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Local Times and Related Properties of Multidimensional Iterated Brownian Motion

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Let $\{W(t), t \in \mathbf{R}\}$ and $\{B(t), t \geqslant 0\}$ be two independent Brownian motions in $\mathbf{R}$ with $W(0) = B(0) = 0$ and let

$$Y(t) = W(B(t)) \qquad (t \geqslant 0)$$

be the iterated Brownian motion. Define d-dimensional iterated Brownian motion by

$$X(t) = (X_1(t),..., X_d(t)) \qquad (t \geqslant 0)$$

where $X_1,..., X_d$ are independent copies of Y. In this paper, we investigate the existence, joint continuity and Hölder conditions in the set variable of the local time

$$L = \{L(x, B): x \in \mathbf{R}^d, B \in \mathscr{B}(\mathbf{R}_+)\}$$

of $X(t)$, where $\mathscr{B}(\mathbf{B}_+)$ is the Borel σ-algebra of $\mathbf{R}_+$. These results are applied to study the irregularities of the sample paths and the uniform Hausdorff dimension of the image and inverse images of $X(t)$.

KEY WORDS: Iterated Brownian motion; Local times; Hölder conditions; Level set; Hausdorff dimension.

1. INTRODUCTION

Let $\{W(t), t \in \mathbf{R}\}$ and $\{B(t), t \in \mathbf{R}_+\}$ be two independent Brownian motions in $\mathbf{R}$ with $W(0) = B(0)$ and let

$$Y(t) = W(B(t)), \qquad t \in \mathbf{R}_+$$

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Using the terminology of Burdzy,[7] $Y(t)$ $(t \in \mathbf{R})_+$ is called iterated Brownian motion or simply IBM. Recently, there has been a lot of investigations on sample path properties of IBM [see Bertoin[6]; Burdzy[7,8]; Burdzy and Khoshnevisan[9]; Csáki *et al.*[10,11]; Deheuvels and Mason[12]: Hu *et al.*[18]; Hu and Shi[19]; Khoshnevisan and Lewis[25,26]; and Shi[32] and the references therein].

It is easy to verify that $Y(t)$ $(t \in \mathbf{R}_+)$ has stationary increments and is a self-similar process of index 1/4, that is, for any $a > 0$

$$Y(a\,\cdot) \overset{\mathrm{d}}{=} a^{1/4} Y(\cdot) \tag{1.1}$$

where $X \overset{\mathrm{d}}{=} Y$ means that the two processes X and Y have the same finite dimensional distributions. The notion of self-similarity was first initiated by Kolmogorov in 1940. Self-similar processes have gained much attention due to their role in numerous physical theories. We refer to Kôno[24] and Taqqu[33] for more identical comments and a bibliographical guide on self-similar processes, especially self-similar stable (including Gaussian) processes. IBM offers an interesting example of nonstable self-similar processes.

The existence and joint continuity of the local time of $Y(t)$ have been proved independently and at the same time by Burdzy and Khoshnevisan[9] and by Csáki *et al.*[11] In particular, they proved that for any $T > 0$, almost surely

$$\limsup_{h \to 0} \sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbf{R}} \frac{l(x, t+h) - l(x, t)}{h^{3/4} (\log 1/h)^{5/4}} \leqslant K \qquad \text{a.s.} \tag{1.2}$$

By using (1.2) and a capacity argument, Burdzy and Khoshnevisan[9] also proved the following uniform Hausdorff dimension result for the level sets of IBM: almost surely

$$\dim(Y^{-1}(x) \cap [0, t]) = \tfrac{3}{4} \tag{1.3}$$

for all $t \geqslant 0$ and all x in the interior of $Y([0, t])$, where

$$Y^{-1}(x) = \{s \geqslant 0 \colon Y(s) = x\}$$

is the x-level set of $Y(t)$ and $\dim E$ is the Hausdorff dimension of E. We refer to Falconer[14] for definition and more properties of Hausdorff measure and Hausdorff dimension.

Associate to IBM $Y(t)$ $(t \geqslant 0)$, we define an $\mathbf{R}^d$ valued process by

$$X(t) = (X_1(t), \ldots, X_d(t)) \tag{1.4}$$

where $X_1,..., X_d$ are independent copies of Y. We will call $X(t)$ $(t \geqslant 0)$ d-dimensional iterated Brownian motion. By (1.1) we see that $X(t)$ $(t \geqslant 0)$ is also self-similar of index 1/4. Furthermore for every $U \in \mathrm{SO}(d)$

$$UX(\cdot) \stackrel{\mathrm{d}}{=} X(\cdot) \tag{1.5}$$

The purpose of this paper is twofold. Our first objective is to investigate the existence, joint continuity and Hölder conditions in the set variable of the local times of $X(t)$ $(t \geqslant 0)$. This is motivated by the papers of Burdzy and Khoshnevisan[9] and Csáki *et al.*[11] In particular, our Theorem 3 improves Proposition 2.4 in Burdzy and Khoshnevisan[9] and Theorem 3.1 in Csáki *et al.*[11] We also apply the results on local time of IBM to derive other sample path properties such as nowhere differentiability and the Hausdorff measure of the graph of $X(t)$. The second objective of this paper is to prove some results on the Hausdorff dimension of the image and inverse image of $X(t)$. In particular, we prove that almost surely for every $x \in \mathbf{R}^d \backslash \{0\}$

$$\dim X^{-1}(x) = 1 - \frac{d}{4}$$

This generalizes (1.3) to iterated Brownian motion in $\mathbf{R}^d$.

We say a few words about the methods used in this paper. The arguments of Burdzy and Khoshnevisan[9] and Csáki *et al.*[11] in the study of local time of iterated Brownian motion depend on the properties of the local time of Brownian motion in $\mathbf{R}$. Since Brownian motion in $\mathbf{R}^d$ has no local time for $d > 1$, their methods can not be carried over to the current case. Instead, we will use the methods developed by Berman[3–5]; Pitt[30]; Ehm[13]; and Xiao[35] to prove the results on local time of d-dimensional iterated Brownian motion. Uniform Hausdorff dimension results for Brownian motion were proved by Kaufman[21, 22]; see also Perkins and Taylor[29]; for locally nondeterministic Gaussian random fields by Monrad and Pitt.[27] We will follow the same line to study the Hausdorff dimension of the image set and the inverse image of $X(t)$.

The paper is organized as follows. In Section 2 we study the existence, joint continuity and Hölder conditions in the set variable of the local time of iterated Brownian motion in $\mathbf{R}^d$. The results are applied to derive partially the results on the modulus nondifferentiability of the sample path proved by Hu and Shi[19] and the Hausdorff measure of the graph of $X(t)$. In Section 3, we prove some uniform Hausdorff dimension results for the image set and the inverse image of $X(t)$.

We will use K to denote an unspecified positive finite constant which may not necessarily be the same in each occurrence.

2. THE LOCAL TIME OF IBM

Let $X(t)$ $(t \in \mathbf{R}_+)$ be iterated Brownian motion in $\mathbf{R}^d$ defined by (1.4). In this section, we investigate the existence, joint continuity and Hölder conditions in the set variable of the local time

$$L = \{L(x, B): x \in \mathbf{R}^d, B \in \mathscr{B}(\mathbf{R}_+)\}$$

of $X(t)$, where $\mathscr{B}(\mathbf{R}_+)$ is the Borel σ-algebra of $\mathbf{R}_+$. These results are applied to study the irregularities of the sample paths of $X(t)$.

We recall briefly the definition of local time. For a comprehensive survey on local times of both random and nonrandom vector fields, we refer to Geman and Horowitz[16] [see also Geman, *et al.*[17]]. Let $X(t)$ be any Borel function on $\mathbf{R}$ with values in $\mathbf{R}^d$. For any Borel set $B \subset \mathbf{R}$, the occupation measure of X is defined by

$$\mu_B(A) = \lambda_1\{t \in B: X(t) \in A\} \tag{2.1}$$

for all Borel set $A \subseteq \mathbf{R}^d$, where λ_1 is the one-dimensional Lebesgue measure. If μ_B is absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbf{R}^d$, we say that $X(t)$ has a local time on B and define its local time $L(x, B)$ to be the Radon–Nikodym derivative of μ_B. If $B = [0, t]$, we simply write $L(x, B)$ as $L(x, t)$.

