm{e}^{-\delta_n s}\mathrm{d}s\int_0^{n-s} \mathrm{e}^{-\gamma u}\chi_{n,\psi_n}(x,u,s)\mathrm{d}u. \tag{5.8}$$

In the above

$$\psi_n(x,s) = \frac{1}{F_n}\phi(x)\widetilde{h}(\frac{s}{n}), \qquad \widetilde{h}(s) = \int_s^1 h(t)\mathrm{d}t,$$

$$V_{n,\psi_n}(x,t,r) = 1 - \mathbb{E}_x\Big[\exp\Big(-\int_0^t \big\langle N_n(s), \psi_n(\cdot, r+s)\big\rangle\,\mathrm{d}s\Big)\Big],$$

and

$$\chi_{n,\psi_n}(x,u,s)=\mathbb{E}_x\Big[\Big(1-\mathrm{e}^{-\int_0^u \psi_n(\vec{\xi}_n(v),s+v)\,\mathrm{d}v}\Big)\psi_n(\vec{\xi}_n(u),s+u)\Big].$$

For $I_1(n,\psi_n)$ we have the following result.

Lemma 5.1 *If $F_n^2=n^\kappa$, then as $n\to\infty$,*

$$I_1(n,\psi_n)\to\frac{\gamma}{2\theta(2\pi)^N}\int_{\mathbb{R}^N}\frac{|\widehat{\phi}(z)|^2}{(\sum_{k=1}^N|z_k|^{\alpha_k})^2}\,\mathrm{d}z\int_0^1\int_0^1 h(u)h(v)\,\mathrm{d}u\mathrm{d}v. \tag{5.9}$$

Proof Similarly to (3.19) in Li and Xiao [16], we write

$$I_1(n,\psi_n)=I_{11}(n,\psi_n)+I_{12}(n,\psi_n), \tag{5.10}$$

where

$$I_{11}(n,\psi_n)=\frac{\gamma}{2}\int_{\mathbb{R}^N}\mathrm{d}x\int_0^n \mathrm{e}^{-\delta_n s}J^2_{n,\psi_n}(x,n-s,s)\,\mathrm{d}s, \tag{5.11}$$

$$I_{12}(n,\psi_n)=\frac{\gamma}{2}\int_{\mathbb{R}^N}\mathrm{d}x\int_0^n \mathrm{e}^{-\delta_n s}\Big(V^2_{n,\psi_n}(x,n-s,s)-J^2_{n,\psi_n}(x,n-s,s)\Big)\,\mathrm{d}s, \tag{5.12}$$

and where

$$J_{n,\psi_n}(x,t,r)=\int_0^t f_n(s)\mathbb{E}_x\Big(\psi_n(\vec{\xi}_n(s),r+s)\Big)\,\mathrm{d}s.$$

Recalled that $\vec{\xi}_n$ is the Lévy process in the Introduction and $f_n(s)$ is defined in (1.2).
From (3.27) in Li and Xiao [16], we derive that

$$\lim_{n\to\infty}I_{11}(n,\psi_n)=\lim_{n\to\infty}\Big\{\frac{n^3\gamma/2}{(2\pi)^N F_n^2}\int_0^1 \mathrm{e}^{-n\delta_n s}\mathrm{d}s\int_{\mathbb{R}^N}|\widehat{\phi}(z)|^2 \times\Big[\int_s^1 \mathrm{e}^{-n(u-s)(\delta_n+\sum_{k=1}^N|z_k|^{\alpha_k})}\widetilde{h}(u)\mathrm{d}u\Big]^2\mathrm{d}z\Big\}. \tag{5.13}$$

