Harmonizable Fractional Stable Fields: Local Nondeterminism and Joint Continuity of the Local Times

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Abstract

By applying a Fourier analytic argument, we prove that, for every $\alpha \in (0,2)$, the N-parameter harmonizable fractional α-stable field (HFαSF) is locally nondeterministic. When $0 < \alpha < 1$, this solves an open problem in [15]. Also, it allows us to establish the joint continuity of the local times of an (N,d)-HFαSF for an arbitrary $\alpha \in (0,2)$, and to obtain new results concerning its sample paths.

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1 Introduction

For any given $\alpha \in (0,2)$ and $H \in (0,1)$, let $X = \{X(t), t \in \mathbb{R}^N\}$ be the real-valued harmonizable fractional α-stable field (HFαSF or HFSF, for brevity) with Hurst index H, defined by:

$$X(t) := \kappa \,\mathrm{Re} \int_{\mathbb{R}^N} \frac{e^{it\cdot\xi} - 1}{|\xi|^{H+N/\alpha}} \,\widetilde{M}_\alpha(d\xi), \tag{1.1}$$

where, $t \cdot \xi$ denotes the usual inner product of t and ξ, $|\xi|$ the Euclidian norm of ξ, κ is the positive normalizing constant given by

$$\kappa := 2^{-1/2}\left(\int_{\mathbb{R}^N} \frac{\big(1-\cos\xi_1\big)^{\alpha/2}}{|\xi|^{\alpha H+N}}\, d\xi\right)^{-1/\alpha}, \tag{1.2}$$

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and $\widetilde{M}_\alpha$ a complex-valued rotationally invariant α-stable random measure with Lebesgue control measure. We refer to [18, Chapter 6] for a detailed presentation on such random measures as well as the corresponding stochastic integrals.

For any real-valued α-stable random variable Y, denote by $\|Y\|_\alpha$ its scale parameter. By the choice of κ in (1.2) we have

$$\|X(t)\|_\alpha = |t|^H, \qquad \forall\, t \in \mathbb{R}^N. \tag{1.3}$$

By using (1.1), (2.3) and (2.5) in Section 2, one can verify that the stable field X is H-self-similar and has stationary and isotropic increments. Hence X is an α-stable analogue of fractional Brownian motion (another one is the linear fractional stable field whose properties are different from HFSF).

Several authors have studied various sample path properties of HFSF and its local times, as well as those of stable stochastic fields/processes related to it. We mention the pioneering work of Nolan [15] who established, for $1 \le \alpha < 2$, the joint continuity of the local times of a d-dimensional harmonizable fractional α-stable field (its components are i.i.d. real-valued HFSF, see (4.1)). Kôno and Shieh [12] and Shieh [19] studied existence and joint continuity of the intersection local times of stable processes including the harmonizable fractional one. Recently, Xiao [24] studied several classes of anisotropic stable random fields connected with HFSF, and further extended the results in [15]. The keystone of the aforementioned articles is the concept of *local nondeterminism (LND)* of Nolan [15], which is an extension, to the frame of stable fields, of the local nondeterminism of Berman [4] for Gaussian processes and Pitt [17] for Gaussian random fields. Roughly speaking, the concept of LND provides a way to characterize the dependence structure of the stable random variables $X(s^1), \ldots, X(s^r)$, provided $s^1, \ldots, s^r \in \mathbb{R}^N$ are close enough. See Section 2 for the definition of LND.

When $1 \le \alpha < 2$, Nolan [15] proved that the HFSF X in (1.1) has the property of local nondeterminism and pointed [15, p. 406] that his method can not be extended to the case of $0 < \alpha < 1$. One of the main difficulties in this latter case is that the classical Hölder inequality: $\left|\int fg\right| \le \left(\int |f|^\alpha\right)^{1/\alpha}\left(\int |g|^{\alpha_*}\right)^{1/\alpha_*}$, $1/\alpha + 1/\alpha_* = 1$, fails.

The problem of proving that HFSF X is locally nondeterministic for the case of $0 < \alpha < 1$ had remained open. The main objective of the present article is to resolve this problem. Our method is based on Fourier analytic arguments.

The rest of the paper is organized as follows. In Section 2, we recall the definition of the local nondeterminism of Nolan [15] and state our main results, Theorems 2.2 and 2.4. Section 3 is devoted to the proof of Theorem 2.2. The new idea is to bound the $L^\alpha(\mathbb{R}^N)$ (quasi) norm related to LND from below by the $L^2(\mathbb{R}^N)$ norm of another function that is constructed by using an appropriate Fourier transform [see (3.14) and (3.18) for details]. In Section 4, we apply the property of local nondeterminism to prove the joint continuity of the local times of an (N, d)-HFSF $\vec{X}$ and to study its fractal properties.

We end the introduction with some notation. For any integer $p \geq 1$, a point (or vector) $t \in \mathbb{R}^p$ is written in terms of its coordinates as $(t_1, \ldots, t_p)$, or as $\langle c \rangle$, if $t_1 = \cdots = t_p = c$. For any $s,\, t \in \mathbb{R}^p$ such that $s_j < t_j$ $(j = 1, \ldots, p)$, we define the closed interval (or rectangle) $[s,\, t] = \prod_{j=1}^{p} [s_j,\, t_j]$. The Lebesgue measure in $\mathbb{R}^p$ is often denoted by λ_p.

2 Property of Local Nondeterminism

Nolan [15] defined the concept of local nondeterminism for stable random fields in a much more general setting than that of HFSF. Namely he considers an arbitrary real-valued stable random field $Y = \{Y(t), t \in \mathbb{R}^N\}$ having a stochastic integral representation of the form:

$$Y(t) = \mathrm{Re} \int_{\mathbb{R}^N} g(t, \xi)\, \widetilde{S}_\alpha(d\xi), \tag{2.1}$$

where $\widetilde{S}_\alpha$ is a complex-valued rotationally invariant α-stable random measure on $\mathbb{R}^N$ with control measure Δ, and $g(t, \cdot) : \mathbb{R}^N \to \mathbb{C}$ $(t \in \mathbb{R}^N)$ is a family of deterministic complex-valued measurable functions on $\mathbb{R}^N$ belonging to the space $L^\alpha(\mathbb{R}^N, \Delta)$, that is:

$$\int_{\mathbb{R}^N} |g(t, \xi)|^\alpha\, \Delta(d\xi) < \infty, \qquad \forall\ t \in \mathbb{R}^N. \tag{2.2}$$