The following existence theorem for the local time of iterated Brownian motion in $\mathbf{R}^d$ is easily proved by using Fourier analysis [see, e.g., Berman[3] or Kahane[20]].

Proposition 1. If $d < 4$, then for any $T > 0$, with probability 1, $X(t)$ $(0 \leqslant t \leqslant T)$ has a square integrable local time $L(x, T)$.

Proof. Let $\mu_{[0, T]}$ be the occupation measure defined by (2.1). Then the Fourier transform of $\mu_{[0, T]}$ is

$$\hat{\mu}(u, T) = \int_0^T \exp(i \langle u, X(t) \rangle)\, dt$$

where $\langle \cdot, \cdot \rangle$ is the ordinary scalar product in $\mathbf{R}^d$ and we will use $|\cdot|$ to denote the Euclidean norm. It follows from Fubini's theorem that

$$E \int_{\mathbf{R}^d} |\hat{\mu}(u, T)|^2\, du = \int_0^T \int_0^T \int_{\mathbf{R}^d} E \exp(i \langle u, X(t) - X(s) \rangle)\, du\, dt\, ds \tag{2.2}$$

To evaluate the characteristic function in (2.2), we assume $0 < s < t$ (the case $0 < t < s$ is similar) and denote the density of $(B(t), B(s))$ by $p_{t,s}(x, y)$. Since $X_1, ..., X_d$ are independent copies of $W(B(t))$, we have

$$
\begin{aligned}
&E \exp(i \langle u, X(t) - X(s) \rangle) \\
&= \prod_{k=1}^{d} E \exp(iu_k(W(B(t)) - W(B(s)))) \\
&= \prod_{k=1}^{d} \int_{\mathbf{R}} \int_{\mathbf{R}} E \exp(iu_k(W(x) - W(y)))\, p_{t,s}(x, y)\, dx\, dy \\
&= \prod_{k=1}^{d} \int_{\mathbf{R}} \int_{\mathbf{R}} \exp\left(-\frac{u_k^2}{2} |x - y|\right) \\
&\quad \times \frac{1}{2\pi \sqrt{s\,|t-s|}} \exp\left(-\frac{y^2}{2s} - \frac{(x-y)^2}{2\,|t-s|}\right) dx\, dy \\
&= \prod_{k=1}^{d} \sqrt{\frac{2}{\pi}} \exp\left(\frac{1}{8} |t-s|\, u_k^4\right) \int_{(1/2)\, u_k^2 \sqrt{|t-s|}}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx
\end{aligned} \tag{2.3}
$$

by substitutions. Putting (2.3) into (2.2), we have

$$
\begin{aligned}
&E \int_{\mathbf{R}^d} |\hat{\mu}(u, T)|^2\, du \\
&= \int_0^T \int_0^T \int_{\mathbf{R}^d} \prod_{k=1}^{d} \sqrt{\frac{2}{\pi}} \exp\left(\frac{1}{8} |t-s|\, u_k^4\right) \\
&\quad \times \int_{(1/2)\, u_k^2 \sqrt{|t-s|}}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx\, du\, ds\, dt \\
&= \left(\frac{2}{\pi}\right)^{d/2} \left[\int_{\mathbf{R}} \exp\left(\frac{1}{8} u^4\right) \int_{(1/2)\, u^2}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx\, du\right]^d \cdot \int_0^T \int_0^T \frac{ds\, dt}{|t-s|^{d/4}} \\
&< \infty
\end{aligned}
$$

since $d < 4$. That is, almost surely $\hat{\mu}(u, T) \in L^2(\mathbf{R}^d)$. Therefore with probability 1, $\mu_{[0,T]}$ is absolutely continuous with respect to λ_d and its density belongs to $L^2(\mathbf{R}^d)$. The proof is completed. □

We can express the local times $L(x, t)$ as the inverse Fourier transform of $\hat{\mu}(u, t)$, namely

$$L(x, t) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{R}^d} \exp(-i\langle u, x\rangle)\, \hat{\mu}(u, t)\, du$$
$$= \left(\frac{1}{2\pi}\right)^d \int_0^t \int_{\mathbf{R}^d} \exp(-i\langle u, x\rangle) \exp(i\langle u, X(s)\rangle)\, du\, ds \tag{2.4}$$

It follows from (2.4) that for any $x, w \in \mathbf{R}^d$, $B \in \mathscr{B}(\mathbf{R}_+)$ and any integer $n \geqslant 1$, we have

$$E[L(x, B)]^n = (2\pi)^{-nd} \int_{B^n} \int_{\mathbf{R}^{nd}} \exp\left(-i \sum_{j=1}^n \langle u_j, x\rangle\right)$$
$$\times E \exp\left(i \sum_{j=1}^n \langle u_j, X(t_j)\rangle\right) d\bar{u}\, d\bar{t} \tag{2.5}$$

and

$$E[L(x+w, B) - L(x, B)]^n$$
$$= (2\pi)^{-nd} \int_{B^n} \int_{\mathbf{R}^{nd}} \prod_{j=1}^n (\exp(-i\langle u_j, x+w\rangle) - \exp(-i\langle u_j, x\rangle))$$
$$\times E \exp\left(i \sum_{j=1}^n \langle u_j, X(t_j)\rangle\right) d\bar{u}\, d\bar{t} \tag{2.6}$$

where $\bar{u} = (u_1, ..., u_n)$, $\bar{t} = (t_1, ..., t_n)$, and each $u_j \in \mathbf{R}^d$, $t_j \in B$ $(j = 1, ..., n)$. In the coordinate notation we then write $u_j = (u_j^1, ..., u_j^d)$. The equalities (2.5) and (2.6) have also been given by (25.5) and (25.7) in Geman and Horowitz.[16]

For a fixed $T > 0$, if we can choose $L(x, t)$ to be a continuous function of (x, t), $x \in \mathbf{R}^d$, $0 \leqslant t \leqslant T$, then X is said to have a jointly continuous local time on $[0, T]$. Under these conditions, $L(x, \cdot)$ can be extended to be a finite measure supported on the level set

$$X_T^{-1}(x) = \{t \in [0, T] \colon X(t) = x\}$$

see Adler[2] [Thm. 8.6.1]. This fact has been used by Berman[4]; Adler[1]; Ehm[13]; Monrad and Pitt,[27 just mention a few, to study the Hausdorff dimension of the level sets and inverse image of stochastic processes.

In order to study the joint continuity and Hölder conditions in the set variable of the local time of d-dimensional iterated Brownian motion, we will use the methods similar to those used by Ehm[13]; Geman *et al.*[17]; and Xiao.[35] We first prove some lemmas.

Lemma 1. Let $0<\alpha<1$. Then for any integer $n\geqslant 1$ and any $x_1,\ldots, x_n\in\mathbf{R}$, we have

$$\int_{\mathbf{R}} \frac{\exp(-x^2/2)}{\min\{|x-x_j|^\alpha, j=1,\ldots, n\}}\, dx \leqslant Kn^\alpha \tag{2.7}$$

and for any $\gamma>0$

$$\int_{\mathbf{R}} \frac{|x|^\gamma \exp(-x^2/2)}{\min\{|x-x_j|^\alpha, j=1,\ldots, n\}}\, dx \leqslant K2^{\gamma/2}\Gamma\left(\frac{\gamma+1}{2}\right) n^\alpha \tag{2.8}$$

where $K>0$ is a finite constant depending only on α.