(Recall that $\widetilde{h}(u)=\int_u^1 h(t)\mathrm{d}t$ for all $u\in[0,1]$.) Applying the substitution $s'=n^{1-\kappa}s$, $u'=n^{1-\kappa}u$ to (5.13) we have that

$$\lim_{n\to\infty}I_{11}(n,\psi_n)=\lim_{n\to\infty}\Big\{\frac{n^\kappa\gamma/2}{(2\pi)^N F_n^2}\int_{\mathbb{R}^N}|\widehat{\phi}(z)|^2\mathrm{d}z\int_0^{n^{1-\kappa}}\mathrm{e}^{-n^\kappa\delta_n s}\mathrm{d}s \times\Big[\int_s^{n^{1-\kappa}}\widetilde{h}(\frac{u}{n^{1-\kappa}})\mathrm{e}^{-n^\kappa(u-s)(\delta_n+\sum_{k=1}^N|z_k|^{\alpha_k})}n^\kappa\mathrm{d}u\Big]^2\Big\}. \tag{5.14}$$

Note that, as $n\to\infty$,

$$\begin{aligned}\frac{\widetilde{h}(\frac{s}{n^{1-\kappa}})}{\sum_{k=1}^N|z_k|^{\alpha_k}}&\ge\int_s^{n^{1-\kappa}}\widetilde{h}(\frac{u}{n^{1-\kappa}})\mathrm{e}^{-n^\kappa(u-s)(\delta_n+\sum_{k=1}^N|z_k|^{\alpha_k})}n^\kappa\mathrm{d}u\\&=\int_0^{n-n^\kappa s}\tilde{h}(\frac{v}{n}+\frac{s}{n^{1-\kappa}})\mathrm{e}^{-v(\delta_n+\sum_{k=1}^N|z_k|^{\alpha_k})}\mathrm{d}v\to\frac{\widetilde{h}(0)}{\sum_{k=1}^N|z_k|^{\alpha_k}}.\end{aligned}$$

Substituting $F_n^2 = n^\kappa$ into (5.14), by the dominated convergence theorem, as $n \to \infty$ we derive that

$$\begin{aligned} I_{11}(n,\psi_n) \to & \ \frac{\gamma}{2(2\pi)^N} \int_{\mathbb{R}^N} |\widehat{\phi}(z)|^2 \mathrm{d}z \int_0^\infty \mathrm{e}^{-\theta s} \Big(\frac{\widetilde{h}(0)}{\sum_{k=1}^N |z_k|^{\alpha_k}} \Big)^2 \mathrm{d}s \\ = & \ \frac{\gamma}{2\theta(2\pi)^N} \int_{\mathbb{R}^N} \frac{|\widehat{\phi}(z)|^2}{(\sum_{k=1}^N |z_k|^{\alpha_k})^2} \mathrm{d}z \int_0^1 \int_0^1 h(u)h(v)\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{5.15}$$

As shown in Remark 2.3 in Li and Xiao [16], the right hand side of (5.15) is finite.

To determine the limit of $I_{12}(n,\psi_n)$, we use the formula (3.40) in [16] to deduce that

$$|I_{12}(n,\psi_n)| \le \gamma I_{121}(n,\psi_n) + \gamma I_{122}(n,\psi_n) + \frac{\gamma^2}{2} I_{123}(n,\psi_n), \tag{5.16}$$

where $I_{121}(n,\psi_n), I_{122}(n,\psi_n)$ and $I_{123}(n,\psi_n)$ are defined by (3.41), (3.42) and (3.43) in [16], respectively. From the formulas (3.45), (3.47) and (3.48) in [16] we have that

$$I_{121}(n,\psi_n) \le \frac{2\delta_n}{\gamma(\gamma-\delta_n)} I_{11}(n,\psi_n), \tag{5.17}$$

and that

$$\begin{aligned} I_{122}(n,\psi_n) \le & \frac{Cn^4}{F_n^3} \int_{\mathbb{R}^{2N}} |\widehat{\phi}(z)\widehat{\phi}(z')| \mathrm{d}z\mathrm{d}z' \int_0^1 \mathrm{e}^{-n\delta_n s} \mathrm{d}s \int_0^{1-s} \mathrm{d}u \int_0^{1-s} \mathrm{d}v \\ & \times \int_0^{1-s-u} \mathrm{e}^{-n(u+v)\sum_{k=1}^N |z_k|^{\alpha_k}} \mathrm{e}^{-nw \sum_{k=1}^N |z_k'|^{\alpha_k}} \mathrm{d}w. \end{aligned} \tag{5.18}$$