It follows from [18, Proposition 6.2.1] that, under condition (2.2), the stochastic integral in (2.1) is well-defined and, moreover, for any integer $m \geq 0$ and $t^0, \ldots, t^m \in \mathbb{R}^N$, the characteristic function of the joint distribution of $Y(t^0), \ldots, Y(t^m)$ is given by:

$$\mathbb{E} \exp\Big(i \sum_{n=0}^{m} b_n Y(t^n) \Big) = \exp\Big(- \Big\| \sum_{n=0}^{m} b_n g(t^n, \cdot) \Big\|_{L^\alpha(\mathbb{R}^N, \Delta)}^\alpha \Big), \tag{2.3}$$

where the real numbers b_n $(0 \leq n \leq m)$ are arbitrary, and $\| \cdot \|_{L^\alpha(\mathbb{R}^N, \Delta)}$ is the usual (quasi) norm on $L^\alpha(\mathbb{R}^N, \Delta)$, defined by

$$\|f\|_{L^\alpha(\mathbb{R}^N, \Delta)}^\alpha := \int_{\mathbb{R}^N} |f(\xi)|^\alpha\, \Delta(d\xi), \qquad \forall\ f \in L^\alpha(\mathbb{R}^N, \Delta). \tag{2.4}$$

A straightforward consequence of (2.3) is that the scale parameter of the real-valued symmetric α-stable random variable $\sum_{n=0}^{m} b_n Y(t^n)$ is given by

$$\Big\| \sum_{n=0}^{m} b_n Y(t^n) \Big\|_\alpha^\alpha = \Big\| \sum_{n=0}^{m} b_n g(t^n, \cdot) \Big\|_{L^\alpha(\mathbb{R}^N, \Delta)}^\alpha = \int_{\mathbb{R}^N} \Big| \sum_{n=0}^{m} b_n g(t^n, \xi) \Big|^\alpha \Delta(d\xi). \tag{2.5}$$

Now let us turn to the definition of local nondeterminism of the field Y. To this end we need to introduce some additional notations. For any integer $m \geq 1$ and $t^0, t^1, \ldots, t^m \in \mathbb{R}^N$, let $M^m := M(t^1, \ldots, t^m)$ be the subspace of $L^\alpha(\mathbb{R}^N, \Delta)$ spanned by the set of functions

$\{g(t^1,\cdot),\ldots,g(t^m,\cdot)\}$, and denote by $\big\|g(t^0,\cdot)\big|M^m\big\|_\alpha$ the $L^\alpha(\mathbb{R}^N,\Delta)$-distance from $g(t^0,\cdot)$ to M^m. That is,

$$\big\|g(t^0,\cdot)\big|M^m\big\|_\alpha = \inf\left\{\Big\|g(t^0,\cdot)-\sum_{n=1}^m b_n g(t^n,\cdot)\Big\|_{L^\alpha(\mathbb{R}^N,\Delta)} :\ \forall\, b_1,\,\ldots,\,b_m\in\mathbb{R}\right\}. \tag{2.6}$$

Since M^m has finite dimension, the infimum in (2.6) is attained. In order to draw analogy with the Gaussian case, we abuse the notation and, from now on, write that for all $t^1,\ldots,t^m\in\mathbb{R}^N$

$$\big\|Y(t^0)\big|Y(t^1),\ldots,Y(t^m)\big\|_\alpha := \big\|g(t^0,\cdot)\big|M^m\big\|_\alpha. \tag{2.7}$$

It can be viewed as the L^α-error of predicting $Y(t^0)$, given $Y(t^1),\ldots,Y(t^m)$.

For any integer $r\ge 2$ and points $s^1,\ldots,s^r\in\mathbb{R}^N$, the notation $s^1\preccurlyeq s^2\preccurlyeq\cdots\preccurlyeq s^r$ means that:

$$|s^j-s^{j-1}|\le|s^j-s^i| \quad \text{for all}\ \ 1\le i<j\le r. \tag{2.8}$$

Note that the partial order defined by (2.8) is not unique. For any r points in $\mathbb{R}^N$ (including the case $N=1$), there are at least r different ways to order them using (2.8). For example, one can pick any point and label it as s^r, then label the one which is the closest to s^r as s^{r-1}, and so on.

The following definition of local nondeterminism is from Nolan [15].

Definition 2.1 *Let $Y=\{Y(t),t\in\mathbb{R}^N\}$ be a real-valued α-stable random field with representation (2.1) and let $I\subset\mathbb{R}^N$ be a closed interval. Then Y is said to be locally nondeterministic on I if*

$$\big\|Y(t)\big\|_\alpha>0 \quad \forall\ t\in I \quad \textit{and} \quad \big\|Y(s)-Y(t)\big\|_\alpha>0 \tag{2.9}$$

for all $s,t\in I$, $s\ne t$ with $|s-t|$ sufficiently small, and for every integer $r\ge 2$

$$\liminf \frac{\big\|Y(s^r)\big|Y(s^1),\ldots,Y(s^{r-1})\big\|_\alpha}{\|Y(s^r)-Y(s^{r-1})\|_\alpha}>0, \tag{2.10}$$

where the $\liminf$ *is taken over all the r points $s^1,\ldots,s^r\in I$ that satisfy $s^1\preccurlyeq s^2\preccurlyeq\cdots\preccurlyeq s^r$ with $|s^r-s^{r-1}|\to 0$.*

Notice that the HFSF $X=\{X(t),t\in\mathbb{R}^N\}$ defined in (1.1) is a special case of (2.1) with $g(t,\xi)=\kappa(e^{it\cdot\xi}-1)$ and $\Delta(d\xi)=|\xi|^{-\alpha H-N}\,d\xi$, where $d\xi$ is the Lebesgue measure on $\mathbb{R}^N$. For the sake of simplicity, we let $I=[\varepsilon,1]^N$, where $\varepsilon\in(0,1)$ is an arbitrary constant, throughout the rest of this article. The following theorem is our main result.

Theorem 2.2 *For any $\alpha\in(0,2)$ and $H\in(0,1)$, let $X=\{X(t),\,t\in\mathbb{R}^N\}$ be a harmonizable fractional α-stable field with values in $\mathbb{R}$ defined by (1.1). For any integer $m\ge 1$, there* *exists a constant $c_1 = c_1(m) > 0$, depending on α, H, N, m and I only, such that for all $t^0, t^1, \ldots, t^m \in I$, we have*

$$\big\| X(t^0) \big| X(t^1), \ldots, X(t^m) \big\|_\alpha \geq c_1 \min\left\{ |t^n - t^0|^H : 1 \leq n \leq m \right\}. \tag{2.11}$$

The proof of Theorem 2.2 is based on a Fourier analytic argument and will be given in Section 3. The key new ingredient is to bound the $L^\alpha(\mathbb{R}^N)$ (quasi) norm in (3.14) from below by the $L^2(\mathbb{R}^N)$ norm of a suitably constructed function [see (3.18)]. This allows us to overcome the difficulty in the case of $0 < \alpha < 1$ that is caused by the unavailability of the ordinary Hölder inequality.

We also mention that, in the case where $1 \leq \alpha < 2$, Xiao [24, Theorem 3.2] proved a stronger conclusion: the constant c_1 is independent of m. The method in [24] is different and it is not applicable when $0 < \alpha < 1$.