Proof. It suffices to prove (2.7). We observe that the left-hand side of (2.7) can be written as

$$\begin{aligned}\sum_{l\in\mathbf{Z}} &\int_l^{l+1} \frac{\exp(-x^2/2)}{\min\{|x-x_j|^\alpha, j=1,\ldots, n\}}\, dx \\ &\leqslant 2\sum_{l\in\mathbf{Z}} \exp(-l^2/2)\int_0^1 \frac{dx}{\min\{|x+l-x_j|^\alpha, j=1,\ldots, n\}}\end{aligned} \tag{2.9}$$

where $\mathbf{Z}$ is the set of the integers. It is clear that (2.7) follows from (2.9) and Lemma 2. □

Lemma 2. Let $0<\alpha<1$. Then for any integer $n\geqslant 1$ and any $x_1,\ldots, x_n\in\mathbf{R}$, we have

$$\int_0^1 \frac{dx}{\min\{|x-x_j|^\alpha, j=1,\ldots, n\}} \leqslant Kn^\alpha \tag{2.10}$$

where $K>0$ is a finite constant depending only on α.

Proof. The easiest case is that $x_i\in[0,1]$ $(i=1,\ldots, n)$. In this case, the proof of (2.10) is implied in the following. If x_i $(i=1,\ldots, n)$ are not all in $[0,1]$, we may and will assume $x_1, x_2,\ldots, x_n\in[-1,2]$ and

$$x_1\leqslant x_2\leqslant \cdots \leqslant x_n$$

Since decreasing the number of x_i's does not increase the value of the integral in (2.10), we will further assume $x_1 \in [-1, 0)$ and

$$|x_1| < x_2 < \cdots < x_{n-1} \leqslant 1 < x_n \leqslant 2 - x_{n-1}$$

In this case

$$\begin{aligned}
&\int_0^1 \frac{dx}{\min\{|x-x_j|^\alpha, j=1,..., n\}} \\
&\quad = \int_0^{(x_1+x_2)/2} \frac{dx}{|x-x_1|^\alpha} + \int_{(x_1+x_2)/2}^{(x_2+x_3)/2} \frac{dx}{|x-x_2|^\alpha} + \cdots + \int_{(x_{n-1}+x_n)/2}^{1} \frac{dx}{|x-x_n|^\alpha} \\
&\quad \leqslant \frac{2^\alpha}{1-\alpha} [(x_2-x_2)^{1-\alpha} + (x_3-x_2)^{1-\alpha} + \cdots + (x_n-x_{n-1})^{1-\alpha}] \\
&\quad \leqslant \frac{2^\alpha n}{1-\alpha} \left[\frac{1}{n} \sum_{j=2}^{n} (x_j - x_{j-1})\right]^{1-\alpha} \\
&\quad \leqslant K n^\alpha
\end{aligned}$$

by Jensen's inequality. This proves (2.10). □

Lemma 3 is a special case of a lemma of Kôno[23], see also Ehm.[13]

Lemma 3. Let $0 < \alpha < 1$ and $h > 0$. Then for any $n \geqslant 1$,

$$\begin{aligned}
&\int_{0<t_1<t_2\cdots<t_n<h} \frac{1}{\prod_{j=1}^{n} (t_j - t_{j-1})^\alpha} dt_1 \cdots dt_n \qquad (t_0 = 0) \\
&\quad \leqslant \left(\frac{2}{1-\alpha}\right)^n h^{(1-\alpha)n} (n!)^{\alpha-1}
\end{aligned}$$

The following lemma gives the basic estimates for the moments of the local time of iterated Brownian motion in $\mathbf{R}^d$.

Lemma 4. Let $X(t)$ $(t \geqslant 0)$ be d-dimensional iterated Brownian motion with $d < 4$. For any $h > 0$, $B = [0, h]$, $x, w \in \mathbf{R}^d$, any even integer $n \geqslant 2$ and any $0 < \gamma < 1/6$, we have

$$E[L(x, B)]^n \leqslant K^n h^{(1-d/4)n} (n!)^{3d/4} \tag{2.11}$$

$$E[L(x+w, B) - L(x, B)]^n \leqslant K^n |w|^{n\gamma} h^{(1-(d+\gamma)/4)n} (n!)^{4d} \tag{2.12}$$

where $K > 0$ is a finite constant depending on d only.

Proof. Since $X_1,..., X_d$ are independent copies of $Y(t)=W(B(t))$, it follows (2.5) that

$$E[L(x, B)]^n \leqslant (2\pi)^{-nd} \int_{B^n} \prod_{k=1}^{d} \left[\int_{\mathbf{R}^n} E \exp\left(i \sum_{j=1}^{n} u_j^k Y(t_j) \right) dU^k \right] d\bar{t} \tag{2.13}$$

where $U^k = (u_1^k,..., u_n^k) \in \mathbf{R}^n$. For any $n \geqslant 1$, let $S(n)$ be the family of all permutations of $\{1,..., n\}$. For any permutation $\pi \in S(n)$, let

$$\Gamma_\pi = \{(t_1,..., t_n) \in \mathbf{R}^n \colon t_{\pi(1)} < t_{\pi(2)} < \cdots < t_{\pi(n)}\}$$

For simplicity, we consider distinct $t_1,..., t_n \in (0, h]$ with

$$0 = t_0 < t_1 < t_2 < \cdots < t_n \tag{2.14}$$

Denote the density of $(B(t_1), B(t_2),..., B(t_n))$ by $p_{t_1,..., t_n}(y_1,..., y_n)$, i.e.,

$$p_{t_1,..., t_n}(y_1,..., y_n) = \frac{1}{(2\pi)^{n/2} \prod_{j=1}^{n} (t_j - t_{j-1})^{1/2}} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}} \right)$$

Then by conditioning, we have

$$\begin{aligned}
&E \exp\left(i \sum_{j=1}^{n} u_j W(B(t_j)) \right) \\
&\quad = \int_{\mathbf{R}^n} E\left\{ \exp\left(i \sum_{j=1}^{n} u_j W(B(t_j)) \right) \middle| B(t_j) = y_j,\ 1 \leqslant j \leqslant n \right\} \\
&\qquad \times p_{t_1,..., t_n}(y_1,..., y_n)\, dy_1 \cdots dy_n \\
&\quad = \int_{\mathbf{R}^n} E \exp\left(i \sum_{j=1}^{n} u_j W(y_j) \right) p_{t_1,..., t_n}(y_1,..., y_n)\, dy_1 \cdots dy_n \\
&\quad = \int_{\mathbf{R}^n} \exp(-\tfrac{1}{2} U R(y_1,..., y_n) U')\, p_{t_1,..., t_n}(y_1,..., y_n)\, dy_1 \cdots dy_n
\end{aligned} \tag{2.15}$$

where $R(y_1,..., y_n)$ is the covariance matrix of $W(y_1),..., W(y_n)$ and U' is the transpose of $U \in \mathbf{R}^n$. We claim that for any $y_1,..., y_n \in \mathbf{R}$

$$\int_{\mathbf{R}^n} \exp\left(-\frac{1}{2} U R(y_1,..., y_n) U' \right) du \leqslant \frac{(2\pi)^{n/2}}{(\det(R(y_1,..., y_n)))^{1/2}} \tag{2.16}$$

were $\det(R)$ denotes the determinant of R. If $\det(R(y_1,..., y_n)) = 0$, then (2.16) is trivial. Otherwise, we see that the following function

$$(2\pi)^{-n/2} (\det(R(y_1,..., y_n)))^{1/2} \exp(-\tfrac{1}{2} U R(y_1,..., y_n) U')$$

is the density function of the Gaussian vector with mean zero and covariance matrix $R^{-1}(y_1,..., y_n)$. This implies (2.16) immediately.