$$\begin{aligned} I_{123}(n,\psi_n) \le & \frac{Cn^3}{F_n^3} \int_{\mathbb{R}^{2N}} \frac{|\widehat{\phi}(z+z')|}{\sum_{k=1}^N |z_k + z_k'|^{\alpha_k}} \frac{|\widehat{\phi}(z')|}{\sum_{k=1}^N |z_k'|^{\alpha_k}} |\widehat{\phi}(z)| \mathrm{d}z\mathrm{d}z' \int_0^1 \mathrm{e}^{-n\delta_n s} \mathrm{d}s \\ & \times \int_0^{1-s} \mathrm{e}^{-nu\sum_{k=1}^N |z_k|^{\alpha_k}} \mathrm{d}u \int_0^{1-s} \mathrm{e}^{-nv\sum_{k=1}^N |z_k|^{\alpha_k}} \mathrm{d}v. \end{aligned} \tag{5.19}$$

Firstly, letting $\delta_n \to 0$, we deduce from (5.15) and (5.16) that

$$I_{121}(n,\psi_n) \to 0. \tag{5.20}$$

Secondly, applying the substitution $s' = n^{1-\kappa}s$, $u' = n^{1-\kappa}u$, $v' = n^{1-\kappa}v$ and $w' = n^{1-\kappa}w$ to (5.18) we derive that

$$\begin{aligned} I_{122}(n,\psi_n) \le & \ \frac{Cn^{4\kappa}}{F_n^3} \int_{\mathbb{R}^{2N}} |\widehat{\phi}(z)\widehat{\phi}(z')| \mathrm{d}z\mathrm{d}z' \int_0^\infty \mathrm{e}^{-n^\kappa \delta_n s} \mathrm{d}s \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v \\ & \times \int_0^\infty \mathrm{e}^{-n^\kappa (u+v)\sum_{k=1}^N |z_k|^{\alpha_k} - n^\kappa w \sum_{k=1}^N |z_k'|^{\alpha_k}} \mathrm{d}w. \end{aligned} \tag{5.21}$$

Note that $n^\kappa \delta_n \to \theta \in (0,\infty)$ and $F_n^2 = n^\kappa$, (5.21) implies that

$$0 \le I_{122}(n,\psi_n) \le \frac{C}{\delta_n n^{3\kappa/2}} \int_{\mathbb{R}^{2N}} \frac{|\widehat{\phi}(z)|}{\big(\sum_{k=1}^N |z_k|^{\alpha_k}\big)^2} \frac{|\widehat{\phi}(z')|}{\sum_{k=1}^N |z_k'|^{\alpha_k}} \mathrm{d}z\mathrm{d}z' \to 0. \tag{5.22}$$

Here we have used the finiteness of $\int_{\mathbb{R}^N} \frac{|\widehat{\phi}(z)|}{\left(\sum_{k=1}^N |z_k|^{\alpha_k}\right)^2} \mathrm{d}z \int_{\mathbb{R}^N} \frac{|\widehat{\phi}(z')|}{\sum_{k=1}^N |z'_k|^{\alpha_k}} \mathrm{d}z'$, which follows from Remark 2.3 in [16].

At last, by the same discussion as above, from (5.19) we have

$$I_{123}(n, \psi_n) \le \frac{C}{\delta_n n^{3\kappa/2}} \int_{\mathbb{R}^{2N}} \frac{|\widehat{\phi}(z+z')|}{\sum_{k=1}^N |z_k + z'_k|^{\alpha_k}} \frac{|\widehat{\phi}(z')|}{\sum_{k=1}^N |z'_k|^{\alpha_k}} \frac{|\widehat{\phi}(z)| \mathrm{d}z \mathrm{d}z'}{(\sum_{k=1}^N |z_k|^{\alpha_k})^2} \tag{5.23}$$

for some constant $C > 0$. Using again Remark 2.3 in [16], we know

$$\int_{\mathbb{R}^N} \frac{|\widehat{\phi}(z)|^2}{(\sum_{k=1}^N |z_k|^{\alpha_k})^2} \mathrm{d}z < \infty,$$