Remark 2.3 The conclusion of Theorem 2.2 can be extended. A careful inspection of the proof of Theorem 2.2, shows that it can be extended to any arbitrary stable field Y of the form (2.1) with $g(t, \xi) = e^{it\cdot\xi} - 1$ and $\Delta(d\xi) = \delta(\xi)\, d\xi$; where δ is a nonnegative continuous even function on $\mathbb{R}^N \setminus \{0\}$, satisfying the following two properties.

(i) There exists a constant $c > 0$, such that, for all $\xi \in \mathbb{R}^N \setminus \{0\}$, one has, $\delta(\xi) \leq c\, |\xi|^{-\alpha H - N}$.

(ii) There are two constants $c' > 0$ and $R > 0$, such that, the inequality $\delta(\xi) \geq c'\, |\xi|^{-\alpha H - N}$, holds for any $\xi \in \mathbb{R}^N$ with $|\xi| \geq R$.

Further extensions can be achieved by applying the methods in Luan and Xiao [13] and the comparison theorems of Nolan and Sinkala [16].

As a consequence of Theorem 2.2, we show that, for every $\alpha \in (0, 2)$, the harmonizable fractional α-stable field X has the property of local nondeterminism in Definition 2.1. This solves an open problem in Nolan [15, pages 406-407], and allows us to study, in Section 4, some fine properties of the sample functions of X.

Theorem 2.4 *For any $\alpha \in (0, 2)$ and $H \in (0, 1)$, the harmonizable fractional α-stable field $X = \{X(t),\, t \in \mathbb{R}^N\}$ is locally nondeterministic on I. Consequently, for any integer $r \geq 2$, there exists a constant $c_2 = c_2(r) > 0$, depending on α, H, N, r and I only, such that for all $s^1, \ldots, s^r \in I$ which are close enough and satisfy $s^1 \preccurlyeq s^2 \preccurlyeq \cdots \preccurlyeq s^r$ (i.e. satisfy (2.8)), the following inequality*

$$\Big\| b_1 X(s^1) + \sum_{j=2}^r b_j \big(X(s^j) - X(s^{j-1}) \big) \Big\|_\alpha \geq c_2 \Big(\big\| b_1 X(s^1) \big\|_\alpha + \sum_{j=2}^r \big\| b_j \big(X(s^j) - X(s^{j-1}) \big) \big\|_\alpha \Big) \tag{2.12}$$

holds for all $b_j \in \mathbb{R}$ $(j = 1, \ldots, r)$.

Proof Since X has stationary increments, using (1.3), we have $\|X(s)-X(t)\|_\alpha = |s-t|^H$ for all $s,t \in \mathbb{R}^N$. Thus condition (2.9) is satisfied. For any integer $r \geq 2$ and points $s^1,\ldots,s^r \in I$ which satisfy (2.8), Theorem 2.2 (in which one takes $m = r-1$ and $t^0 = s^r, t^1 = s^{r-1},\ldots,t^{r-1} = s^1$) gives

$$\big\|X(s^r)\big|X(s^1),\ldots,X(s^{r-1})\big\|_\alpha \geq c_1\,|s^r - s^{r-1}|^H.$$

Hence, for all sequence $s^1,\ldots,s^r \in I$ which satisfy (2.8) and $|s^r - s^{r-1}| \to 0$, we have

$$\liminf \frac{\big\|X(s^r)\big|X(s^1),\ldots,X(s^{r-1})\big\|_\alpha}{\|X(s^r)-X(s^{r-1})\|_\alpha} \geq c_1 > 0.$$

This proves that X is locally nondeterministic on I. This and Theorem 3.2 of Nolan [15] imply the second conclusion (2.12). □

3 Proof of Theorem 2.2

In order to prove Theorem 2.2, by (2.5), (2.6) and (2.7), it is sufficient to show that there exists a constant $c_1(m) > 0$ such that the inequality:

$$\Big\|X(t^0) - \sum_{n=1}^m b_n X(t^n)\Big\|_\alpha^\alpha \geq c_1^\alpha(m)\,\min\big\{|t^n - t^0|^{\alpha H} : 1 \leq n \leq m\big\} \tag{3.1}$$

holds for all $t^0,\, t^1,\ldots,t^m \in I = [\varepsilon,1]^N$ and all real numbers $b_1,\ldots,b_m$.

As an intermediate step, we first prove (3.1) under an extra condition (3.2). For clarity of presentation, we replace $b_1,\ldots,b_m$ in (3.1) by $a_1,\ldots,a_m$.

Proposition 3.1 *For any integer $m \geq 1$, there is a constant $c_3 = c_3(m) > 0$, depending on α, H, N, m and I only, such that for all t^0, $t^1,\ldots,t^m \in I$ and all real numbers $a_1,\ldots,a_m$ verifying*

$$\max\{|a_n| \,:\, 1 \leq n \leq m\} \leq 2, \tag{3.2}$$

one has

$$\Big\|X(t^0) - \sum_{n=1}^m a_n X(t^n)\Big\|_\alpha^\alpha \geq c_3(m)\,\min\big\{|t^n - t^0|^{\alpha H} \,:\, 1 \leq n \leq m\big\}. \tag{3.3}$$

For proving Proposition 3.1, we will make use of the following two lemmas.

Lemma 3.2 *We denote by h the step function from $\mathbb{R}$ to $\mathbb{R}$ defined by*

$$h := 4^{-1}\big(\mathbb{1}_{[0,1]} - \mathbb{1}_{[-1,0)}\big). \tag{3.4}$$

Let $(\tau_q)_{q\geq 1}$ be the sequence of the functions defined through the convolution products

$$\tau_1 := -(h * h) \;\; \text{and} \;\; \tau_{q+1} := \tau_q * \tau_1, \;\; \forall\, q \geq 1. \tag{3.5}$$

Then, for every $q \geq 1$, the following statements hold.

(i) The function τ_q is function on $\mathbb{R}$ with a compact support included in $[-2q, 2q]$.

(ii) Let $\widehat{\tau}_q$ be the Fourier transform of τ_q, defined, for every $v \in \mathbb{R}$, as $\widehat{\tau}_q(v) := \int_{\mathbb{R}} e^{-ivx}\tau(x)\,dx$; then, it satisfies,

$$\widehat{\tau}_q(0) = 0 \text{ and } \widehat{\tau}_q(v) = v^{-2q}\sin^{4q}\left(2^{-1}v\right), \ \ \forall\, v \in \mathbb{R}\setminus\{0\}. \tag{3.6}$$

(iii) One has $\tau_q(0) > 0$.

The following straightforward consequence of Lemma 3.2 is crucial for establishing Proposition 3.1.

Lemma 3.3 *Let*

$$q_0 := \lfloor H + N/\alpha + N/2 \rfloor + 1 \tag{3.7}$$

and $L_0 := 2q_0N^{1/2}$, where $\lfloor x \rfloor$ denotes the largest integer that is at most x. Let G be the continuous and compactly supported function from $\mathbb{R}^N$ to $\mathbb{R}$, defined as the following tensor product:

$$G(s) := \prod_{n=1}^{N} \tau_{q_0}(L_0 s_n), \quad \forall s = (s_1, \ldots, s_N) \in \mathbb{R}^N, \tag{3.8}$$

where the function τ_{q_0} is defined in Lemma 3.2. Then, the following statements hold.