It is well known that

$$\det(R(y_1,..., y_n)) = \operatorname{Var}(W(y_1)) \prod_{j=2}^{n} \operatorname{Var}(W(y_j) \mid W(y_1),..., W(y_{j-1})) \quad (2.17)$$

where Var Y and $\operatorname{Var}(Y \mid Z)$ denote the variance of Y and the conditional variance of Y given Z respectively, and for any $y_1,..., y_n \in \mathbf{R}$

$$\begin{aligned} &\operatorname{Var}(W(y_n) \mid W(y_1),..., W(y_{n-1})) \\ &\quad \geqslant c \min\{|y_n - y_n|, j = 0, 1,..., n-1\} \qquad (y_0 = 0) \end{aligned} \quad (2.18)$$

where c is an absolute constant. In fact, (2.18) holds for a large class of locally nondeterministic Gaussian processes (see e.g., Xiao[35] and the references therein). We make the change of variables

$$y_j = \sum_{l=1}^{j} (t_l - t_{l-1})^{1/2} z_l \qquad (j = 1,..., n)$$

then it follows from (2.15)–(2.18) and Lemma 1 that

$$\begin{aligned} &\left| \int_{\mathbf{R}^n} E \exp\left(i \sum_{j=1}^{n} u_y Y(t_j) \right) dU \right| \\ &\quad \leqslant \int_{\mathbf{R}^n} \frac{1}{(\det(R(y_1,..., y_n)))^{1/2}} \cdot \exp\left(-\frac{1}{2} \sum_{j=1}^{n} z_j^2 \right) dz_1 \cdots dz_n \\ &\quad \leqslant K^n \int_{\mathbf{R}^n} \frac{1}{\prod_{j=1}^{n} \min\{|y_j - y_i|^{1/2} : 0 \leqslant i \leqslant j-1\}} \\ &\qquad \times \exp\left(-\frac{1}{2} \sum_{j=1}^{n} z_j^2 \right) dz_1 \cdots dz_n \\ &\quad \leqslant K^n \prod_{j=1}^{n} \frac{1}{(t_j - t_{j-1})^{1/4}} \int_{\mathbf{R}^n} \frac{1}{\prod_{j=1}^{n} \min\{|z_j - x_i|^{1/2} : 0 \leqslant i \leqslant j-1\}} \\ &\qquad \times \exp\left(-\frac{1}{2} \sum_{j=1}^{n} z_j^2 \right) dz_1 \cdots dz_n \\ &\quad \leqslant K^n (n!)^{1/2} \cdot \prod_{j=1}^{n} \frac{1}{(t_j - t_{j-1})^{1/4}} \end{aligned} \quad (2.19)$$

where

$$x_i = -\sum_{l=i+1}^{j-1} \sqrt{\frac{t_l - t_{l-1}}{t_j - t_{j-1}}}\, z_l$$

By Lemma 3 and (2.19) we see that (2.13) is at most

$$\begin{aligned} K^n (n!)^{d/2} &\sum_{\pi \in S(n)} \int_{\Gamma_\pi \cap [0,h]^n} \prod_{j=1}^n \frac{1}{(t_{\pi(j)} - t_{\pi(j-1)})^{d/4}}\, dt_1 \cdots dt_n \\ &\leqslant K^n (n!)^{3d/4}\, h^{(1-d/4)\,n} \end{aligned}$$

This proves (2.11).

Now we turn to the proof of (2.12). By (2.6) and the elementary inequality

$$|e^{iu} - 1| \leqslant 2^{1-\gamma} |u|^\gamma \qquad \text{for any} \quad u \in \mathbf{R},\ 0 < \gamma < 1$$

we see that for any even integer $n \geqslant 2$ and any $0 < \gamma < 1$,

$$\begin{aligned} &E[L(x+w, B) - L(x, B)]^n \\ &\quad \leqslant (2\pi)^{-nd}\, 2^{n(1-\gamma)}\, |w|^{n\gamma} \int_{B^n} \int_{\mathbf{R}^{nd}} \prod_{j=1}^n |u_j|^\gamma\, E \exp\left(i \sum_{j=1}^n \langle u_j, X(t_j) \rangle\right) d\bar{u}\, d\bar{t} \end{aligned} \tag{2.20}$$

By making the change of variables $t_j = hs_j$, $j = 1,..., n$ and $u_j = h^{-1/4} v_j$, $j = 1,..., n$ and changing the letters s, v back to t, u, we see that the right-hand side of (2.20) equals

$$\begin{aligned} &(2\pi)^{-nd}\, 2^{n(1-\gamma)}\, |w|^{n\gamma}\, h^{n(1-(d+\gamma)/4)} \\ &\quad \times \int_{[0,1]^n} \int_{\mathbf{R}^{nd}} \prod_{j=1}^n |u_j|^\gamma\, E \exp\left(i \sum_{j=1}^n \langle u_j, X(t_j) \rangle\right) d\bar{u}\, d\bar{t} \end{aligned} \tag{2.21}$$

We fix any distinct $t_1,..., t_n \in [0, 1]$ satisfying (2.14) and let

$$J = \int_{\mathbf{R}^{nd}} \prod_{j=1}^n |u_j|^\gamma\, E \exp\left(i \sum_{j=1}^n \langle u_j, X(t_j) \rangle\right) d\bar{u}$$

Since for any $0 < \gamma < 1$, $|a + b|^\gamma \leqslant |a|^\gamma + |b|^\gamma$, we have

$$\prod_{j=1}^n |u_j|^\gamma \leqslant \sum{}' \prod_{j=1}^n |u_j^{\beta_j}|^\gamma \tag{2.22}$$

where the summation $\sum'$ is taken over all $(\beta_1,..., \beta_n) \in \{1,..., d\}^n$. It follows from (2.22) that

$$J \leqslant \sum' \int_{\mathbf{R}^{nd}} \prod_{j=1}^{n} |u_j^{\beta_j}|^\gamma \, E \exp\left(i \sum_{j=1}^{n} \langle u_j, X(t_j) \rangle \right) d\bar{u} \tag{2.23}$$

Now fix a sequence $(\beta_1,..., \beta_n) \in \{1,..., d\}^n$, for $1 \leqslant k \leqslant d$, let

$$\Lambda_k = \{ j: \beta_j = k \}$$

Then Λ_k $(k = 1,..., d)$ are disjoint and $\sum_k \# \Lambda_k = n$. By the independence of $X_1,..., X_d$, we have

$$\begin{aligned} &\int_{\mathbf{R}^{nd}} \prod_{j=1}^{n} |u_j^{\beta_j}|^\gamma \, E \exp\left(i \sum_{j=1}^{n} \langle u_j, X(t_j) \rangle \right) d\bar{u} \\ &\quad = \prod_{k=1}^{d} \left[\int_{\mathbf{R}^n} \prod_{j \in \Lambda_k} |u_j^k|^\gamma \, E \exp\left(i \sum_{j=1}^{n} u_j^k X_k(t_j) \right) dU^k \right] \end{aligned} \tag{2.24}$$

Fix a $k \in \{1,..., d\}$ and consider the integral in (2.24). To simplify the notations, we will omit the superscript (subscript) k. Similar to (2.15), we have

$$\begin{aligned} &\int_{\mathbf{R}^n} \prod_{j \in \Lambda} |u_j|^\gamma \, E \exp\left(i \sum_{j=1}^{n} u_j W(B(t_j)) \right) dU \\ &\quad = \int_{\mathbf{R}^n} \prod_{j \in \Lambda} |u_j|^\gamma \left[\int_{\mathbf{R}^n} \exp\left(-\tfrac{1}{2} \operatorname{Var}\left(\sum_{j=1}^{n} u_j W(y_j) \right) \right) \right. \\ &\quad\quad \left. \times p_{t_1,..., t_n}(y_1,..., y_n) \, dy_1 \cdots dy_n \right] dU \\ &\quad = \sum_{\pi \in S(n)} \int_{\mathbf{R}^n \cap \Gamma_\pi} \left[\int_{\mathbf{R}^n} \prod_{j \in \Lambda} |u_j|^\gamma \exp\left(-\tfrac{1}{2} \operatorname{Var}\left(\sum_{j=1}^{n} u_j W(y_j) \right) \right) dU \right] \\ &\quad\quad \times p_{t_1,..., t_n}(y_1,..., y_n) \, dy_1 \cdots dy_n \end{aligned} \tag{2.25}$$