and hence by Hölder inequality,

$$\int_{\mathbb{R}^N} \frac{|\widehat{\phi}(z')||\widehat{\phi}(z+z')|}{\sum_{k=1}^N |z_k + z'_k|^{\alpha_k} \sum_{k=1}^N |z'_k|^{\alpha_k}} \mathrm{d}z'$$

is bounded for all $z \in \mathbb{R}^N$. Therefore

$$\int_{\mathbb{R}^{2N}} \frac{|\widehat{\phi}(z+z')|}{\sum_{k=1}^N |z_k + z'_k|^{\alpha_k}} \frac{|\widehat{\phi}(z')|}{\sum_{k=1}^N |z'_k|^{\alpha_k}} \frac{|\widehat{\phi}(z)|}{(\sum_{k=1}^N |z_k|^{\alpha_k})^2} \mathrm{d}z \mathrm{d}z' < \infty.$$

Consequently, (5.23) implies that as $n \to \infty$,

$$I_{123}(n, \psi_n) \to 0. \tag{5.24}$$

Combining (5.16) with (5.20), (5.22) and (5.24) yields that

$$I_{12}(n, \psi_n) \to 0.$$

This, (5.10) and (5.15) imply (5.9). □

For the limits of $I_2(n, \psi_n)$ and $I_3(n, \psi_n)$, we have the following lemmas whose proofs are similar to those of Lemma 3.3 and Lemma 3.4 in [16] and are omitted.

Lemma 5.2 *If $F_n^2 = n^\kappa$, then*

$$\lim_{n\to\infty} I_2(n, \psi_n) = \frac{1}{\theta(2\pi)^N} \int_0^1 h(t) \mathrm{d}t \int_0^1 h(t') \mathrm{d}t' \int_{\mathbb{R}^N} \frac{|\widehat{\phi}(z)|^2}{\sum_{k=1}^N |z_k|^{\alpha_k}} \mathrm{d}z. \tag{5.25}$$

Lemma 5.3 *If $F_n \to \infty$ as $n \to \infty$, then*

$$\lim_{n\to\infty} I_3(n, \psi_n) = 0. \tag{5.26}$$

We are ready to prove Theorem 5.1. Since the main arguments are similar to those in Li and Xiao [16], as well as in Bojedcki *et al* [7, 8, 9] and Li [14]. For simplicity, we will omit some common steps which can be referred to [16] and focus on places where modifications are needed.

Proof of Theorem 5.1 We employ the space-time method formulated in Bojdecki et al. [7]. Following Bojdecki et al. [9], it suffices to show the following statements:

(i) $\langle \widetilde{X}_n, \psi\rangle$ converges in distribution to $\langle \widetilde{X}, \psi\rangle$ for all $\psi \in \mathcal{S}(\mathbb{R}^{N+1})$ as $n \to \infty$.

(ii) $\{\langle X_n, \phi\rangle; n \geq 1\}$ is tight in $C([\varepsilon, 1], \mathbb{R})$ for all $\phi \in \mathcal{S}(\mathbb{R}^N)$, where the theorem of Mitoma [20] is used.

To prove (i), as pointed out by Bojidecki et al. [8], it is sufficient to show that as $n \to \infty$

$$\mathbb{E}(\mathrm{e}^{-\langle \widetilde{X}_n, \psi\rangle}) \to \exp\left\{\frac{1}{2}\int_0^1\int_0^1 \mathrm{Cov}(\langle X(s), \psi(\cdot, s)\rangle, \langle X(t), \psi(\cdot, t)\rangle)\mathrm{d}s\mathrm{d}t\right\} \tag{5.27}$$

for every non-negative $\psi \in \mathcal{S}(\mathbb{R}^{N+1})$.