(i) The support of G is included in the interval $\big[-N^{-1/2}, N^{-1/2}\big]^N$ and $G(0) > 0$.

(ii) The Fourier transform of G, denoted by $\widehat{G}$, takes its values in $[0,1]$. Moreover, there exists a finite positive constant c_4', only depending on N, H, and α, such that

$$\widehat{G}(\eta) \le c_4' \min\left\{|\eta|^{2q_0}, |\eta|^{-2q_0}\right\}, \quad \forall\, \eta \in \mathbb{R}^N. \tag{3.9}$$

Proof of Lemma 3.2 Let us first show that part (i) holds. We proceed by induction. By using (3.4) and the definition of τ_1, we can verify that for all $x \in \mathbb{R}$,

$$\tau_1(x) = 16^{-1}\sum_{k=0}^{2}(-1)^{k+1}\binom{2}{k}\max\left\{0, 1 - |x-1+k|\right\},$$

where $\binom{2}{k}$'s are the usual binomial coefficients. Therefore τ_1 is a continuous function on $\mathbb{R}$, supported by $[-2,2]$. Next we assume that for some $q \ge 1$, τ_q is a continuous function on $\mathbb{R}$ and $\mathrm{supp}(\tau_q) \subseteq [-2q, 2q]$. Then, it follows from the second equality in (3.5) that

$$\tau_{q+1}(x) = (\tau_q * \tau_1)(x) = \int_{\mathbb{R}} \tau_q(y)\tau_1(x-y)\,dx, \ \forall\, x \in \mathbb{R}. \tag{3.10}$$

By the Dominated Convergence Theorem, we see that τ_{q+1} is a continuous function on $\mathbb{R}$. Moreover the information on the supports of τ_1 and τ_q implies $\mathrm{supp}(\tau_{q+1}) \subseteq [-2(q+1), 2(q+1)]$.

To prove part (*ii*), we observe that (3.5) can be expressed in the Fourier domain as

$$\widehat{\tau}_1(v) = -\big(\widehat{h}(v)\big)^2 \text{ and } \widehat{\tau}_{q+1}(v) = \widehat{\tau}_q(v)\widehat{\tau}_1(v), \ \forall\, v \in \mathbb{R} \text{ and } q \geq 1. \tag{3.11}$$

It follows from (3.4) that, for all $v \in \mathbb{R} \setminus \{0\}$,

$$\widehat{h}(v) := \int_{\mathbb{R}} e^{-ivx} h(x)\, dx = -\frac{i}{2}\int_0^1 \sin(vx)\, dx = -iv^{-1}\sin^2\big(2^{-1}v\big). \tag{3.12}$$

Combining (3.11) and (3.12) we get (3.6); notice that the equality $\widehat{\tau}_q(0) = 0$ follows from the continuity of the function $\widehat{\tau}_q$. Finally, part (*iii*) follows directly from the Fourier inversion formula and (3.6). □

Now, we are ready to go into the heart of the proof of Proposition 3.1.

Proof of Proposition 3.1 For any arbitrary $m+1$ vectors $s^0, s^1, \ldots, s^m \in \mathbb{R}^N$, we set $\overline{s} := (s^0, s^1, \ldots, s^m) \in \mathbb{R}^{(m+1)N}$. For all $(\overline{s}, a, \xi) \in \mathbb{R}^{(m+1)N} \times \mathbb{R}^m \times \mathbb{R}^N$, define

$$\begin{aligned} F(\overline{s}, a, \xi) &:= e^{-is^0\cdot\xi}\Big(e^{is^0\cdot\xi} - 1 - \sum_{n=1}^m a_n\big(e^{is^n\cdot\xi} - 1\big)\Big) \\ &= 1 - \sum_{n=1}^m a_n e^{i(s^n - s^0)\cdot\xi} - \Big(1 - \sum_{n=1}^m a_n\Big)e^{-is^0\cdot\xi}, \end{aligned} \tag{3.13}$$

where $a_1, \ldots, a_m$ are the coordinates of the vector $a \in \mathbb{R}^m$. Then we derive from (1.1), (2.5), and (3.13) that, for any $\overline{t} \in I^{m+1}$ and $a \in \mathbb{R}^m$, the left-hand side of (3.3) can be written as

$$\Big\| X(t^0) - \sum_{n=1}^m a_n X(t^n)\Big\|_\alpha^\alpha = \kappa^\alpha \int_{\mathbb{R}^N} \Big(\big|F(\overline{t}, a, \xi)\big||\xi|^{-H-N/\alpha}\Big)^\alpha d\xi. \tag{3.14}$$

For proving (3.3), there is no loss of generality to assume $\min\big\{|t^n - t^0| \,:\, 1 \leq n \leq m\big\} > 0$. Otherwise the inequality holds automatically. Let $\Upsilon_{\overline{t},\varepsilon}$ be the positive number defined by

$$\Upsilon_{\overline{t},\varepsilon}^{-1} = \varepsilon \min\big\{|t^n - t^0| \,:\, 1 \leq n \leq m\big\}. \tag{3.15}$$

Making the change of variable $\eta = \Upsilon_{\overline{t},\varepsilon}^{-1}\xi$ in (3.14), we see that it is enough to show that there exists a constant $c_4 > 0$, depending on α, H, N, m and ε only, such that the inequality

$$\int_{\mathbb{R}^N} \Big(\big|F(\Upsilon_{\overline{t},\varepsilon}\overline{t}, a, \eta)\big||\eta|^{-H-N/\alpha}\Big)^\alpha d\eta \geq c_4 \tag{3.16}$$

holds for all $a \in \mathcal{K} := [-2, 2]^m$.

Let G be the function defined in Lemma 3.3. Since its Fourier transform $\widehat{G}$ takes values in $[0,1]$, we have

$$\int_{\mathbb{R}^N} \Big(\big|F(\Upsilon_{\bar{t},\varepsilon}\bar{t},a,\eta)\big||\eta|^{-H-N/\alpha}\Big)^{\alpha} d\eta \geq \int_{\mathbb{R}^N} \Big(\big|F(\Upsilon_{\bar{t},\varepsilon}\bar{t},a,\eta)\big||\eta|^{-H-N/\alpha}\sqrt{\widehat{G}(\eta)}\Big)^{\alpha} d\eta. \quad (3.17)$$