For any fixed $(y_1,..., y_n) \in \mathbf{R}^n \cap \Gamma_\pi$, by the independence of increments of $W(\cdot)$, we have

$$\operatorname{Var}\left(\sum_{j=1}^{n} u_j W(y_j) \right) = \sum_{j=1}^{n} v_j^2 (y_{\pi(j)} - y_{\pi(j-1)}) \tag{2.26}$$

where

$$v_j = \sum_{j=j}^{n} u_{\pi(l)}$$

It follows from (2.26) that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \prod_{j \in A} |u_j|^\gamma \exp\left(-\tfrac{1}{2}\operatorname{Var}\left(\sum_{j=1}^{n} u_j W(y_j)\right)\right) du_1 \cdots du_n \\
&\quad = \int_{\mathbf{R}^n} \prod_{j \in \pi^{-1}(A)} |v_j - v_{j+1}|^\gamma \exp\left(-\tfrac{1}{2}\sum_{j=1}^{n} v_j^2 (y_{\pi(j)} - y_{\pi(j-1)})\right) dv_1 \cdots dv_n
\end{aligned} \tag{2.27}$$

Using again the inequality $|a+b|^\gamma \leqslant |a|^\gamma + |b|^\gamma$ $(0 < \gamma < 1)$, we see that (2.27) is at most

$$\begin{aligned}
&\sum{}'' \int_{\mathbf{R}^n} \prod_{j=1}^{n} |v_j|^{\eta_j \gamma} \exp\left(-\frac{1}{2}\sum_{j=1}^{n} v_j^2 (y_{\pi(j)} - y_{\pi(j-1)})\right) dv_1 \cdots dv_n \\
&\quad \leqslant \sum{}'' \frac{K^n}{\prod_{j=1}^{n} (y_{\pi(j)} - y_{\pi(j-1)})^{(1+\eta_j\gamma)/2}}
\end{aligned} \tag{2.28}$$

by simple substitutions, where the summation $\sum''$ is taken over all $(\eta_1, ..., \eta_n) \in \{0, 1, 2\}^n$ with $\sum \eta_j = \# A$. It follows from (2.27) and (2.28) that

$$\begin{aligned}
&\int_{\mathbf{R}^n \cap \Gamma_\pi} \left[\int_{\mathbf{R}^n} \prod_{j \in A} |u_j|^\gamma \exp\left(-\frac{1}{2}\operatorname{Var}\left(\sum_{j=1}^{n} u_j W(y_j)\right)\right) du_1 \cdots du_n\right] \\
&\qquad \times p_{t_1, ..., t_n}(y_1, ..., y_n)\, dy_1 \cdots dy_n \\
&\quad \leqslant \sum{}'' K^n \int_{\mathbf{R}^n \cap \Gamma_\pi} \frac{1}{\prod_{j=1}^{n} (y_{\pi(j)} - y_{\pi(j-1)})^{(1+\eta_j\gamma)/2}}\, p_{t_1, ..., t_n}(y_1, ..., y_n)\, dy_1 \cdots dy_n \\
&\quad = \sum{}'' K^n \int_{\mathbf{R}^n \cap \Gamma_\pi} \frac{1}{(\det(R(y_1, ..., y_n)))^{(1+2\gamma)/2}} \\
&\qquad \times \prod_{j=1}^{n} (y_{\pi(j)} - y_{\pi(j-1)})^{(2-\eta_j)\gamma/2}\, p_{t_1, ..., t_n}(y_1, ..., y_n)\, dy_1 \cdots dy_n
\end{aligned} \tag{2.29}$$

We make the substitution $y_j - y_{j-1} = (t_j - t_{j-1})^{1/2} z_j$ $(j = 1,..., n)$ in (2.29) and notice that

$$\prod_{j=1}^{n} (y_{\pi(j)} - y_{\pi(j-1)})^{(2-\eta_j)\gamma/2} \leqslant \prod_{j=1}^{n} \left(\sum_{l=1}^{n} |z_l|^{(2-\eta_j)\gamma/2} \right)$$
$$\leqslant \sum{}''' \prod_{j=1}^{n} |z_{\delta_j}|^{(2-\eta_j)\gamma/2} \tag{2.30}$$

where the summation $\sum'''$ is taken over all $(\delta_1,..., \delta_n) \in \{1,..., n\}^n$. For any such a fixed sequence $(\delta_1,..., \delta_n)$, we can write

$$\prod_{j=1}^{n} |z_{\delta_j}|^{(2-\eta_j)\gamma/2} = \prod_{j=1}^{n} |z_j|^{\tau_j} \tag{2.31}$$

with $\tau_j \geqslant 0$ and

$$\sum_{j=1}^{n} \tau_j = \sum_{j=1}^{n} \frac{(2-\eta_j)\gamma}{2} = \left(n - \frac{\# \Lambda}{2} \right) \gamma \tag{2.32}$$

By (2.30), we have that each summand in (2.29) is at most

$$K^n \sum{}''' \int_{\mathbf{R}^n} \frac{1}{(\det(R(y_1,..., y_n)))^{(1+2\gamma)/2}} \prod_{j=1}^{n} |z_{\delta_j}|^{(2-\eta_j)\gamma/2}$$
$$\times \exp\left(-\frac{1}{2} \sum_{j=1}^{n} z_j^2 \right) dz_1 \cdots dz_n$$
$$= \frac{K^n}{\prod_{j=1}^{n} (t_j - t_{j-1})^{(1+2\gamma)/4}}$$
$$\times \sum{}''' \int_{\mathbf{R}^n} \frac{1}{\prod_{j=1}^{n} \min\{|z_j - x_i|^{(1+2\gamma)/2}, i = 0, 1,..., j-1\}}$$
$$\times \prod_{j=1}^{n} |z_{\delta_j}|^{(2-\eta_j)\gamma/2} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} z_j^2 \right) dz_1 \cdots dz_n \tag{2.33}$$

It follows from (2.31), (2.32) and Lemma 1 that the integral in the summand of (2.33) is at most

$$\int_{\mathbf{R}^n} \prod_{j=1}^{n} \frac{|z_j|^{\tau_j}}{\min\{|z_j - x_i|^{(1+2\gamma)/2}, i = 0, 1,..., j-1\}} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} z_j^2 \right) dz_1 \cdots dz_n$$
$$\leqslant K^n (n!)^{(1+2\gamma)/2} \prod_{j=1}^{n} 2^{\tau_j/2} \Gamma\left(\frac{\tau_j + 1}{2} \right) \leqslant K^n (n!)^{1+2\gamma} \tag{2.34}$$

Combining (2.23)–(2.34), we have

$$J \leqslant \frac{K^n}{\prod_{j=1}^n (t_j - t_{j-1})^{(1+2\gamma)\,d/4}} (n!)^{(3+2\gamma)\,d} \tag{2.35}$$

It follows from (2.20), (2.21), (2.35) and Lemma 3 that for $0 < \gamma < 1/6$

$$\begin{aligned} E[L(x+w, B) - L(x, B)]^n &\leqslant K^n \, |w|^{n\gamma} \, h^{n(1-(d+\gamma)/4)} (n!)^{(3+2\gamma)\,d} \\ &\quad \times \sum_{\pi \in S(n)} \int_{[0,1]^n \cap \Gamma_\pi} \frac{1}{\prod_{j=1}^n (t_j - t_{j-1})^{(1+2\gamma)\,d/2}} \, d\bar{t} \\ &\leqslant K^n \, |w|^{n\gamma} \, h^{n(1-(d+\gamma)/4)} (n!)^{4d} \end{aligned}$$

This proves (2.12).

Remark 1. We believe a better inequality than (2.12) with $(n!)^{4d}$ replaced by $(n!)^{3(d+\gamma)/4}$ hold. However, (2.12) is enough for our purpose.

Since $X(t)$ has stationary increments, the above arguments also prove Lemma 5.