To this end, we assume $\psi(x, t) = \phi(x)h(t)$, where $\phi \in \mathcal{S}(\mathbb{R}^N)$ and $h \in \mathcal{S}(\mathbb{R})$ are nonnegative functions. It follows from (5.5), Lemmas 5.1–5.3 that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}(\mathrm{e}^{-\langle \widetilde{X}_n, \psi\rangle}) \\
&= \exp\left\{\frac{1}{\theta}\int_{\mathbb{R}^N}\left(\frac{\gamma}{2}\frac{1}{(\sum_{k=1}^N |z_k|^{\alpha_k})^2} + \frac{1}{\sum_{k=1}^N |z_k|^{\alpha_k}}\right)\widehat{\phi}_1(z)\overline{\widehat{\phi}_2(z)}\mathrm{d}z\Big]\Big(\int_0^1 h(t)\mathrm{d}t\Big)^2\right\} \\
&= \exp\left(\frac{1}{2}\int_0^1\int_0^1 \mathrm{Cov}(\langle X(s), \psi(\cdot, s)\rangle, \langle X(t), \psi(\cdot, t)\rangle\mathrm{d}s\mathrm{d}t\right),
\end{aligned}$$

where X is the limit process in Theorem 5.1. This proves (5.27).

For general ψ, the proof of (5.27) is the same with slightly more complicated notation, and hence is omitted.

Now we prove (ii). Note that by the same argument as those used in Bojdecki *et al* [11], from the proof of (i), we get that X_n converges to X in finite-dimensional distributions. This implies the tightness of the sequence $\{\langle X_n(\varepsilon), \phi\rangle\}$. According to Theorem 12.3 in Billingsley [6] and the proof of Proposition 3.3 in Bojdecki *et al* [10], the remainder is to prove that for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, $\varepsilon \leq t_1 < t_2 \leq 1$ and $\eta > 0$, there exist constants $a \geq 1$, $b > 0$ and $K > 0$, which is independent of t_1, t_2, such that for all $n \geq 1$.

$$\int_0^{1/\eta}\Big(1 - \mathrm{Re}\Big(\mathbb{E}\big(\exp\{-i\omega\langle \tilde{X}_n, \phi h\rangle\}\big)\Big)\Big)\mathrm{d}\omega \leq \frac{K}{\eta^a}(t_2 - t_1)^{1+b}, \tag{5.28}$$

where $h \in \mathcal{S}(R)$ is an approximation of $\mathbf{1}_{\{t_2\}}(t) - \mathbf{1}_{\{t_1\}}(t)$ supported on $[t_1, t_2]$ such that $\tilde{h}(t)$ satisfies

$$\tilde{h} \in \mathcal{S}(R), \qquad 0 \leq \tilde{h} \leq \mathbf{1}_{[t_1, t_2]}.$$

The proof of (5.28) is almost the same as the corresponding part of Theorem 2.1 in [14]. The details are omitted here and the proof of Theorem 5.1 is complete. □

Remark 5.2 If we replace the branching particle systems with constant splitting rate γ with $(N, \vec{\alpha}, \delta_n, \gamma_n)$-branching particle systems, where $\gamma_n \to 0$ and $\delta_n/\gamma_n \to 0$ as $n \to \infty$ and (5.1) holds, then from the proof of Theorem 5.1, we can prove that a similar conclusion holds, where the limit process X is a centered constant (in time) Gaussian process valued in $\mathcal{S}'(\mathbb{R}^N)$, with covariance function

$$\mathrm{Cov}(\langle X(s), \phi_1\rangle, \langle X(t), \phi_2\rangle) = \frac{2}{\theta(2\pi)^N}\int_{\mathbb{R}^N}\frac{\widehat{\phi}_1(z)\overline{\widehat{\phi}_2(z)}}{\sum_{k=1}^N |z_k|^{\alpha_k}}\mathrm{d}z \tag{5.29}$$

for all ϕ_1 and ϕ_2 in $\mathcal{S}(\mathbb{R}^N)$. In this way, we can construct the pseudo-fractional Brownian sheet Y_1 directly from the limit process.

References

[1] Adler, R. J., *The Geometry of Random Fields.* Wiley, New York, 1981.

[2] Adler, R. J. and Taylor, J. E., *Random Fields and Geometry.* Springer, New York, 2007.

[3] Ayache, A., Leger, S. and Pontier, M., Drap Brownien fractionnaire. *Potential Theory* **17** (2002), 31–43.