Next, we set

$$c_5 := \sup\Big\{\big|F(\bar{s},a,\eta)\big||\eta|^{-H-N/\alpha}\sqrt{\widehat{G}(\eta)} : \ (\bar{s},a,\eta)\in\mathbb{R}^{(m+1)N}\times\mathcal{K}\times\mathbb{R}^N\Big\},$$

with the convention that $|0|^{-H-N/\alpha}\sqrt{\widehat{G}(0)} := 0$. It follows from (3.13) and Lemma 3.3 that the constant c_5 depends on α, H, N, m and ε only, and satisfies $0 < c_5 < +\infty$. Thus,

$$0 \leq c_5^{-1}\big|F(\bar{s},a,\eta)\big||\eta|^{-H-N/\alpha}\sqrt{\widehat{G}(\eta)} \leq 1$$

for all $(\bar{s},a,\eta)\in\mathbb{R}^{(m+1)N}\times\mathcal{K}\times\mathbb{R}^N$ which, in turn, entails that

$$\begin{aligned}\int_{\mathbb{R}^N} &\Big(\big|F(\Upsilon_{\bar{t},\varepsilon}\bar{t},a,\eta)\big||\eta|^{-H-N/\alpha}\sqrt{\widehat{G}(\eta)}\Big)^{\alpha} d\eta \\ &\geq c_5^{\alpha-2}\int_{\mathbb{R}^N}\Big(\big|F(\Upsilon_{\bar{t},\varepsilon}\bar{t},a,\eta)\big||\eta|^{-H-N/\alpha}\sqrt{\widehat{G}(\eta)}\Big)^{2} d\eta.\end{aligned} \quad (3.18)$$

By using the Cauchy-Schwarz inequality, we get that

$$c_6\int_{\mathbb{R}^N}\Big(\big|F(\Upsilon_{\bar{t},\varepsilon}\bar{t},a,\eta)\big||\eta|^{-H-N/\alpha}\sqrt{\widehat{G}(\eta)}\Big)^{2} d\eta \geq \Big|\int_{\mathbb{R}^N} F(\Upsilon_{\bar{t},\varepsilon}\bar{t},a,\eta)\widehat{G}(\eta)\,d\eta\Big|^2, \quad (3.19)$$

where c_6 is the positive and finite constant (thanks to Lemma 3.3) defined by

$$c_6 := \int_{\mathbb{R}^N}\Big(|\eta|^{H+N/\alpha}\sqrt{\widehat{G}(\eta)}\Big)^2 d\eta.$$

On the other hand, in view of (3.13) and the Fourier inversion formula, we have,

$$\begin{aligned}&\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} F(\Upsilon_{\bar{t},\varepsilon}\bar{t},a,\eta)\widehat{G}(\eta)\,d\eta \\ &\quad = G(0) - \sum_{n=1}^{N} a_n G\big(\Upsilon_{\bar{t},\varepsilon}(t^n-t^0)\big) - \Big(1-\sum_{n=1}^{m} a_n\Big)G\big(-\Upsilon_{\bar{t},\varepsilon}t^0\big).\end{aligned} \quad (3.20)$$

Observe that (3.15) implies that

$$\big|\Upsilon_{\bar{t},\varepsilon}(t^n-t^0)\big| \geq \varepsilon^{-1} > 1,\ \forall\, n=1,\ldots,m \text{ and } \big|\Upsilon_{\bar{t},\varepsilon}t^0\big| > 1. \quad (3.21)$$

Thus, in view of (3.21) and Part (i) of Lemma 3.3, the equation (3.20) reduces to

$$\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} F(\Upsilon_{\bar{t},\varepsilon}\bar{t},a,\eta)\widehat{G}(\eta)\,d\eta = G(0). \quad (3.22)$$

By taking $c_4 = c_5^{\alpha-2}c_6^{-1}(2\pi)^{2N}\big|G(0)\big|^2$, we see that (3.16) follows from (3.17), (3.18), (3.19) and (3.22). □

Finally we can finish the proof of Theorem 2.2.

Proof of (3.1) In view of Proposition 3.1, we may and will assume that

$$\max\{|b_n| \,:\, 1 \le n \le m\} > 2. \tag{3.23}$$

In order to show that (3.1) holds for all $\{b_n\}$ which satisfy (3.23), we will argue by induction. First we suppose $m = 1$, then using the inequality $|b_1| > 2$ (see (3.23)) and Proposition 3.1, in the case where $m = 1$ and t^0 and t^1 are interchanged, one gets

$$\left\|X(t^0) - b_1 X(t^1)\right\|_\alpha^\alpha = |b_1|^\alpha \left\|X(t^1) - \frac{1}{b_1} X(t^0)\right\|_\alpha^\alpha > c_3(1)|t^0 - t^1|^{\alpha H}.$$

Hence, for $m = 1$, (3.1) is valid with $c_1^\alpha(1) = c_3(1)$. Next we suppose that (3.1) holds when m is replaced by $m-1$ and the corresponding constant $c_1(m-1)$. In order to prove (3.1), we let c_7 be the constant defined as

$$c_7 := 2^{-1}\, c_1^\alpha(m-1) \tag{3.24}$$

and let n_0 be an element of $\{1, \ldots, m\}$ such that

$$|b_{n_0}| = \max\{|b_n| \,:\, 1 \le n \le m\}. \tag{3.25}$$

In the sequel, we will distinguish the following two cases:

$$|b_{n_0}|^\alpha \min_{0 \le n \le m,\, n \ne n_0} |t^n - t^{n_0}|^{\alpha H} \ge c_7 \min_{1 \le n \le m} |t^n - t^0|^{\alpha H} \tag{3.26}$$

or

$$|b_{n_0}|^\alpha \min_{0 \le n \le m,\, n \ne n_0} |t^n - t^{n_0}|^{\alpha H} < c_7 \min_{1 \le n \le m} |t^n - t^0|^{\alpha H}. \tag{3.27}$$

First, we assume that (3.26) holds. Let $a_0 = b_{n_0}^{-1}$ and $a_n = -b_n b_{n_0}^{-1}$ for all $n \in \{1, \ldots, m\} \setminus \{n_0\}$. In view of (3.23) and (3.25), one has

$$|a_n| \le 1 \quad \text{for every } n \in \{0, \ldots, m\} \setminus \{n_0\}; \tag{3.28}$$

also observe that

$$\left\|X(t^0) - \sum_{n=1}^m b_n X(t^n)\right\|_\alpha^\alpha = |b_{n_0}|^\alpha \left\|X(t^{n_0}) - \sum_{n=0, n \ne n_0}^m a_n X(t^n)\right\|_\alpha^\alpha. \tag{3.29}$$

Thanks to (3.28) we can apply Proposition 3.1 to $\left\|X(t^{n_0}) - \sum_{n=0, n\ne n_0}^m a_n X(t^n)\right\|_\alpha^\alpha$ (notice that in this case t^0 and t^{n_0} are interchanged) to obtain

$$\left\|X(t^{n_0}) - \sum_{n=0, n\ne n_0}^m a_n X(t^n)\right\|_\alpha^\alpha \ge c_3(m) \min\left\{|t^n - t^{n_0}|^{\alpha H} \,:\, 0 \le n \le m \text{ and } n \ne n_0\right\}. \tag{3.30}$$

Putting together (3.29), (3.30) and (3.26), one gets

$$\Big\| X(t^0) - \sum_{n=1}^{m} b_n X(t^n) \Big\|_{\alpha}^{\alpha} \geq c_7 c_3(m) \, \min\big\{ |t^n - t^0|^{\alpha H} \, : \, 1 \leq n \leq m \big\}. \tag{3.31}$$