Lemma 5. For any $\tau \geqslant 0$, let $B = (\tau, \tau + h]$ with $h > 0$. Then for any $x, w \in \mathbf{R}^d$, any even integer $n \geqslant 2$ and any $0 < \gamma < 1/6$

$$E[L(x + X(\tau), B)]^n \leqslant K^n h^{(1-d/4)\,n} (n!)^{3d/4} \tag{2.36}$$

$$E[L(x + w + X(\tau), B) - L(x + X(\tau), B)]^n \leqslant K^n \, |w|^{n\gamma} \, h^{(1-(d+\gamma)/4)\,n} (n!)^{4d} \tag{2.37}$$

The Lemma 6 is a consequence of Lemma 5 and Chebyshev's inequality.

Lemma 6. With the notations of Lemma 5, for any $\lambda > 0$, there exists a finite constant $A > 0$, depending on λ and d only, such that for any $u > 0$

$$P\{L(x + X(\tau), B) \geqslant A h^{1-d/4} u^{3d/4}\} \leqslant \exp(-\lambda u) \tag{2.38}$$

$$P\{|L(x + w + X(\tau), B) - L(x + X(\tau), B)| \geqslant A \, |w|^\gamma \, h^{1-(d+\gamma)/4} u^{4d}\} \leqslant \exp(-\lambda u) \tag{2.39}$$

In particular, (2.38) and (2.39) hold for $\tau = 0$.

Proof. We only prove (2.38); inequality (2.39) can be proved in the same way. Let

$$A = \frac{L(x + X(\tau), B)}{h^{1-d/4}}$$

By (2.36), Jensen's inequality and Stirling's formula, we have

$$E(A^{4/(3d)})^n \leqslant K^n n! \tag{2.40}$$

Then (2.38) follows from (2.40) and Chebyshev's inequality as in the proof of Lemma 3.14 in Geman, *et al.*[17] □

Now we prove the main results of this section.

Theorem 1. If $d < 4$, then almost surely $X(t)$ $(t \geqslant 0)$ has a jointly continuous local time $L(x, t)$ $(x \in \mathbf{R}^d, t \geqslant 0)$ and for any $B \in \mathscr{B}(\mathbf{R}_+)$, $A \in \mathscr{B}(\mathbf{R}^d)$

$$\mu_B(A) = \int_A L(x, B)\, dx \tag{2.41}$$

Proof. It follows from Lemma 4 that for any $0 < \gamma < 1/6$ and any even integer $n \geqslant 2$

$$E[L(x, [0, s]) - L(y, [0, t])]^n \leqslant K^n\, |(x, s) - (y, t)|^{n\gamma} \tag{2.42}$$

Then the joint continuity of $L(x, t)$ follows immediately from (2.42) and a continuity lemma of Garsia.(15) See also Geman *et al.*[17] for a more general result.

To prove Theorem 2, we will need Lemma 7, which can be derived from LIL of Brownian motion (see Burdzy[7]).

Lemma 7. Let $Y(t)$ $(t \geqslant 0)$ be iterated Brownian motion in $\mathbf{R}$. For any $\tau \in \mathbf{R}$, with probability 1,

$$\limsup_{h \to 0} \sup_{\tau \leqslant t \leqslant \tau + h} \frac{|Y(t) - Y(\tau)|}{h^{1/4}(\log\log 1/h)^{3/4}} \leqslant K$$

Theorem 2. If $d < 4$, then there exits a positive and finite constant K such that for any $\tau \geqslant 0$ with probability 1

$$\limsup_{h \to 0} \sup_{x \in \mathbf{R}^d} \frac{L(x, \tau + h) - L(x, \tau)}{h^{1-d/4}(\log\log 1/h)^{3d/4}} \leqslant K \tag{2.43}$$

Proof. The proof is similar to the proof of Theorem 1.1 in Xiao[35] [see also Geman *et al.*[17] and Ehm[13]]. For any fixed $\tau \geqslant 0$, let $B_n = [\tau, \tau + 2^{-n}]$ $(n = 1, 2,...)$. It follows from Lemma 7 that almost surely there exists $n_1 = n_1(\omega)$ such that

$$\sup_{t \in B_n} |X(t) - X(\tau)| \leqslant K(2^{-n})^{1/4} (\log\log 2^n)^{3/4} \qquad \text{for} \quad n \geqslant n_1 \tag{2.44}$$

Fix a γ with $0 < \gamma < 1/6$ and for any integer $n \geqslant 1$, let

$$\theta_n = (2^{-n})^{1/4} / (\log\log 2^n)^{13d/(4\gamma)}$$

and let

$$G_n = \{x \in \mathbf{R}^d \colon |x| \leqslant K(2^{-n})^{1/4} (\log\log 2^n)^{3/4}, x = \theta_n p \text{ for some } p \in \mathbf{Z}^d\}$$

where $\mathbf{Z}^d$ is the integer lattice in $\mathbf{R}^d$. The cardinality of G_n satisfies

$$\# G_n \leqslant K(\log n)^\alpha$$

at least when n is large enough, where $\alpha > 0$ is a constant depending on γ and d only. Denote

$$\phi(s) = s^{1-d/4} (\log\log 1/s)^{3d/4}$$

It follows from Lemma 6 that for some constant $A > 1$,

$$\begin{aligned} &P\{L(x + X(\tau), B_n) \geqslant A\phi_1(2^{-n}) \text{ for some } x \in G_n\} \\ &\quad \leqslant K(\log n)^\alpha \exp(-2\log n) \\ &\quad = K(\log n)^\alpha n^{-2} \end{aligned}$$

Hence by the Borel–Cantelli lemma there is $n_2 = n_2(\omega)$ such that almost surely

$$\sup_{x \in G_n} L(x + X(\tau), B_n) \leqslant A\phi_1(2^{-n}) \qquad \text{for} \quad n \geqslant n_2 \tag{2.45}$$

For any fixed integers n with $n^2 > 2^d$, $m \geqslant 1$ and any $x \in G_n$, define

$$F(n, m, x) = \left\{ y \in \mathbf{R}^d \colon y = x + \theta_n \sum_{j=1}^m \varepsilon_j 2^{-j} \text{ for } \varepsilon_j \in \{0, 1\}^d \right\}$$

A pair of points y_1, $y_2 \in F(n, m, x)$ is said to be linked if $y_2 - y_1 = \theta_n \varepsilon 2^{-m}$ for some $\varepsilon \in \{0, 1\}^d$. Then by (2.39) we have

$$
\begin{aligned}
P\{ &|L(y_1 + X(\tau), B_n) - L(y_2 + X(\tau), B_n)| \\
&\quad \geqslant A\,|y_1 - y_2|^{\gamma}\, 2^{-n(1-(d+\gamma)/4)} (m \log\log 2^n)^{4d} \text{ for some } x \in G_n,\ m \geqslant 1 \\
&\quad \text{and some linked pair } y_1,\ y_2 \in F(n, m, x)\} \\
&\leqslant \# G_n \sum_{m=1}^{\infty} 2^{md} \exp(-2m \log n) \\
&\leqslant K(\log n)^{\alpha} \frac{2^d/n^2}{1 - 2^d/n^2}
\end{aligned}
$$