[4] Ayache, A. and Xiao, Y., Asymptotic properties and hausdorff dimensions of fractional Brownian sheets. *J. Fourier Anal. Appl.* **11**(2005), 407–439.

[5] Biermé, H., Meerschaert, M. M. and Scheffler, H. P., Operator scaling stable random fields. *Stochastic Process. Appl.* **117** (2007), 312–332.

[6] Billingsley, P., *Convergence of Probability Meausre.* Wiley, New York, 1968.

[7] Bojdecki, T., Gorostiza, L, and Ramaswami, S., Convergence of $\mathcal{S}'$-valued processes and space-time random fields. *J. Funct. Anal.* **66** (1986), 21–41.

[8] Bojdecki, T., Gorostiza, L, and Talarczyk, A., Limit theorem for occupation time fluctuations of branching systems I: Long-range dependence. *Stochastic Process. Appl.* **116** (2006), 1–18.

[9] Bojdecki, T., Gorostiza, L, and Talarczyk, A., Limit theorem for occupation time fluctuations of branching systems II: Critical and large dimensions. *Stochastic Process. Appl.* **116** (2006), 19–35.

[10] Bojdecki, T., Gorostiza, L, and Talarczyk, A. A long range dependence stable process and an infinite variance branching system. *Ann. Probab.*, 35 (2007): 500-527.

[11] Bojdecki, T., Gorostiza, L, and Talarczyk, A. Occupation time fluctuations of an infinite-variance branching system in large dimension. *Bernoulli*, 13 (2007): 20-39.

[12] Herbin, E., From N parameter fractional Brownian motions to N parameter multifractional Brownian motions. *Rocky Mount. J. Math.* **36** (2006), 1249–1284.

[13] Kamont, A., On the fractional anisotropic Wiener field. *Probab. Math. Statist.* **16** (1996), 85–98.

[14] Li, Y., Fluctuation limits of strongly degenerate branching systems. *Sci. China Math*, 54 (2011): 2669-2682.

[15] Li, Y. and Xiao, Y., Multivariate operator-self-similar random fields. *Stochastic Process. Appl.* **121** (2011), 1178–1200.

[16] Li, Y. and Xiao, Y., Occupation time fluctuations of weakly degenerate branching systems. *J. Theoretic. Probab.*, Doi: 10.1007/s10959-011-0358-3

[17] Marcus, M. and Rosen, J., *Markov Processes, Gaussian Processes, and Local Times.* Cambridge University Press, Cambridge, 2006.

[18] Mason, D. M. and Shi, Z., Small deviations for some multi-parameter Gaussian processes. *J. Theoret. Probab.* **14** (2001), 213–239.

[19] Meerschaert, M., Wang, W. and Xiao, Y., Fernique-type inequalities and moduli of continuity of anisotropic Gaussian random fields. *Trans. Amer. Math. Soc.*, (to appear). Available at http://www.stt.msu.edu/˜mcubed/MWX10.pdf2010

[20] Mitoma, I., Tightness of probability on $C([0,1];\mathscr{S}')$ and $D([0,1];\mathscr{S}')$. *Ann. Probab.* **11** (1983), 989–999.

[21] Potthoff, J., Sample properties of random fields III: differentiability. *Comm. Stoch. Anal.* **4** (2010), 335–353.

[22] Samorodnitsky, G. and Taqqu, M., *Stable Non-Gaussian Random Processes.* Chapman and Hall, New York, 1994.

[23] Talagrand, M., Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. Probab.* **23** (1995), 767–775.

[24] Wu, D. and Xiao, Y., Geometric properties of fractional Brownian sheets. *J. Fourier Anal. Appl.* **13** (2007), 1–37.

[25] Xiao, Y., Sample path properties of anisotropic Gaussian random fields. In: *A Minicourse on Stochastic Partial Differential Equations*, Khoshnevisan, D., Rassoul-Agha, F. (Eds.), pp. 145–212, Lecture Notes in Math. **1962**, Springer, New York, 2009.

[26] Xue, Y. and Xiao, Y., Fractal and smoothness properties of anisotropic Gaussian models. *Frontiers Math. China* **6** (2011), 1217–1246.