Next we consider the case when (3.27) holds. Let $n_1 \in \{0, \ldots, m\} \setminus \{n_0\}$ be such that

$$|t^{n_1} - t^{n_0}|^{\alpha H} = \min\big\{ |t^n - t^{n_0}|^{\alpha H} \, : \, 0 \leq n \leq m \text{ and } n \neq n_0 \big\}. \tag{3.32}$$

From now on we restrict to $\alpha \in (0,1]$, the case where $\alpha \in (1,2)$ can be treated in the same way, but one has to replace $\|\cdot\|_{\alpha}^{\alpha}$ by $\|\cdot\|_{\alpha}$ in order to be able to use the triangle inequality. Using the latter inequality, the fact that $\{X(t), \, t \in \mathbb{R}^N\}$ has stationary increments, and $X(0) = 0$, one gets

$$\begin{aligned}
&\Big\| X(t^0) - \sum_{n=1}^{m} b_n X(t^n) \Big\|_{\alpha}^{\alpha} \\
&= \Big\| X(t^0) - \sum_{n=1, n \neq n_0, n_1}^{m} b_n X(t^n) - (b_{n_1} + b_{n_0}) X(t^{n_1}) - b_{n_0}\big(X(t^{n_0}) - X(t^{n_1}) \big) \Big\|_{\alpha}^{\alpha} \\
&\geq \Big\| X(t^0) - \sum_{n=1, n \neq n_0, n_1}^{m} b_n X(t^n) - (b_{n_1} + b_{n_0}) X(t^{n_1}) \Big\|_{\alpha}^{\alpha} - \Big\| b_{n_0}\big(X(t^{n_0}) - X(t^{n_1}) \big) \Big\|_{\alpha}^{\alpha} \\
&\geq \Big\| X(t^0) - \sum_{n=1, n \neq n_0, n_1}^{m} b_n X(t^n) - (b_{n_1} + b_{n_0}) X(t^{n_1}) \Big\|_{\alpha}^{\alpha} - |b_{n_0}|^{\alpha} \big\| X(t^{n_0} - t^{n_1}) \big\|_{\alpha}^{\alpha},
\end{aligned} \tag{3.33}$$

with the convention that $\sum_{n=1, n \neq n_0, n_1}^{m} b_n X(t^n) = 0$ when $m = 2$. Next notice that the induction hypothesis entails

$$\begin{aligned}
&\Big\| X(t^0) - \sum_{n=1, n \neq n_0, n_1}^{m} b_n X(t^n) - (b_{n_1} + b_{n_0}) X(t^{n_1}) \Big\|_{\alpha}^{\alpha} \\
&\geq c_1^{\alpha}(m-1) \, \min\big\{ |t^0 - t^n|^{\alpha H} \, : \, 1 \leq n \leq m \text{ and } n \neq n_0 \big\} \\
&\geq c_1^{\alpha}(m-1) \, \min\big\{ |t^0 - t^n|^{\alpha H} \, : \, 1 \leq n \leq m \big\}.
\end{aligned} \tag{3.34}$$

Recall from (1.3) that $\big\| X(t^{n_0} - t^{n_1}) \big\|_{\alpha}^{\alpha} = |t^{n_0} - t^{n_1}|^{\alpha H}$. Putting together (3.33), (3.34), (3.32), (3.24) and (3.27) yields

$$\Big\| X(t^0) - \sum_{n=1}^{m} b_n X(t^n) \Big\|_{\alpha}^{\alpha} \geq c_7 \min\big\{ |t^0 - t^n|^{\alpha H} \, : \, 1 \leq n \leq m \big\}. \tag{3.35}$$

Finally, setting $c_1^{\alpha}(m) = \min\big\{ c_3(m), \, c_7 c_3(m), \, c_7 \big\}$, in view of (3.31) and (3.35), it follows that (3.1) holds. □

4 Joint continuity of the local times

In this section we consider an $\mathbb{R}^d$-valued harmonizable fractional α-stable field $\vec{X} = \{\vec{X}(t), t \in \mathbb{R}^N\}$ defined by

$$\vec{X}(t) = \big(X_1(t), \dots, X_d(t)\big), \qquad t \in \mathbb{R}^N, \tag{4.1}$$

where $X_1, \dots, X_d$ are independent copies of a real-valued HFαSF X defined in (1.1). The i.i.d. assumption on the coordinate processes $X_1, \dots, X_d$ can be relaxed. For example, the results in this section can be extended to the case where $X_1, \dots, X_d$ are assumed to be independent, but may have different stability and self-similarity indices (α_j, H_j) for $j = 1, \dots, d$. However, removing the independence assumption on $X_1, \dots, X_d$ would require more work. In particular, one needs to develop an appropriate notion of local nondeterminism for vector-valued stable random fields.

We apply the property of the local nondeterminism to establish the joint continuity and Hölder conditions for the local times of HFSF $\vec{X}$, and to prove a uniform Hausdorff dimension result for the inverse image of $\vec{X}$.

For any fixed closed interval (or rectangle) $T \subset \mathbb{R}^N$, the occupation measure of $\vec{X}$ on T, denoted by μ_T, is the Borel measure on $\mathbb{R}^d$ defined by $\mu_T(\bullet) = \lambda_N\big\{t \in T : \vec{X}(t) \in \bullet\big\}$. If μ_T is almost surely absolutely continuous with respect to the Lebesgue measure λ_d, then $\vec{X}$ is said to have local times on T, and its local time $L(\cdot, T)$ is defined as the Radon–Nikodým derivative of μ_T with respect to λ_d, i.e.,

$$L(x, T) = \frac{d\mu_T}{d\lambda_d}(x), \qquad \forall x \in \mathbb{R}^d.$$

In the above, x is the *space variable*, and T is the *time variable*. Sometimes, we write $L(x, t)$ in place of $L(x, [0, t])$.

Express the interval T as $T = \prod_{i=1}^N [a_i, a_i + h_i]$ where $a_i \in \mathbb{R}$ and $h_i \in \mathbb{R}_+$, for all $i = 1, \dots, N$. Then $\vec{X}$ is said to have a jointly continuous local time on T if we can choose a version of the local time, still denoted by $L\big(x, \prod_{i=1}^N [a_i, a_i + t_i]\big)$, such that it is a continuous function of $(x, t_1, \cdots, t_N) \in \mathbb{R}^d \times \prod_{i=1}^N [0, h_i]$. We refer to Geman and Horowitz [9] and Dozzi [7] for further information on local times of multivariate random fields.

It is shown in Adler [1] that, when a local time is jointly continuous, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on the level set $\vec{X}_T^{-1}(x) = \{t \in T : \vec{X}(t) = x\}$. This makes local times useful in studying the fractal properties of level sets and inverse images of the random field $\vec{X}$. See Berman [3], Ehm [8], Monrad and Pitt [14], and Xiao [21, 22] for previous results.