Since

$$
\sum_{n=[2^{d/2}]+1}^{\infty} (\log n)^{\alpha} \frac{2^d/n^2}{1 - 2^d/n^2} < \infty
$$

there exists $n_3 = n_3(\omega)$ such that for almost surely for $n \geqslant n_3$

$$
\begin{aligned}
&|L(y_1 + X(\tau), B_n) - L(y_2 + X(\tau), B_n)| \\
&\qquad \leqslant A\,|y_1 - y_2|^{\gamma}\, 2^{-n(1-(d+\gamma)/4)} (m \log\log 2^n)^{4d}
\end{aligned} \tag{2.46}
$$

for all $x \in G_n$, $m \geqslant 1$ and any linked pair y_1, $y_2 \in F(n, m, x)$. Let Ω_0 be the event that (2.44)–(2.46) hold eventually. Then $P(\Omega_0) = 1$. Fix an $n \geqslant n_4 = \max\{n_1, n_2, n_3\}$ and any $y \in \mathbf{R}^d$ with $|y| \leqslant K 2^{-n/4} (\log\log 2^n)^{3/4}$. We represent y in the form $y = \lim_{m \to \infty} y_m$, where

$$
y_m = x + \theta_n \sum_{j=1}^{m} \varepsilon_j 2^{-j} \qquad (y_0 = x,\ \varepsilon_j \in \{0, 1\}^d)
$$

for some $x \in G_n$. Then each pair y_{m-1}, y_m is linked, so by (2.46) and the continuity of $L(\cdot, B_n)$ we have

$$
\begin{aligned}
&|L(y + X(\tau), B_n) - L(x + X(\tau), B_n)| \\
&\qquad \leqslant \sum_{m=1}^{\infty} A\,|\theta_n 2^{-m}|^{\gamma}\, 2^{-n(1-(d+\gamma)/4)} (m \log\log 2^n)^{4d} \\
&\qquad \leqslant K 2^{-n(1-(d+\gamma)/4)} (\log\log 2^n)^{4d}\, \theta_n^{\gamma} \sum_{m=1}^{\infty} 2^{-m\gamma} m^{4d} \\
&\qquad = K\phi_1(2^{-n})
\end{aligned} \tag{2.47}
$$

It follows from (2.45) and (2.47) that almost surely for $n \geqslant n_4$

$$L(y + X(\tau), B_n) \leqslant K\phi_1(2^{-n})$$

for any $y \in \mathbf{R}^d$ with $|y| \leqslant K2^{-n/4}(\log\log 2^n)^{3/4}$. Therefore

$$\sup_{x \in \mathbf{R}^d} L(x, B_n) = \sup_{x \in \overline{X(B_n)}} L(x, B_n) \leqslant K\phi_1(2^{-n}) \tag{2.48}$$

Finally for any $r > 0$ small enough, there exists $n \geqslant n_4$ such that $2^{-n-1} \leqslant r < 2^{-n}$. Hence by (2.48) we have

$$\sup_{x \in \mathbf{R}^d} L(x, B(\tau, r)) \leqslant K\phi_1(r)$$

This completes the proof of Theorem 2. □

Remark 1. If $d = 1$, Csáki *et al.*[11] [Thm. 4.1] proved that almost surely

$$\limsup_{h \to 0} \sup_{x \in \mathbf{R}^d} \frac{L(x, \tau + h) - L(x, \tau)}{h^{1-d/4}(\log\log 1/h)^{3d/4}} \geqslant K$$

We believe this inequality also holds for $1 < d < 4$, but we have not been able to prove it.

Theorem 3. If $d < 4$, then for any $T > 0$, there exists a positive finite constant K such that almost surely

$$\limsup_{h \to 0} \sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbf{R}^d} \frac{L(x, t + h) - L(x, t)}{h^{1-d/4}(\log 1/h)^{3d/4}} \leqslant K \tag{2.49}$$

Proof. The proof, using Lemma 6, is very similar to that of Xiao[35] [Thm. 1.2]. Hence we will omit the details.

Remark 2. If $d = 1$, (2.49) becomes

$$\limsup_{h \to 0} \sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbf{R}^d} \frac{L(x, t + h) - L(x, t)}{h^{3/4}(\log 1/h)^{3/4}} \leqslant K$$

This improves Proposition 2.4 in Burdzy and Khoshnevisan[9] and Theorem 3.1 in Csáki *et al.*[11] as mentioned in (1.2).

The Hölder conditions for the local time of a stochastic process $X(t)$ are closely related to the irregularity of the sample paths of $X(t)$ (cf. Berman[4]). In the following, we will apply Theorems 2 and 3 to derive results about the degree of oscillation of the sample paths and the Hausdorff measure of the graph set of $X(t)$.

Theorem 4. Let $X(t)(t \in \mathbf{R}_+)$ be iterated Brownian motion in $\mathbf{R}^d$ with $d < 4$. For any $\tau \in \mathbf{R}_+$, there is a finite constant $K > 0$ such that

$$\liminf_{r \to 0} \sup_{s \in B(\tau, r)} \frac{|X(s) - X(\tau)|}{r^{1/4}/(\log \log 1/r)^{3/4}} \geqslant K \qquad \text{a.s.} \tag{2.50}$$

For any interval $T \subseteq \mathbf{R}_+$

$$\liminf_{r \to 0} \inf_{t \in T} \sup_{s \in B(t, r)} \frac{|X(s) - X(t)|}{r^{1/4}/(\log 1/r)^{3/4}} \geqslant K \qquad \text{a.s.} \tag{2.51}$$

In particular, $X(t)$ is almost surely nowhere differentiable in $\mathbf{R}_+$.

Proof. For any interval $Q \subseteq \mathbf{R}_+$,

$$\begin{aligned} \lambda_1(Q) &= \int_{\overline{X(Q)}} L(x, Q)\, dx \\ &\leqslant L^*(Q) \cdot \big(\sup_{s, t \in Q} |X(s) - X(t)|\big)^d \end{aligned} \tag{2.52}$$

Let $Q = B(\tau, r)$. Then (2.50) follows immediately from (2.43) and (2.52). Similarly, (2.51) follows from (2.49) and (2.52). □

Remark 3. Theorem 4 recovers partially the results obtained by Khoshnevisan and Lewis[26]; Hu *et al.*[18]; and Hu and Shi,[19] respectively.

Now let

$$X([0, 1]) = \{X(t) \colon t \in [0, 1]\}$$

and

$$GrX([0, 1]) = \{(t, X(t)) \colon t \in [0, 1]\}$$

be the image and graph set of iterated Brownian motion $X(t)$ in $\mathbf{R}^d$. It follows from standard methods that with probability 1

$$\dim X([0, 1]) = \min\{d, 4\} \tag{2.53}$$

$$\dim GrX([0, 1]) = \min\{4, 1 + 3d/4\} \tag{2.54}$$

It would be interesting to study the exact Hausdorff measure of the image and graph of iterated Brownian motion in $\mathbf{R}^d$. We end this section by presenting a result on the lower bound of the Hausdorff measure of the graph as an application of Theorem 2.

Theorem 5. Let $X(t)$ $(t \in \mathbf{R}_+)$ be iterated Brownian motion in $\mathbf{R}^d$ with $d < 4$. Then almost surely

$$\phi_3\text{-}m(GrX([0, 1])) \geqslant K \tag{2.55}$$

where ϕ_3-m is the ϕ_3-Hausdorff measure, K is a positive finite constant depending on d only and

$$\phi_3(r) = r^{1+3d/4}(\log\log 1/r)^{3d/4}$$

Proof. We define a random Borel measure μ in $GrX([0, 1]) \subseteq \mathbf{R}^{1+d}$ by

$$\mu(B) = \lambda_1\{t \in [0, 1]: (t, X(t)) \in B\} \qquad \text{for any} \quad B \subseteq \mathbf{R}^{1+d}$$

Then $\mu(\mathbf{R}^{1+d}) = \mu(GrX([0, 1])) = 1$. It follows from Theorem 2 that for any fixed $t_0 \in [0, 1]$ almost surely

$$\mu(B((t_0, X(t_0)),)) \leqslant \int_{B(X(t_0), r)} L(x, B(t_0, r))\, dx \leqslant K\phi_3(r) \tag{2.56}$$

By Fubini's theorem, we see that (2.56) holds almost surely for λ_1 *a.e.* $t_0 \in [0, 1]^N$. Then the lower bound in (2.55) follows from (2.56) and an upper density theorem for Hausdorff measure due to Rogers and Taylor.[31]

3. UNIFORM DIMENSION RESULTS

It follows from a capacity argument and the following Lemma 8 that for any Borel set $E \subseteq \mathbf{R}_+$ almost surely

$$\dim X(E) = \min\{d, 4 \dim E\} \tag{3.1}$$

The exceptional null set in (3.1) depends on E. In the following, we will prove a uniform Hausdorff dimension result: if $d \geqslant 4$, then outside a single null probability set, (3.1) holds simultaneously for every Borel set $E \subseteq \mathbf{R}_+$. This is not true for $d < 4$. The Hausdorff dimension of the inverse image $X^{-1}(F)$ for any Borel set $F \subseteq \mathbf{R}^d$ will also be considered. In particular, we will generalize (1.3) to d-dimensional iterated Brownian motion. Hausdorff dimension results of this nature for Brownian motion were proved by Kaufman[21, 22]; see also Perkins and Taylor[29]; for locally nondeterministic Gaussian random fields by Monrad and Pitt.[27] We will follow the same line.