By using a Fourier analytic argument (cf. [9, Theorem 21.9]), it is easy to prove the following existence result. Note that the condition is exactly the same as for fractional Brownian motion.

Proposition 4.1 *Let $\vec{X} = \{\vec{X}(t), t \in \mathbb{R}^N\}$ be the $\mathbb{R}^d$-valued harmonizable fractional α-stable field defined above, and let $\mathbb{P}$ be the probability measure on the underlying probability space. Then $\vec{X}$ has a local time $L(x,T) \in L^2(\mathbb{P} \times \lambda_d)$ if and only if $N > Hd$.*

The main purpose of this section is to establish the joint continuity of local times of $\vec{X}$. The following theorem extends the results of Pitt [17] for fractional Brownian motion (i.e., $\alpha = 2$) and Nolan [15] for $1 \le \alpha < 2$.

Theorem 4.2 *Let $\vec{X} = \{\vec{X}(t), t \in \mathbb{R}^N\}$ be the $\mathbb{R}^d$-valued harmonizable fractional α-stable field defined in (4.1). If $N > Hd$, then for any closed interval $T \subset \mathbb{R}^N$, $\vec{X}$ has a jointly continuous local time on T.*

Proof Similar to [4, 17, 9, 15], the proof of Theorem 4.2 is based on a multiparameter version of Kolmogorov continuity theorem (cf. [10]) and the following two estimates:

(i). For all integers $n \ge 1$, there exists a finite constant c_8, which depends on n, such that for all hypercubes $B = [a, a + \langle r \rangle] \subseteq T$ with side-length $r \in (0,1)$ and all $x \in \mathbb{R}^d$,

$$\mathbb{E}\big[L(x,B)^n\big] \le c_8\, r^{n(N-Hd)}. \tag{4.2}$$

(ii). For all even integers $n \ge 2$ and $\gamma \in (0,\, 1 \wedge \frac{1}{2}(\frac{N}{Hd} - 1))$, there exists a finite constant c_9, which depends on n and γ, such that for all hypercubes $B = [a, a + \langle r \rangle] \subseteq T$ with side-length $r \in (0,1)$ and all $x, y \in \mathbb{R}^d$ with $|x - y| \le 1$,

$$\mathbb{E}\Big[\big(L(x,B) - L(y,B)\big)^n\Big] \le c_9\, |x-y|^{n\gamma}\, r^{n(N-H(d+\gamma))}. \tag{4.3}$$

For proving (i) and (ii), let us recall the following identities from Geman and Horowitz [9] (see also [17]): For all $x, y \in \mathbb{R}^d$, $B \subseteq T$ and all integers $n \ge 1$,

$$\begin{aligned}\mathbb{E}\Big[L(x,B)^n\Big] \;&=\; (2\pi)^{-nd} \int_{B^n} \int_{\mathbb{R}^{nd}} \exp\Big(-i \sum_{j=1}^n u^j \cdot x\Big) \\ &\qquad \times \mathbb{E}\exp\Big(i \sum_{j=1}^n u^j \cdot \vec{X}(t^j)\Big)\, d\overline{u}\, d\overline{t}\end{aligned} \tag{4.4}$$

and for all even integers $n \ge 2$,

$$\begin{aligned}\mathbb{E}\Big[\big(L(x,B) - L(y,B)\big)^n\Big] &= (2\pi)^{-nd} \int_{B^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \Big[e^{-iu^j \cdot x} - e^{-iu^j \cdot y}\Big] \\ &\qquad \times \mathbb{E}\exp\Big(i \sum_{j=1}^n u^j \cdot \vec{X}(t^j)\Big)\, d\overline{u}\, d\overline{t},\end{aligned} \tag{4.5}$$

where $\overline{u} = (u^1, \ldots, u^n)$, $\overline{t} = (t^1, \ldots, t^n)$, and each $u^j \in \mathbb{R}^d$, $t^j \in B \subseteq (0, \infty)^N$. In the coordinate notation we then write $u^j = (u_1^j, \ldots, u_d^j)$.

In order to apply the property of local nondeterminism in Theorem 2.4 to bound the integrals in (4.4) and (4.5), we observe that for any constant $c > 0$ and $t^0 \in (0, \infty)^N$,

$$\mathbb{E} \exp\left(i \sum_{j=1}^n u^j \cdot \vec{X}(t^j)\right) = \mathbb{E} \exp\left(i \sum_{j=1}^n c^H\, u^j \cdot (\vec{X}((t^j + t^0)/c) - \vec{X}(t^0/c))\right)$$

thanks to the stationarity of increments and self-similarity of $\vec{X}$. It can be seen that, given any compact interval $T \subset \mathbb{R}^N$, we can choose a positive constant $c > 0$ and $t^0 \in (0, \infty)^N$ such that $t^0/c \in I = [\varepsilon, 1]^N$ and $(t + t^0)/c \in I$ for each $t \in T$. The change of variables

$$v^j = c^H u^j, \quad s^j = \frac{t^j + t^0}{c}, \quad j = 1, \ldots, n$$

only induces a constant factor to the right-hand sides of (4.4) and (4.5). Moreover, the extra term $\vec{X}(t^0/c)$ does not cause any inconvenience for our estimation if we write $s^0 = t^0/c$ and

$$\sum_{j=1}^n v^j \cdot (\vec{X}(s^j) - \vec{X}(s^0)) = \sum_{j=1}^n w^j \cdot (\vec{X}(s^j) - \vec{X}(s^{j-1})),$$

where $w^j = \sum_{k=j}^n v^k$. Hence, without loss of generality, we may and will assume $T = [\varepsilon, 1]^N$.

With the above reduction, the rest of the proof of (i) and (ii) is essentially the same as in [17, 15] (see also [24]), where the fact that the coordinate processes of $\vec{X}$ are independent copies of X and Theorem 2.4 play important roles. We will not reproduce the details here. $\square$

The following theorem provides the uniform Hölder condition for the local time $L(x, \bullet)$.

Theorem 4.3 *Let $\vec{X} = \{\vec{X}(t), t \in \mathbb{R}^N\}$ be the $\mathbb{R}^d$-valued harmonizable fractional α-stable field defined in (4.1). If $N > Hd$, then for any compact interval $T \subset \mathbb{R}^N$ and any $\eta > 0$, there is a constant c_{10} such that with probability 1,*

$$\sup_{a \in T} \sup_{x \in \mathbb{R}^d} L(x,\, [a, a + \langle r \rangle]) \leq c_{10}\, r^{N - Hd - \eta} \tag{4.6}$$

for $r > 0$ sufficiently small.

Proof The proof is standard; it is based on the moment estimates (4.2) and (4.3) and a chaining argument as in Ehm [8] (see also [21]). We omit the details. $\square$

Observe that (4.6) remains valid when $[a, a + \langle r\rangle]$ is replaced by

$$U_T(a, r) := \{s \in T : |s - a| \le r\}.$$

It is known that the Hölder condition for the local times of $\vec{X}$ is closely related to the irregularity of the sample paths of $\vec{X}$ (cf. [3, 8, 21]). As a corollary of Theorem 4.3 we have the following result for real-valued HFSF X.