We need several lemmas. Lemma 8 is a direct consequence of a result of Khoshnevisan and Lewis[25] and Lemma 9 is from Monrad and Pitt.[27]

Lemma 8. Let $Y(t)$ be iterated Brownian motion in $\mathbf{R}$, then with probability 1

$$\lim_{h \to 0} \sup_{0 \leqslant t \leqslant 1} \sup_{0 \leqslant s \leqslant h} \frac{|Y(t+s) - Y(t)|}{h^{1/4}(\log 1/h)^{3/4}} = 1 \tag{3.1}$$

Lemma 9. If for any $\varepsilon > 0$, $X(t)\colon [0, 1] \to \mathbf{R}^d$ satisfies a uniform Hölder condition of order $\alpha - \varepsilon$, then

$$\dim X(E) \leqslant \frac{1}{\alpha} \dim E \tag{3.2}$$

If, in addition, $X(t)$ also has a bounded local time $L(x, [0, 1])$, then for every closed set $F \subseteq \mathbf{R}^d$,

$$\dim X^{-1}(F) \leqslant 1 - \frac{d}{4} + \frac{1}{4} \dim F \tag{3.3}$$

For $n = 1, 2,...$ and $k \in \{1, 2,..., 2^n\}$, let $I_{nk} = [(k-1)\, 2^{-n}, k 2^{-n}]$.

Lemma 10. If $d \geqslant 4$, then almost surely for n large enough and for any ball D in $\mathbf{R}^d$ of radius $2^{-n/4} n^{3/4}$, $X^{-1}(D)$ can intersect at most n^{2d+2} intervals I_{nk}.

By Lemma 8, we see that Lemma 10 is a corollary of Lemma 11.

Lemma 11. If $d \geqslant 4$, then almost surely for n large enough, any ball D of radius $2^{1-n/4} n^{3/4}$ in $\mathbf{R}^d$ can contain at most n^{2d+2} points $X(k 2^{-n})$.

Proof. Let A_n be the event that there exists a ball D of radius $2^{1-n/4} n^{3/4}$ in $\mathbf{R}^d$ such that it contains at least n^{2d+2} points $X(k 2^{-n})$. Consider n distinct points $t_j = k_j 2^{-n}$ ($k_j \in \{1,..., 2^n\}$) satisfying

$$t_1 < t_2 < \cdots < t_n$$

and $X(t_j) \in D$ for $j = 1,..., n$. Denote by N_n the number of such n-tuples $(t_1,..., t_n)$.

Since $X_1,..., X_d$ are independent copies of $W(B(\cdot))$, we have

$$P\{X(t_j)\in D,\ j=1,..., n\} \leqslant \prod_{l=1}^{d} P\{W(B(t_j))\in D_l,\ j=1,..., n\} \tag{3.4}$$

where D_l is the orthogonal projection of D in the lth axis. For each fixed $1\leqslant l\leqslant d$, similar to (2.15) and (2.19) we have

$$\begin{aligned}
&P\{W(B(t_j))\in D_l,\ j=1,..., n\}\\
&\quad=\int_{\mathbf{R}^n} P\{W(y_j)\in D_l,\ j=1,..., n\}\cdot p_{t_1,..., t_n}(y_1,..., y_n)\, dy_1\cdots dy_n\\
&\quad\leqslant \sum_{\pi\in S(n)}\int_{\mathbf{R}^n\cap\Gamma_\pi}\prod_{j=1}^{n}\frac{2^{2-n/4}n^{3/4}}{(y_{\pi(j)}-y_{\pi(j-1)})^{1/2}}\cdot p_{t_1,..., t_n}(y_1,..., y_n)\, dy_1\cdots dy_n\\
&\quad\leqslant (2^{2-n/4}n^{3/4})^n\int_{\mathbf{R}^n}\frac{1}{\det(R(y_1,..., y_n))^{1/2}}\exp\left(-\frac{1}{2}\sum_{j=1}^{n} z_j^2\right) dz_1\cdots dz_n\\
&\quad\leqslant K^n(2^{2-n/4}n^{3/4})^n\,(n!)^{1/2}\prod_{j=1}^{n}\frac{1}{(t_j-t_{j-1})^{1/4}}
\end{aligned}\tag{3.5}$$

where the last inequality follows from Lemma 1. It follows from (3.4) and (3.5) that

$$P\{X(t_j)\in D,\ j=1,..., n\}\leqslant K^n n^{3nd/4}(n!)\prod_{j=1}^{n}\frac{1}{(k_j-k_{j-1})^{d/4}}$$

Therefore,

$$\begin{aligned}
E(N_n)&=K^n n^{3nd/4}(n!)^{d/2}\sum_{k_1<k_2<\cdots<k_n\leqslant 2^n}\frac{1}{\prod_{j=1}^{n}(k_j-k_{j-1})^{d/4}}\\
&\leqslant\begin{cases}K^n n^{3nd/4}(n!)^{d/2} & \text{if}\quad d>4\\ K^n n^{3n}(n!)^2\, n^n & \text{if}\quad d=4\end{cases}
\end{aligned}$$

We can simply write for $d\geqslant 4$

$$E(N_n)\leqslant K^n n^{2nd} \tag{3.6}$$

On the other hand, if the event A_n occurs, then for n large enough

$$N_n\geqslant\binom{n^{2d+2}}{n}\geqslant n^{(2d+1)n}e^{n/2} \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$P(A_n) \leqslant P(N_n \geqslant n^{(2d+1)n} e^{n/2}) \leqslant K^n n^{-n}$$

Since $\sum_n P(A_n) < \infty$, the proof of Lemma 11 is completed by the Borel–Cantelli lemma. □

Now we prove the main theorem of this section.

Theorem 6. Let $X(t)$ be iterated Brownian motion in $\mathbf{R}^d$ with $d \geqslant 4$, then with probability 1,

$$\dim X(E) = 4 \dim E \qquad \text{for every Borel set} \quad E \subseteq \mathbf{R}_+ \tag{3.8}$$

Proof. It suffices to prove (3.8) for $E \subseteq [0, 1]$. Then the upper bound in (3.8) follows from Lemmas 8 and (3.2). The lower bound follows from Lemma 10 in a standard way. □

Theorem 7. Let $X(t)$ be iterated Brownian motion in $\mathbf{R}^d$ with $d < 4$, then with probability 1, for every closed set $F \subseteq \mathbf{R}^d \backslash \{0\}$

$$\dim X^{-1}(F) = 1 - \frac{d}{4} + \frac{1}{4} \dim F \tag{3.9}$$

In particular, for every $x \in \mathbf{R}^d \backslash \{0\}$

$$\dim X^{-1}(x) = 1 - \frac{d}{4}$$

Proof. The upper bound in (3.9) follows from Lemma 8, the continuity of the local time and (3.3). The proof of the lower bound

$$\dim X^{-1}(F) \geqslant 1 - \frac{d}{4} + \frac{1}{4} \dim F$$

for every closed set $F \subseteq O = \cup_{[s,t]} \{x\colon L(x, [s, t] > 0\}$, is the same as that of Theorem 1 in Monrad and Pitt,[21] using Theorem 3. The open set O is almost surely nonempty. It follows from the self-similarly of $X(t)$ and (1.5) that $O = \mathbf{R}^d \backslash \{0\}$.

Remark 4. The packing dimension [see Taylor and Tricot[34]] of the image and inverse image of $X(t)$ can also be discussed. Results similar to (3.8) and (3.9) also hold.

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