Corollary 4.4 *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued harmonizable fractional α-stable field with Hurst index $H \in (0, 1)$ (see (1.1)). For any compact interval $T \subseteq \mathbb{R}^N$ and any $\eta > 0$, there is a finite constant $c_{11} > 0$ such that*

$$\liminf_{r\to 0} \inf_{t\in T} \sup_{s\in U_T(t,r)} \frac{|X(s) - X(t)|}{r^{H+\eta}} \ge c_{11} \qquad a.s. \tag{4.7}$$

In particular, the sample function (or sample path) $t \mapsto X(t, \omega)$ is almost surely nowhere differentiable in $\mathbb{R}^N$.

Proof In view of the definition of local time of X, we have, for all $t \in T$ and $r > 0$,

$$\begin{aligned}\lambda_N(U_T(t, r)) &= \int_{\overline{X((U_T(t,r))}} L(x, U_T(t, r))\, dx \\ &\le \max_{x\in\mathbb{R}} L(x, U_T(t, r)) \cdot \sup_{s',s''\in U_T(t,r)} |X(s') - X(s'')|.\end{aligned}$$

Then applying Theorem 4.3 with $d = 1$, we see that (4.7) follows from (4.6), in which $[a, a + \langle r\rangle]$ is replaced by $U_T(a, r)$. □

Remark 4.5 Compared with the exact uniform modulus of continuity and laws of the iterated logarithm for the local times of fractional Brownian motion in [2, 6, 21], (4.6) is not sharp. It is an open problem to establish the results in [2, 6, 21] for HFSF $\vec{X}$. Similarly, (4.7) is not sharp either. Both Chung-type law of the iterated logarithm and its uniform analogue have not been established for HFSF. We believe that, besides the strong local nondeterminism in [24], other significantly new tools will be needed for solving these problems.

We end this section with some fractal properties of $\vec{X}$. The Hausdorff dimensions of the image and graph of stable random fields have been studied in Xiao [20]. Applying the uniform modulus of continuity of HFSF in [11, 5, 23] and Theorem 3.1 in Xiao [20], we have that for any Borel set $E \subset \mathbb{R}^N$,

$$\dim_{\mathrm{H}} \vec{X}(E) = \min\left\{d, \frac{1}{H}\dim_{\mathrm{H}} E\right\}, \qquad \text{a.s.}$$

and for $\mathrm{Gr}\vec{X}(E) = \{(t, \vec{X}(t)) : t \in E\}$,

$$\dim_{\mathrm{H}} \mathrm{Gr}\vec{X}(E) = \min\left\{\dim_{\mathrm{H}} E + (1-H)d,\ \frac{1}{H}\dim_{\mathrm{H}} E\right\}, \quad \text{a.s.}$$

By applying Theorem 4.3 and Frostman's theorem (cf. [10]), one can show that for any Borel set $F \subset \mathbb{R}^d$,

$$\dim_{\mathrm{H}} \vec{X}^{-1}(F) = N - Hd + H\dim_{\mathrm{H}} F \quad \text{with positive probability.} \tag{4.8}$$

In the following, we prove a uniform version of (4.8) for $\vec{X}$; namely, under a general condition on F, (4.8) holds except on a null probability event which does not depend on F. This extends a result of Monrad and Pitt [14] for fractional Brownian motion.

Theorem 4.6 *Let $\vec{X} = \{\vec{X}(t), t \in \mathbb{R}^N\}$ be the $\mathbb{R}^d$-valued harmonizable fractional α-stable field defined in (4.1). If $N > Hd$, then with probability 1,*

$$\dim_{\mathrm{H}} \vec{X}^{-1}(F) = N - Hd + H\dim_{\mathrm{H}} F \tag{4.9}$$

for all Borel sets $F \subset \mathcal{O} := \bigcup_{a,b\in\mathbb{Q}_+^N : a<b}\{x \in \mathbb{R}^d : L(x, [a,b]) > 0\}$.

Proof For any compact interval $T \subset \mathbb{R}^N$, it follows from Theorem 4.2 that the local time of $\vec{X}$ on T, $L(x,T)$, is continuous and bounded in x. Moreover, by [5, 23], $\vec{X}(t)$ satisfies a.s. a uniform Hölder condition of any order $< H$ on T. Hence, by the proof of Lemma 3.1 in Monrad and Pitt [14], we have

$$\mathbb{P}\{\dim_{\mathrm{H}}\big(\vec{X}^{-1}(F)\cap T\big) \le N - Hd + H\dim_{\mathrm{H}} F \quad \text{for all Borel sets} \ \ F \subset \mathbb{R}^d\} = 1.$$

Since T is arbitrary, this proves the uniform upper bound in (4.9).

In order to prove that, for any Borel set $F \subset \mathcal{O}$,

$$\dim_{\mathrm{H}} \vec{X}^{-1}(F) \ge N - Hd + H\dim_{\mathrm{H}} F, \tag{4.10}$$

we assume $\dim_{\mathrm{H}} F > 0$ (the case $\dim_{\mathrm{H}} F = 0$ is trivial, we take an $x \in F$, (4.10) follows from Theorem 4.3 and Frostman's theorem). For any $\gamma \in (0, \dim_{\mathrm{H}} F)$, by Frostman's lemma, there is a probability measure σ on (a compact subset of) F such that $\sigma(U_F(x,r)) \le c_{12}\, r^\gamma$ for all $x \in \mathbb{R}^d$, where $c_{12} > 0$ is a constant. As in [14], we define a random measure μ on $\mathbb{R}^N$ by

$$\mu(B) = \int_{\mathbb{R}^d} L(x,B)\,\sigma(dx), \quad \forall \ \text{Borel set} \ \ B \subset \mathbb{R}^N.$$

Since $F \subset \mathcal{O}$, we can verify that μ is a positive random measure and is carried by $X^{-1}(F)$. (Here we use the facts that σ is supported on a compact subset of F and $L(x,B) = 0$ if $x \notin \overline{\vec{X}(B)}$.)

Now, let $\eta > 0$ be an arbitrarily small constant. For any $a \in \mathbb{R}^N$, we take $B = [a, a + \langle r \rangle]$. Since $L(x, B) = 0$ when $x \notin \overline{\vec{X}(B)}$, and $\operatorname{diam}\big(\overline{\vec{X}(B)}\big) = \mathcal{O}(r^{H-\eta})$, we apply Theorem 4.3 to obtain

$$\mu([a, a + \langle r \rangle])) \le c_{10}\, r^{N-Hd-\eta} \int_{\overline{\vec{X}(B)}} \sigma(dx) \le c_{13}\, r^{N-Hd+H\gamma-(1+\gamma)\eta}$$

for r small enough. This, together with Frostman's theorem, implies that $\dim_{\mathrm{H}} X^{-1}(F) \ge N - Hd + H\gamma - (1+\gamma)\eta$; thus one gets (4.10), which completes the proof. □

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